element. The slope of the deflection at an interior point is

$$-\frac{dw_h^e}{dx} = -\sum_{j=1}^{4} \Delta_j^e \frac{d\phi_j^e}{dx} \tag{5.2.26}$$

Similarly, the bending moment at any point in the element Ω^e of the beam can be computed from the formula

$$\begin{aligned} M(x) &= \int_A \sigma_x z \, dA = B \int_{-H/2}^{H/2} \sigma_x z \, dz = -BE \int_{-H/2}^{H/2} z^2 \frac{d^2 w}{dx^2} \, dz = -EI \frac{d^2 w}{dx^2} \\ &= -EI \sum_{j=1}^{4} \Delta_j^e \frac{d^2 \phi_j^e}{dx^2} \end{aligned} \tag{5.2.27}$$

where B is the width and H is the height of the beam (for a rectangular cross-section beam). The bending stress is given by

$$\sigma_x(x, z) = -\frac{M(x)z}{I} = Ez \frac{d^2 w}{dx^2} = Ez \sum_{j=1}^{4} \Delta_j^e \frac{d^2 \phi_j^e(x)}{dx^2} \tag{5.2.28}$$

The maximum tensile stress occurs at the bottom ($z = H/2$) and the maximum compressive stress at the top ($z = -H/2$) of the beam.

We close this section with a note that whenever the flexural rigidity EI is a constant in each element, the finite element solution for the generalized displacements at the nodes is exact for any applied transverse load q. Further, the solution is exact at all points if the distributed load is such that the exact solution is ~~a~~ cubic. delete

5.2.7 Numerical Examples

Example 5.2.1

Consider a cantilever beam of length L and subjected to linearly varying distributed load $q(x)$, point load F_0, and moment M_0, as shown in Fig. 5.2.10. We wish to determine the displacement field and bending moment in the beam using two elements ($h = L/2$).

First we note that $q(x) = q_0(1 - x/L)$. There, we must evaluate its contribution to the element load vector by computing [see Eq. (5.2.19)]

$$q_i^e = \int_{x_e}^{x_{e+1}} q_0 \left(1 - \frac{x}{L}\right) \phi_i^e(x) \, dx = \int_0^{h_e} q_0 \left(1 - \frac{\bar{x} + x_e}{L}\right) \phi_i^e(\bar{x}) \, d\bar{x}$$

where $\phi_i^e(\bar{x})$ are given in Eq. (5.2.12). Evaluating the integrals,

$$\{q^e\} = \frac{q_0 h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \frac{q_0 h_e}{60L} \begin{Bmatrix} -(9h_e + 30x_e) \\ h_e(2h_e + 5x_e) \\ -(21h_e + 30x_e) \\ -h_e(3h_e + 5x_e) \end{Bmatrix}$$