The obvious boundary conditions are $U_1 = U_2 = Q_4 = 0$. The effect of the spring is that [see Fig. 5.2.9(*b*)] it exerts a force of kU_3 upward on the beam. Hence, $Q_3 = -kU_3$. Thus, we have

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ -kU_3 \\ 0 \end{Bmatrix}$$

and the condensed equations for the unknown displacements U_3 (deflection) and U_4 (rotation) become

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}$$

whose solution is

$$U_3 = w(L) = \frac{q_0 L^4}{8EI}\frac{1}{\left(1 + \frac{kL^3}{3EI}\right)}, \quad U_4 = \theta(L) = -\frac{q_0 L^3}{6EI}\frac{\left(EI - \frac{kL^3}{24}\right)}{\left(EI + \frac{kL^3}{3}\right)}$$

Note that when $k = 0$, we obtain the deflection $U_3 = q_0 L^4/8EI$ and rotation $U_4 = -q_0 L^3/6EI$ at the free end of a cantilever beam under uniformly distributed load of intensity q_0. When $k \to \infty$, we obtain the deflection $U_3 = 0$ and rotation $U_4 = -q_0 L^3/48EI$ at $x = L$ (where it is simply supported).

Alternatively, the assembly of the beam and spring elements [see Fig. 5.2.9(*c*)] yields the result

$$\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\ -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} + k & \frac{6EI}{L^2} & -k \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L}\ L^2 & \frac{4EI}{L} & 0 \\ 0 & 0 & -k & 0 & k \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 \\ Q_2^2 \end{Bmatrix}$$

Using the boundary conditions, $U_1 = U_2 = U_5 = 0$ and $Q_4^1 = 0$, and the equilibrium condition $Q_3^1 + Q_1^2 = 0$, we obtain the condensed equations

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}$$

which are identical to those obtained earlier.

5.2.6 Postprocessing of the Solution

Once the boundary conditions are imposed, the resulting equations are solved for the unknown nodal displacements and forces. The solution is then given by Eq. (5.2.10) in each