59. ***Big saver.*** Suppose you deposit one cent into your piggy bank on the first day of December and, on each day of December after that, you deposit twice as much as on the previous day. How much will you have in the bank after the last deposit?

60. ***Big family.*** Consider yourself, your parents, your grandparents, your great-grandparents, your great-great-grandparents, and so on, back to your grandparents with the word "great" used in front 40 times. What is the total number of people you are considering?

61. ***Total economic impact.*** In Exercise 43 of Section 14.1 we described a factory that spends \$1 million annually in a community in which 80% of the money received is respent in the community. Economists assume the money is respent again and again at the 80% rate. The total economic impact of the factory is the total of all of this spending. Find an approximation for the total by using the formula for the sum of an infinite geometric series with a rate of 80%.

62. ***Less impact.*** Repeat Exercise 61, assuming money is respent again and again at the 50% rate.

GETTING MORE INVOLVED

63. ***Discussion.*** Which of the following sequences is not a geometric sequence? Explain your answer.

a) $1, 2, 4, \ldots$

b) $0.1, 0.01, 0.001, \ldots$

c) $-1, 2, -4, \ldots$

d) $2, 4, 6, \ldots$

64. ***Discussion.*** The repeating decimal number 0.44444 . . . can be written as

$$\frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \cdots,$$

an infinite geometric series. Find the sum of this geometric series.

65. ***Discussion.*** Write the repeating decimal number 0.24242424 . . . as an infinite geometric series. Find the sum of the geometric series.

14.5 BINOMIAL EXPANSIONS

In this section

- Some Examples
- Obtaining the Coefficients
- The Binomial Theorem

In Chapter 5 you learned how to square a binomial. In this section you will study higher powers of binomials.

Some Examples

We know that $(x + y)^2 = x^2 + 2xy + y^2$. To find $(x + y)^3$, we multiply $(x + y)^2$ by $x + y$:

$$\begin{aligned}(x + y)^3 &= (x^2 + 2xy + y^2)(x + y)\\ &= (x^2 + 2xy + y^2)x + (x^2 + 2xy + y^2)y\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

The sum $x^3 + 3x^2y + 3xy^2 + y^3$ is called the **binomial expansion** of $(x + y)^3$. If we again multiply by $x + y$, we will get the binomial expansion of $(x + y)^4$. This method is rather tedious. However, if we examine these expansions, we can find a pattern and learn how to find binomial expansions without multiplying.

Consider the following binomial expansions:

$$
\begin{aligned}
(x + y)^0 &= 1 \\
(x + y)^1 &= x + y \\
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}
$$

Observe that the exponents on the variable x are decreasing, whereas the exponents on the variable y are increasing, as we read from left to right. Also notice that the sum of the exponents in each term is the same for that entire line. For instance, in the fourth expansion the terms x^4, x^3y, x^2y^2, xy^3, and y^4 all have exponents with a sum of 4. If we continue the pattern, the expansion of $(x + y)^6$ will have seven terms containing x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, and y^6. Now we must find the pattern for the coefficients of these terms.

Obtaining the Coefficients

If we write out only the coefficients of the expansions that we already have, we can easily see a pattern. This triangular array of coefficients for the binomial expansions is called **Pascal's triangle.**

$$
\begin{array}{ccccccccccc|l}
 & & & & & 1 & & & & & & (x + y)^0 = 1 \\
 & & & & 1 & & 1 & & & & & (x + y)^1 = 1x + 1y \\
 & & & 1 & & 2 & & 1 & & & & (x + y)^2 = 1x^2 + 2xy + 1y^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & (x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & (x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \text{Coefficients in } (x + y)^5
\end{array}
$$

Notice that each line starts and ends with a 1 and that each entry of a line is the sum of the two entries above it in the previous line. For instance, $4 = 3 + 1$, and $10 = 6 + 4$. Following this pattern, the sixth and seventh lines of coefficients are

$$
\begin{array}{ccccccccccccccc}
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Pascal's triangle gives us an easy way to get the coefficients for the binomial expansion with small powers, but it is impractical for larger powers. For larger powers we use a formula involving **factorial notation.**

$n!$ (n factorial)

If n is a positive integer, $n!$ (read "n factorial") is defined to be the product of all of the positive integers from 1 through n.

calculator close-up

You can evaluate the coefficients using either the factorial notation or ${}_nC_r$. The factorial symbol and ${}_nC_r$ are found in the MATH menu under PRB.

```
4!/(1!3!)
                4
4 nCr 3
                4
```

For example, $3! = 3 \cdot 2 \cdot 1 = 6$, and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. We also define $0!$ to be 1.

Before we state a general formula, consider how the coefficients for $(x + y)^4$ are found by using factorials:

$$\frac{4!}{4!\,0!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 1 \quad \text{Coefficient of } x^4 \text{ (or } x^4y^0\text{)}$$

$$\frac{4!}{3!\,1!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1} = 4 \quad \text{Coefficient of } 4x^3y$$

$$\frac{4!}{2!\,2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6 \quad \text{Coefficient of } 6x^2y^2$$

$$\frac{4!}{1!\,3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2 \cdot 1} = 4 \quad \text{Coefficient of } 4xy^3$$

$$\frac{4!}{0!\,4!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1 \quad \text{Coefficient of } y^4 \text{ (or } x^0y^4\text{)}$$

Note that each expression has 4! in the numerator, with factorials in the denominator corresponding to the exponents on x and y.

The Binomial Theorem

We now summarize these ideas in the **binomial theorem.**

The Binomial Theorem

In the expansion of $(x + y)^n$ for a positive integer n, there are $n + 1$ terms, given by the following formula:

$$(x + y)^n = \frac{n!}{n!\,0!}x^n + \frac{n!}{(n-1)!\,1!}x^{n-1}y + \frac{n!}{(n-2)!\,2!}x^{n-2}y^2 + \cdots + \frac{n!}{0!\,n!}y^n$$

The notation $\binom{n}{r}$ is often used in place of $\frac{n!}{(n-r)!r!}$ in the binomial expansion. Using this notation, we write the expansion as

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n.$$

Another notation for $\frac{n!}{(n-r)!r!}$ is ${}_nC_r$. Using this notation, we have

$$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_ny^n.$$

EXAMPLE 1 **Calculating the binomial coefficients**

Evaluate each expression.

a) $\dfrac{7!}{4!3!}$ **b)** $\dfrac{10!}{8!2!}$

Solution

a) $\dfrac{7!}{4!3!} = \dfrac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 3 \cdot 2 \cdot 1} = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

b) $\dfrac{10!}{8!2!} = \dfrac{10 \cdot 9 \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 2 \cdot 1} = \dfrac{10 \cdot 9}{2 \cdot 1} = 45$ ■

EXAMPLE 2

Using the binomial theorem

Write out the first three terms of $(x + y)^9$.

Solution

$$(x + y)^9 = \frac{9!}{9!0!}x^9 + \frac{9!}{8!1!}x^8y + \frac{9!}{7!2!}x^7y^2 + \cdots = x^9 + 9x^8y + 36x^7y^2 + \cdots$$ ■

EXAMPLE 3

Using the binomial theorem

Write the binomial expansion for $(x^2 - 2a)^5$.

Solution

We expand a difference by writing it as a sum and using the binomial theorem:

$$\begin{aligned}(x^2 - 2a)^5 &= (x^2 + (-2a))^5 \\ &= \frac{5!}{5!0!}(x^2)^5 + \frac{5!}{4!1!}(x^2)^4(-2a)^1 + \frac{5!}{3!2!}(x^2)^3(-2a)^2 + \frac{5!}{2!3!}(x^2)^2(-2a)^3 \\ &\quad + \frac{5!}{1!4!}(x^2)(-2a)^4 + \frac{5!}{0!5!}(-2a)^5 \\ &= x^{10} - 10x^8a + 40x^6a^2 - 80x^4a^3 + 80x^2a^4 - 32a^5\end{aligned}$$ ■

EXAMPLE 4

Finding a specific term

Find the fourth term of the expansion of $(a + b)^{12}$.

Solution

The variables in the first term are $a^{12}b^0$, those in the second term are $a^{11}b^1$, those in the third term are $a^{10}b^2$, and those in the fourth term are a^9b^3. So

$$\frac{12!}{9!3!}a^9b^3 = 220a^9b^3.$$

The fourth term is $220a^9b^3$. ■

Using the ideas of Example 4, we can write a formula for any term of a binomial expansion.

calculator close-up

Because ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, we have

${}_{12}C_9 = \dfrac{12!}{3!9!}$ and ${}_{12}C_3 = \dfrac{12!}{9!3!}$.

So there is more than one way to compute 12!/(9!3!):

```
12!/(9!3!)
                220
12 nCr 9
                220
12 nCr 3
                220
```

Formula for the kth Term of $(x + y)^n$

For k ranging from 1 to $n + 1$, the kth term of the expansion of $(x + y)^n$ is given by the formula

$$\frac{n!}{(n - k + 1)!(k - 1)!}x^{n-k+1}y^{k-1}.$$

EXAMPLE 5 **Finding a specific term**

Find the sixth term of the expansion of $(a^2 - 2b)^7$.

Solution

Use the formula for the kth term with $k = 6$ and $n = 7$:

$$\frac{7!}{(7 - 6 + 1)!(6 - 1)!}(a^2)^2(-2b)^5 = 21a^4(-32b^5) = -672a^4b^5$$

We can think of the binomial expansion as a finite series. Using summation notation, we can write the binomial theorem as follows.

The Binomial Theorem (Using Summation Notation)

For any positive integer n,

$$(x + y)^n = \sum_{i=0}^{n} \frac{n!}{(n - i)!i!} x^{n-i}y^i \quad \text{or} \quad (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i.$$

EXAMPLE 6 **Using summation notation**

Write $(a + b)^5$ using summation notation.

Solution

Use $n = 5$ in the binomial theorem:

$$(a + b)^5 = \sum_{i=0}^{5} \frac{5!}{(5 - i)!\,i!} a^{5-i}b^i$$

WARM-UPS

True or false? Explain your answer.

1. There are 12 terms in the expansion of $(a + b)^{12}$.
2. The seventh term of $(a + b)^{12}$ is a multiple of a^5b^7.
3. For all values of x, $(x + 2)^5 = x^5 + 32$.
4. In the expansion of $(x - 5)^8$ the signs of the terms alternate.
5. The eighth line of Pascal's triangle is

   1 8 28 56 70 56 28 8 1.
6. The sum of the coefficients in the expansion of $(a + b)^4$ is 2^4.
7. $(a + b)^3 = \sum_{i=0}^{3} \frac{3!}{(3 - i)!\,i!} a^{3-i}b^i$
8. The sum of the coefficients in the expansion of $(a + b)^n$ is 2^n.
9. $0! = 1!$
10. $\frac{7!}{5!2!} = 21$

14.5 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a binomial expansion?

2. What is Pascal's triangle and how do you make it?

3. What does $n!$ mean?

4. What is the binomial theorem?

Evaluate each expression. See Example 1.

5. $\dfrac{5!}{2!\,3!}$

6. $\dfrac{6!}{5!\,1!}$

7. $\dfrac{8!}{5!\,3!}$

8. $\dfrac{9!}{2!\,7!}$

Use the binomial theorem to expand each binomial. See Examples 2 and 3.

9. $(r + t)^5$

10. $(r + t)^6$

11. $(m - n)^3$

12. $(m - n)^4$

13. $(x + 2a)^3$

14. $(a + 3b)^4$

15. $(x^2 - 2)^4$

16. $(x^2 - a^2)^5$

17. $(x - 1)^7$

18. $(x + 1)^6$

Write out the first four terms in the expansion of each binomial. See Examples 2 and 3.

19. $(a - 3b)^{12}$

20. $(x - 2y)^{10}$

21. $(x^2 + 5)^9$

22. $(x^2 + 1)^{20}$

23. $(x - 1)^{22}$

24. $(2x - 1)^8$

25. $\left(\dfrac{x}{2} + \dfrac{y}{3}\right)^{10}$

26. $\left(\dfrac{a}{2} + \dfrac{b}{5}\right)^8$

Find the indicated term of the binomial expansion. See Examples 4 and 5.

27. $(a + w)^{13}$, 6th term

28. $(m + n)^{12}$, 7th term

29. $(m - n)^{16}$, 8th term

30. $(a - b)^{14}$, 6th term

31. $(x + 2y)^8$, 4th term

32. $(3a + b)^7$, 4th term

33. $(2a^2 - b)^{20}$, 7th term

34. $(a^2 - w^2)^{12}$, 5th term

Write each expansion using summation notation. See Example 6.

35. $(a + m)^8$

36. $(z + w)^{13}$

37. $(a - 2x)^5$

38. $(w - 3m)^7$

GETTING MORE INVOLVED

39. ***Discussion.*** Find the trinomial expansion for $(a + b + c)^3$ by using $x = a$ and $y = b + c$ in the binomial theorem.

40. ***Discussion.*** What problem do you encounter when trying to find the fourth term in the binomial expansion for $(x + y)^{120}$? How can you overcome this problem? Find the fifth term in the binomial expansion for $(x - 2y)^{100}$.