where

$$K_{ij}^{12} = \int_{x_a}^{x_b} \varphi_i^{(1)} \frac{d^2\varphi_j^{(2)}}{dx^2} dx, \quad F_i^e = -\int_{x_a}^{x_b} q\varphi_i^{(1)}\, dx,$$
$$K_{ij}^{21} = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2\varphi_j^{(1)}}{dx^2}\, dx, \quad K_{ij}^{22} = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2\varphi_i^{(2)}}{dx^2} dx \tag{14.2.50b}$$

$\lambda \left(\frac{1}{EI}\right)$

The coefficient matrix in Eq. (14.2.50*a*) is *not* symmetric.

The least-squares finite element model is based on the variational statement

$$0 = \delta \int_{x_a}^{x_b} \left[\left(\frac{d^2w}{dx^2} + \frac{M}{EI} \right)^2 + \left(\frac{d^2M}{dx^2} + q \right)^2 \right] dx$$
$$= 2 \int_{x_a}^{x_b} \left[\left(\frac{d^2w}{dx^2} + \frac{M}{EI} \right) \left(\frac{d^2\delta w}{dx^2} + \frac{\delta M}{EI} \right) + \left(\frac{d^2M}{dx^2} + q \right) \frac{d^2\delta M}{dx^2} \right] dx$$

or

$$0 = \int_{x_a}^{x_b} \frac{d^2\delta w}{dx^2} \left(\frac{d^2w}{dx^2} + \frac{M}{EI} \right) dx \tag{14.2.51a}$$
$$0 = \int_{x_a}^{x_b} \left[\frac{\delta M}{EI} \left(\frac{d^2w}{dx^2} + \frac{M}{EI} \right) + EI \frac{d^2\delta M}{dx^2} \left(\frac{d^2M}{dx^2} + q \right) \right] dx \tag{14.2.51b}$$

$\lambda \; EI/g$

Substituting the approximations in (14.2.49) into Eqs. (14.1.51*a*) and (14.2.51*b*), we obtain

$$\begin{bmatrix} \mathbf{A}^e & \mathbf{B}^e \\ (\mathbf{B}^e)^{\mathrm{T}} & \mathbf{D}^e \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Lambda}^e \\ \boldsymbol{\Delta}^e \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^e \\ \mathbf{0} \end{Bmatrix} \tag{14.2.52a}$$

where

$$A_{ij}^e = \int_{x_a}^{x_b} \left(EI \frac{d^2\varphi_i^{(2)}}{dx^2} \frac{d^2\varphi_j^{(2)}}{dx^2} + \varphi_i^{(2)}\varphi_j^{(2)} \right) dx, \quad F_i^e = -\int_{x_a}^{x_b} \frac{d^2\varphi_i^{(2)}}{dx^2} q\, dx$$

$\lambda \frac{1}{(EI)^2}$

$$B_{ij}^e = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2\varphi_j^{(1)}}{dx^2}\, dx, \quad D_{ij}^e = \int_{x_a}^{x_b} \frac{d^2\varphi_i^{(1)}}{dx^2} \frac{d^2\varphi_j^{(1)}}{dx^2}\, dx \tag{14.2.52b}$$

$\lambda \frac{1}{EI}$

14.3 THREE-DIMENSIONAL PROBLEMS

Most of the basic ideas covered in Chapter 8 for two-dimensional problems can be extended to three-dimensional problems. For the sake of completeness, here we discuss finite element formulations of (1) the Poisson equation governing three-dimensional heat transfer, (2) three-dimensional elasticity equations, and (3) equations governing three-dimensional flows of viscous incompressible fluids. In addition, some of the commonly used three-dimensional finite elements will be presented. To keep the size as well as the scope of the book to reasonable limits, only essential steps are presented and only one numerical example is included.