such that the conditions (5.2.7) are satisfied:

$$\begin{aligned}
\Delta_1^e &= w_h^e(x_e) && = c_1^e + c_2^e x_e + c_3^e x_e^2 + c_4^e x_e^3 \\
\Delta_2^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_e} && = -c_2^e - 2c_3^e x_e - 3c_4^e x_e \\
\Delta_3^e &= w_h^e(x_{e+1}) && = c_1^e + c_2^e x_{e+1} + c_3^e x_{e+1}^2 + c_4^e x_{e+1}^3 \\
\Delta_4^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_{e+1}} && = -c_2^e - 2c_3^e x_{e+1} - 3c_4^e x_{e+1}
\end{aligned}$$

or

$$\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{bmatrix} 1 & x_e & x_e^2 & x_e^3 \\ 0 & -1 & -2x_e & -3x_e^2 \\ 1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3 \\ 0 & -1 & -2x_{e+1} & -3x_{e+1}^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \\ c_4^e \end{Bmatrix} \tag{5.2.9}$$

Inverting this matrix equation to express c_i^e in terms of Δ_1^e, Δ_2^e, Δ_3^e, and Δ_4^e, and substituting the result into (5.2.8), we obtain

$$w_h^e(x) = \Delta_1^e\phi_1^e + \Delta_2^e\phi_2^e + \Delta_3^e\phi_3^e + \Delta_4^e\phi_4^e = \sum_{j=1}^{4} \Delta_j^e\phi_j^e \tag{5.2.10}$$

where (with $x_{e+1} = x_e + h_e$)

$$\begin{aligned}
\phi_1^e &= 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3 \\
\phi_2^e &= -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2 \\
\phi_3^e &= 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3 \\
\phi_4^e &= -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]
\end{aligned} \tag{5.2.11}$$

Note that the cubic interpolation functions in (5.2.11) are derived by interpolating w as well as its derivative dw/dx at the nodes. Such polynomials are known as the *Hermite family of interpolation functions*, and ϕ_i^e in (5.2.11) are called the *Hermite cubic interpolation* (or *cubic spline*) *functions*. Plots of the Hermite cubic interpolation functions are shown in Fig. 5.2.4.

Recall that the Lagrange cubic interpolation functions are derived to interpolate a function, but not its derivatives, at the nodes. Hence, a Lagrange cubic element will have four nodes, with the dependent variable, not its derivative, as the nodal degree of freedom. Since the slope (or derivative) of the dependent variable is also required by the weak form to be continuous at the nodes for the Euler–Bernoulli beam theory, the Lagrange cubic