and conditions on the boundary Γ of Ω, then $U_N(\mathbf{x}) = u(\mathbf{x})$, which is the exact solution of the problem. Of course, approximate methods are not about problems for which exact solutions can be determined by some methods of mathematical analysis; the role of approximate methods is to find an approximate solution of problems that do not admit analytical solutions. When the exact solution cannot be determined, the alternative is to find a solution U_N that satisfies the governing equations in an approximate way. In the process of satisfying the governing equations approximately, we obtain (not accidentally but by planning) N algebraic relations among the N parameters $c_1, c_2, \cdots, c_N$. A detailed discussion of these ideas is given in the next few paragraphs in connection with a specific problem.

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = f(x) \quad \text{for } 0 < x < L \tag{2.1.2a}$$

subjected to the boundary conditions

$$u(0) = u_0, \quad \left[a(x)\frac{du}{dx}\right]_{x=\cancel{1}\ L} = Q_0 \tag{2.1.2b}$$

where $a(x)$, $c(x)$, and $f(x)$ are known functions, u_0 and Q_0 are known parameters, and $u(x)$ is the function to be determined. The set $a(x)$, $c(x)$, $f(x)$, u_0, and Q_0 is called the problem *data*. An example of the above problem is given by the heat transfer in an uninsulated rod (see Example 1.2.2): here u denotes the temperature (θ), $f(x)$ is the internal heat generation per unit length (Ag), $a(x)$ is the thermal resistance (kA), $c = \beta P$, u_0 is the specified temperature (θ_0), and Q_0 is the specified heat.

We seek an approximate solution over the entire domain $\Omega = (0, L)$ in the form

$$U_N \equiv \sum_{j=1}^{N} c_j\phi_j(x) + \phi_0(x) \tag{2.1.3}$$

where the c_j are coefficients to be determined and $\phi_j(x)$ and $\phi_0(x)$ are functions chosen such that the specified boundary conditions of the problem are satisfied by the N-parameter approximate solution U_N. Note that the particular form in (2.1.3) has two parts: one containing the unknowns $(\sum c_j\phi_j)$ that is termed the homogeneous part and the other is the nonhomogeneous part (ϕ_0) that has the sole purpose of satisfying the specified boundary conditions of the problem. Since ϕ_0 satisfies the boundary conditions, the sum $\sum c_j\phi_j$ must satisfy, for arbitrary c_j, the homogeneous form of the boundary conditions ($Bu = \hat{u}$ is said to be a nonhomogeneous boundary condition when $\hat{u} \neq 0$, and it is termed a homogeneous boundary condition when $\hat{u} = 0$; here B denotes some operator). Thus, in the present case, the actual boundary conditions are both nonhomogeneous ($B = 1$ and $\hat{u} = u_0$ at $x = 0$, and $B = a(x)(d/dx)$ and $\hat{u} = Q_0$ at $x = L$). The particular form (2.1.3) is convenient in selecting ϕ_0 and ϕ_j. Thus, ϕ_0 and ϕ_j satisfy the conditions

$$B\phi_0 = \hat{u}, \quad B\phi_j = 0 \ \text{ for all } j = 1, 2, \cdots, n \tag{2.1.4}$$

To be more specific, let $L = 1$, $u_0 = 1$, $Q_0 = 0$, $a(x) = x$, $c(x) = 1$, $f(x) = 0$, and $N = 2$. Then we choose the approximate solution in the form

$$U_2 = c_1\phi_1 + c_2\phi_2 + \phi_0 \ \text{ with } \ \phi_0 = 1,\ \phi_1(x) = x^2 - 2x,\ \phi_2 = x^3 - 3x$$