November 15, 2004 11:9 Reddy Red66855˙ch09

We have

$$
\begin{aligned}
S_{11}^{00} &= \frac{1}{64}\left[\left(1+\frac{1}{\sqrt{3}}\right)^4\left(20-\frac{4}{\sqrt{3}}+\frac{5}{\sqrt{3}}\right)+\left(1+\frac{1}{\sqrt{3}}\right)^2\left(1-\frac{1}{\sqrt{3}}\right)^2\left(20-\frac{4}{\sqrt{3}}-\frac{5}{\sqrt{3}}\right)\right.\\
&\quad\left.+\left(1-\frac{1}{\sqrt{3}}\right)^2\left(1+\frac{1}{\sqrt{3}}\right)^2\left(20+\frac{4}{\sqrt{3}}+\frac{5}{\sqrt{3}}\right)+\left(1-\frac{1}{\sqrt{3}}\right)^4\left(20+\frac{4}{\sqrt{3}}-\frac{5}{\sqrt{3}}\right)\right]\\
&= \frac{1}{64}\left[\frac{1120}{9}+\frac{160}{9}+\frac{32}{3\sqrt{3}}\left(-\frac{4}{\sqrt{3}}+\frac{5}{\sqrt{3}}\right)\right]=\frac{1312}{576}=2.27778
\end{aligned}
$$

Similarly, consider the coefficient S_{12}^{11}:

$$
\begin{aligned}
S_{12}^{11} &= \int_{\Omega_e}\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial x}\,dx\,dy\\
&= \int_{-1}^{1}\int_{-1}^{1}\left(J_{11}^*\frac{\partial\psi_1}{\partial\xi}+J_{12}^*\frac{\partial\psi_1}{\partial\eta}\right)\left(J_{11}^*\frac{\partial\psi_2}{\partial\xi}+J_{12}^*\frac{\partial\psi_2}{\partial\eta}\right)J\,d\xi\,d\eta\\
&= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}[-(10+2\xi)(1-\eta)+2\eta(1-\xi)]\,[(10+2\xi)(1-\eta)+2\eta(1-\xi)]\\
&\quad\times\frac{1}{(20+4\xi-5\eta)}\,d\xi\,d\eta\\
&= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}\left[-(10+2\xi)^2(1-\eta)^2+4\eta^2(1-\xi)^2\right]\frac{1}{(20+4\xi-5\eta)}\,d\xi\,d\eta
\end{aligned}
$$

which is a ratio of polynomials. Hence, we do not expect to evaluate the integral exactly. The integrand varies, approximately, as a quadratic polynomial in each ξ and η. Hence, we may use the two-point Gauss integration to evaluate the integral

$$
\begin{aligned}
S_{12}^{11} &= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}\left[-(10+2\xi)^2(1-\eta)^2+4\eta^2(1-\xi)^2\right]\frac{1}{(20+4\xi-5\eta)}\,d\xi\,d\eta\\
&\approx \sum_{i,j=1}^{2}\left[-(10+2\xi_i)^2(1-\eta_j)^2+4\eta_j^2(1-\xi_i)^2\right]\frac{1}{64(20+4\xi_i-5\eta_j)}\\
&= \left[-\left(10-\frac{2}{\sqrt{3}}\right)^2\left(1+\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(1+\frac{1}{\sqrt{3}}\right)^2\right]\frac{1}{64\left(20+\frac{1}{\sqrt{3}}\right)}\\
&\quad+\left[-\left(10+\frac{2}{\sqrt{3}}\right)^2\left(1+\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(1-\frac{1}{\sqrt{3}}\right)^2\right]\frac{1}{64\left(20+\frac{9}{\sqrt{3}}\right)}\\
&\quad+\left[-\left(10-\frac{2}{\sqrt{3}}\right)^2\left(1-\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(1+\frac{1}{\sqrt{3}}\right)^2\right]\frac{1}{64\left(20-\frac{9}{\sqrt{3}}\right)}\\
&\quad+\left[-\left(10+\frac{2}{\sqrt{3}}\right)^2\left(1-\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(1-\frac{1}{\sqrt{3}}\right)^2\right]\frac{1}{64\left(20-\frac{1}{\sqrt{3}}\right)}=-0.36892
\end{aligned}
$$

A three-point integration gives $S_{11}^{00}=-0.36998$.