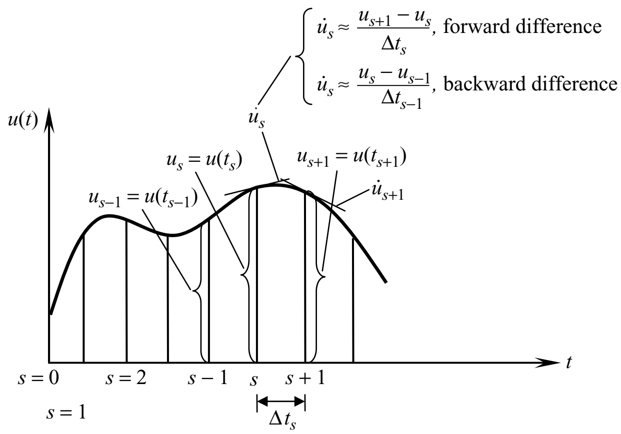


**Figure 6.2.2** Approximation of the derivative of a function.

When $\alpha = 0$, Eq. (6.2.10$a$) gives

$$\dot{u}_s = \frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

which is nothing but the slope of the function $u(t)$ at time $t = t_s$ based on the values of the function at time $t_s$ and $t_{s+1}$. Since the value of the function from a step in front is used, it is termed a *forward difference* approximation. When $\alpha = 1$, we obtain

$$\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \quad \rightarrow \quad \dot{u}_s = \frac{u_s - u_{s-1}}{\Delta t_s}$$

which is termed, for obvious reason, the *backward difference* approximation.

Returning to Eq. (6.2.8), we note that it is valid for all times $t > 0$. In particular, it is valid at times $t = t_s$ and $t = t_{s+1}$. Hence, from Eq. (6.2.8) we have

$$\dot{u}_s = \frac{1}{a}\left(f_s - bu_s\right), \quad \dot{u}_{s+1} = \frac{1}{a}\left(f_{s+1} - bu_{s+1}\right)$$

Substituting the above expressions into Eq. (6.2.10$a$), we arrive at

$$(1 - \alpha)\left(f_s - bu_s\right) + \alpha\left(f_{s+1} - bu_{s+1}\right) = a\left(\frac{u_{s+1} - u_s}{\Delta t_{s+1}}\right)$$

Solving for $u_{s+1}$, we obtain

$$[a + \alpha\, \Delta t_{s+1}\, b]\, u_{s+1} = [a - (1 - \alpha)\, \Delta t_{s+1}\, b]\, u_s + \Delta t_{s+1}\, [\alpha\, f_{s+1} + (1 - \alpha) f_s] \tag{6.2.11a}$$

or

$$u_{s+1} = \frac{a - (1 - \alpha)\, \Delta t_{s+1}\, b}{a + \alpha\, \Delta t_{s+1}\, b} u_s + \Delta t_{s+1} \frac{[\alpha\, f_{s+1} + (1 - \alpha) f_s]}{a + \alpha\, \Delta t_{s+1}\, b} \tag{6.2.11b}$$

Thus, Eq. (6.2.11) can be used repeatedly to march in time and obtain the solution at times $t = t_{s+1}, t_{s+2}, \ldots, t_N$, where N~~time~~ is the number of time steps required to reach the final