

The two roots of this quadratic equation for  $H = 1$  are

$$\bar{\lambda}_{1,2} = \frac{32 \pm \sqrt{(32)^2 - 15 \times 24}}{15} \rightarrow \bar{\lambda}_1 = 0.41545, \bar{\lambda}_2 = 3.85121$$

and the eigenvalues are ( $\lambda = 12\bar{\lambda}$ )

$$\lambda_1 = 4.1545, \lambda_2 = 38.5121$$

The corresponding eigenvector components are (not normalized)

$$\begin{aligned} \{U^{(1)}\} &= \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ 1.0591 \end{Bmatrix} \\ \{U^{(2)}\} &= \begin{Bmatrix} U_2^{(2)} \\ U_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -2.9052 \end{Bmatrix} \end{aligned}$$

The exact eigenfunctions for Set 2 boundary conditions are

$$U_n(x) = \sin \sqrt{\lambda_n} x \quad (6.1.35a)$$

and the eigenvalues  $\lambda_n$  are computed from the equation

$$H \sin \sqrt{\lambda_n} + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} = 0 \quad (6.1.35b)$$

The first two roots of the transcendental equation (6.1.35b) are (for  $H = 1$ )

$$\sqrt{\lambda_1} = 2.0288 \rightarrow \lambda_1 = 4.1160; \quad \sqrt{\lambda_2} = 4.9132 \rightarrow \lambda_2 = 24.1393$$

A comparison of the eigenvalues obtained using meshes of linear and quadratic elements with the exact values is presented in Table 6.1.1. Note that the number of eigenvalues obtained in

**Table 6.1.1** Eigenvalues of the heat conduction equations (6.1.31a) and (6.1.31b) for two sets of boundary conditions.

| Mesh  | $\lambda_1$          | $\lambda_2$          | $\lambda_3$          | $\lambda_4$           | $\lambda_5$          | $\lambda_6$          | $\lambda_7$          |
|-------|----------------------|----------------------|----------------------|-----------------------|----------------------|----------------------|----------------------|
| 2L    | 12.0000<br>(4.4900)* | (36.6529)            |                      |                       |                      |                      |                      |
| 4L    | 10.3866<br>(4.2054)  | 48.0000<br>(27.3318) | 126.756<br>(85.7864) | (177.604)             |                      |                      |                      |
| 8L    | 9.9971<br>(4.1380)   | 41.5466<br>(24.9088) | 99.4855<br>(69.1036) | 192.000<br>(143.530)  | 328.291<br>(257.580) | 507.025<br>(417.701) | 686.512<br>(607.022) |
| 1Q    | 10.000<br>(4.1545)   | (38.5121)            |                      |                       |                      |                      |                      |
| 2Q    | 9.9439<br>(4.1196)   | 40.000<br>(24.8995)  | 128.723<br>(81.4446) | (207.653)             |                      |                      |                      |
| 4Q    | 9.8747<br>(4.1161)   | 39.7754<br>(24.2040) | 91.7847<br>(64.7704) | 160.000<br>(129.261)  | 308.253<br>(240.540) | 514.891<br>(405.254) | 794.794<br>(658.133) |
| Exact | 9.8696<br>(4.1160)   | 39.4784<br>(24.1393) | 88.8264<br>(63.6597) | 157.9137<br>(122.889) | 246.740<br>(201.851) | 355.306<br>(300.550) | 483.611<br>(418.987) |

\* The second line for each mesh corresponds to set 2 boundary conditions.