$$
\begin{aligned}
K_{ij}^{22} &= \int_{x_a}^{x_b} \left(EI \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + GAK_s \psi_i^e \psi_j^e \right) dx \\
M_{ij}^{11} &= \int_{x_a}^{x_b} \rho A \psi_i^e \psi_j^e \, dx, \quad M_{ij}^{22} = \int_{x_a}^{x_b} \rho I \psi_i^e \psi_j^e \, dx \\
F_j^1 &= Q_{2i-1}, \quad F_i^2 = Q_{2i}
\end{aligned} \tag{6.1.48a}
$$

$$
\begin{aligned}
Q_1^e &\equiv -\left[GAK_s \left(S + \frac{dW}{dx} \right) \right]\bigg|_{x_a}, \quad Q_2^e \equiv -\left(EI \frac{dS}{dx} \right)\bigg|_{x_a} \\
Q_3^e &\equiv \left[GAK_s \left(S + \frac{dW}{dx} \right) \right]\bigg|_{x_b}, \qquad Q_4^e \equiv \left(EI \frac{dS}{dx} \right)\bigg|_{x_b}
\end{aligned} \tag{6.1.48b}
$$

For the choice of linear interpolation functions, Eq. (6.1.47) has the explicit form (after rearranging the nodal variables)

$$
[K^e] = \left(\frac{2E_eI_e}{\mu_0 h_e^3} \right) \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5 + 6\Lambda_e) & 3h_e & h_e^2(1.5 - 6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5 - 6\Lambda_e) & 3h_e & h_e^2(1.5 + 6\Lambda_e) \end{bmatrix}
$$

$$
[M^e] = \frac{\rho^e A_e}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2r_e & 0 & r_e \\ 1 & 0 & 2 & 0 \\ 0 & r_e & 0 & 2r_e \end{bmatrix}, \quad r_e = \frac{I_e}{A_e} \tag{6.1.49a}
$$

$$
\Lambda_e = \frac{E_e I_e}{G_e A_e K_s h_e^2}, \quad \mu_0 = 12\Lambda_e \tag{6.1.49b}
$$

For Hermite cubic interpolation of $W(x)$ and related quadratic approximation of $S(x)$ [i.e., *interdependent interpolation element* (IIE)], the resulting mass matrix is cumbersome and depends on the stiffness parameters (E and G). We will not consider it here [see Reddy (1999*b*, 2000)].

Example 6.1.2

Consider a uniform beam of rectangular cross section ($B \times H$), fixed at $x = 0$ and free at $x = L$ (i.e., a cantilever beam). We wish to determine the first four flexural frequencies of the beam.

The boundary conditions of the structure are

$$
W(0) = 0, \quad \frac{dW}{dx} = 0, \quad \left[EI \frac{d^2W}{dx^2} \right]_{x=L} = 0, \quad \left[EI \frac{d^3W}{dx^3} \right]_{x=L} = 0 \tag{6.1.50a}
$$

in the Euler–Bernoulli beam theory and

$$
W(0) = 0, \quad S(0) = 0, \quad \left[EI \frac{dS}{dx} \right]_{x=L} = 0, \quad \left[GAK_s \left(\frac{dW}{dx} + S \right) \right]_{x=L} = 0 \tag{6.1.50b}
$$

in the Timoshenko beam theory. The first two terms in each case are the essential (or geometric) boundary conditions and the last two are the natural (or force) boundary conditions.