have

$$
\begin{aligned}
(Q_1^1)_{\text{def}} &= -\left(EA\frac{du_h^1}{dx}\right)\bigg|_{x=0} = -EA\frac{U_3-U_1}{h_1} = -\frac{EA}{h_1}U_3 = K_{12}^1 U_3 \\
(Q_1^2)_{\text{def}} &= -\left(EA\frac{du_h^2}{dx}\right)\bigg|_{x=0} = K_{12}^2 U_3 \\
(Q_2^3)_{\text{def}} &= \left(EA\frac{du_h^3}{dx}\right)\bigg|_{x=h_1+h_3} = EA\frac{U_4-U_3}{h_3} = -\frac{EA}{h_3}U_3 = K_{21}^3 U_3
\end{aligned}
\tag{3.2.63}
$$

where h_1 and h_3 are the lengths of elements 1 and 3, respectively.

The secondary variables computed using the definitions (3.2.63) are the same as those derived from the assembled equations for the problem in Fig. 3.2.10. This equality is *not to be expected in general*. In fact, when the source vector f is not zero, the secondary variables computed from the definitions (3.2.6) will be in error compared with those computed from the assembled equations. The error decreases as the number of elements or the degree of interpolation is increased.

This completes the basic steps involved in the finite element analysis of the model equation (3.2.1). Next, we consider an example of application of the finite element method. Additional applications of the method to one-dimensional problems of heat transfer, fluid mechanics, and solid mechanics, are presented in Chapter 4.

Example 3.2.1

We wish to use the finite element method to solve the problem described by the following differential equation and boundary conditions (see Example 2.5.1):

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1 \tag{3.2.64}$$

$$u(0) = 0, \quad u(1) = 0 \tag{3.2.65}$$

The differential equation in (3.2.64) is a special case of the model equation (3.2.1) for the following data: $a = 1$, $c = -1$, and $f(x) = -x^2$. Hence, the coefficient matrix is given by

$$
\begin{aligned}
K_{ij}^e &= \int_{x_a}^{x_b}\left(\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} - \psi_i^e\psi_j^e\right)dx \\
f_i^e &= \int_{x_a}^{x_b}(-x^2)\psi_i^e\,dx
\end{aligned}
$$

First we consider a mesh of four linear elements and next a mesh of two quadratic elements to solve the problem.

Linear Elements. The element coefficient matrix is given by [see Eq. (3.2.34), with $a_e = 1$, $c_e = -1$, $h_e = \frac{1}{4}$]

$$[K^e] = \frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 \\ -4.0417 & 3.9167 \end{bmatrix}$$