The boundary and balance conditions are [for boundary conditions in Eq. (4.3.19a)]

$$U_1 = T_0,\ Q_2^1 = 0,\ Q_3^1 + Q_1^2 = 0,\ Q_2^2 = 0,\ Q_3^2 = -\beta A U_5 + \beta A T_\infty$$

Hence, the condensed equations are

$$\left(\frac{kA}{3h}\begin{bmatrix} 16 & -8 & 0 & 0 \\ -8 & 14 & -8 & 1 \\ 0 & -8 & 16 & -8 \\ 0 & 1 & -8 & 7 \end{bmatrix} + \frac{\beta Ph}{30}\begin{bmatrix} 16 & 2 & 0 & 0 \\ 2 & 8 & 2 & -1 \\ 0 & 2 & 16 & 2 \\ 0 & -1 & 2 & 4+\alpha \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{\cancel{7}^{\,8}kA}{3h} - \frac{\cancel{4}^{\,2}\beta Ph}{30}\right)T_0 \\ \cancel{0}\ \left(-\frac{kA}{3h} + \frac{8\beta Ph}{30}\right)T_0 \\ 0 \\ \beta A T_\infty \end{Bmatrix} + \frac{\beta P T_\infty h}{6}\begin{Bmatrix} 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

where $\alpha = \beta A/(\beta Ph/30) = 30A/Ph$.

The solution of the condensed equations for the unknown temperatures is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 82.374^\circ\text{C},\ U_3 = 70.884^\circ\text{C},\ U_4 = 64.380^\circ\text{C},\ U_5 = 62.240^\circ\text{C}$$

The heat input at node 1 is

$$Q_1^1 = -0.05 + 0.091833 \times 100 - 0.10167 \times 82.374 + 0.012333 \times 70.884 = 1.633 \text{ W}$$

The total heat loss from the surface of the fin can be calculated using

$$Q = \sum_{e=1}^{4}\int_{x_a}^{x_b} P\beta\left(T_1^e\psi_1^e + T_2^e\psi_2^e + T_3^e\psi_3^e - T_\infty\right)dx + \beta A(U_5 - T_\infty)$$

$$= \sum_{e=1}^{4}\beta P h_e\left(\frac{T_1^e + 4T_2^e + T_3^e}{6} - T_\infty\right) + \beta A(U_5 - T_\infty)$$

$$= \frac{\beta Ph}{6}(U_1 + 4U_2 + 2U_3 + 4U_4 + U_5 - 12T_\infty) = 1.62756 + 0.00528 = 1.633 \text{ W}$$

Set 2 Boundary Conditions

For Set 2 boundary conditions (4.3.19b), the finite element boundary and balance conditions for the mesh of four linear elements are

$$U_1 = T_0,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 + Q_1^4 = 0,\ U_5 = T_L = 20^\circ\text{C}$$

and the finite element solution is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 73.955^\circ\text{C},\ U_3 = 53.252^\circ\text{C},\ U_4 = 35.842^\circ\text{C},\ U_5 = 20^\circ\text{C}$$

The heats at nodes 1 and 5 are

$$Q_1^1 = -0.075 + (0.077 + 0.0025)\,100 + (-0.077 + 0.00125)\,73.955 = 2.273 \text{ W}$$

$$Q_2^4 = -0.075 + (-0.077 + 0.00125)\,35.842 + (0.077 + 0.0025)\,20 = -1.2 \text{ W}$$