Clearly, the body is in a state of plane strain. For an orthotropic material, with principal material axes (x_1, x_2, x_3) coinciding with the (x, y, z) coordinates, the stress components are given by

$$\sigma_{xz} = \sigma_{yz} = 0, \quad \sigma_{zz} = E_3\left(\frac{\nu_{13}}{E_1}\sigma_{xx} + \frac{\nu_{23}}{E_2}\sigma_{yy}\right) \tag{11.2.3a}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & 0 \\ \bar{c}_{12} & \bar{c}_{22} & 0 \\ 0 & 0 & \bar{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{11.2.3b}$$

~~where $\bar{c}_{ij}$ are the elastic stiffnesses~~

$$\begin{aligned} \cancel{\bar{c}_{11} = \frac{E_1(1-\nu_{12})}{(1+\nu_{12})(1-\nu_{12}-\nu_{21})}} \\ \cancel{\bar{c}_{22} = \frac{E_2(1-\nu_{21})}{(1+\nu_{21})(1-\nu_{12}-\nu_{21})}} \\ \cancel{\bar{c}_{12} = \nu_{12}\bar{c}_{22}, \quad \bar{c}_{66} = G_{12}} \end{aligned} \qquad \cancel{(11.2.4)}$$

and E_1 and E_2 are principal (Young's) moduli in the x and y directions, respectively, G_{12} the shear modulus in the xy plane, and ν_{12} and ν_{21} the Poisson ratio (i.e., the negative of the ratio of the transverse strain in the y direction to the strain in the x direction when stress is applied in the x direction). The Poisson ratio ν_{21} can be computed from the reciprocal relation

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1} \tag{11.2.5}$$

Additional engineering constants E_3, ν_{23}, and ν_{13} are required to compute σ_{zz}. For an isotropic material, we have

$$E_1 = E_2 = E_3 = E, \quad \nu_{12} = \nu_{21} = \nu_{13} = \nu_{23} = \nu, \quad G_{12} = G = \frac{E}{2(1+\nu)} \tag{11.2.6}$$

The equations of motion of three-dimensional linear elasticity, $\sigma_{ij,j} + f_i = \rho\ddot{u}_i$ with the body force components $f_3 = f_z = 0$, $f_1 = f_x = f_x(x, y)$, and $f_2 = f_y = f_y(x, y)$, and ρ the density of the material, reduce to the following two plane strain equations of motion

$$\rho\frac{\partial^2 u_x}{\partial t^2} - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - f_x = 0 \tag{11.2.7}$$

$$\rho\frac{\partial^2 u_y}{\partial t^2} - \frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} - f_y = 0 \tag{11.2.8}$$

An example of a plane strain problem is provided by the long cylindrical member under external loads that are independent of z, as shown in Fig. 11.2.1. For cross sections sufficiently far from the ends, it is clear that the displacement u_z is zero and that u_x and u_y are independent of z, i.e., a state of plane strain exists.

11.2.2 Plane Stress

A state of plane stress is defined as one in which the following stress field exists:

$$\begin{aligned} \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \\ \sigma_{xx} = \sigma_{xx}(x, y), \quad \sigma_{xy} = \sigma_{xy}(x, y), \quad \sigma_{yy} = \sigma_{yy}(x, y) \end{aligned} \tag{11.2.9a}$$