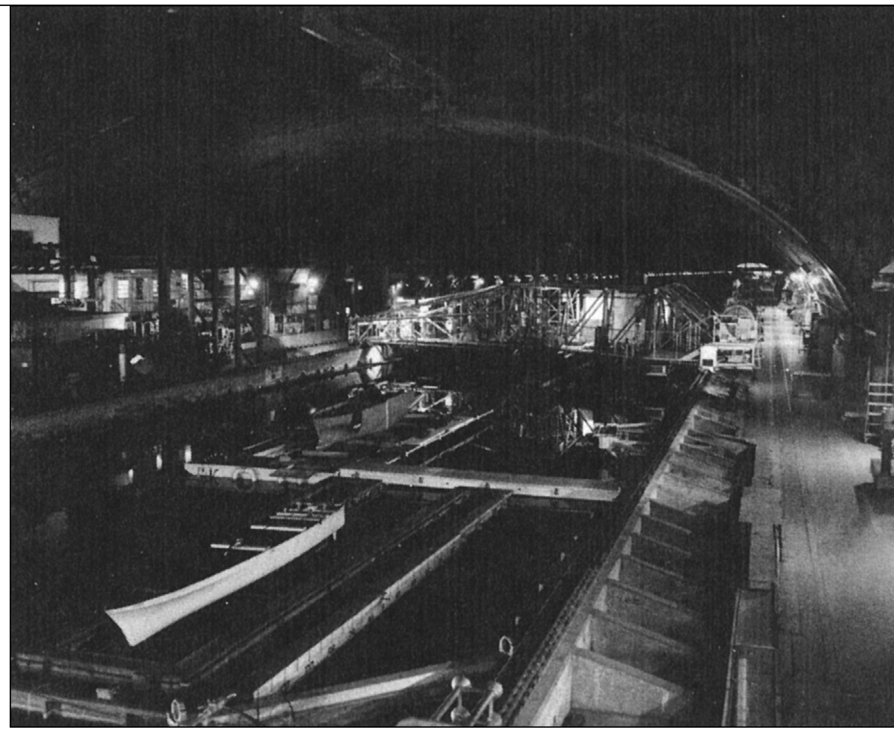

Shown is part of a towing tank facility of the Naval Ship Research and Development Center at Carderock, Maryland. The basin comprises three adjoining sections: (1) A deep water section 22 feet deep, 50.20 feet wide, and 889 feet long. (2) A shallow water section 10 feet deep, 50.96 feet wide, and 303 feet long. The depth of water can be varied. A 32 feet by 5 feet fitting out dock is located here. The photograph is taken for the shallow water section. (3) A turning basin in the form of a *J* in which self-propelled models can be allowed to maneuver. The carriage speed can move models up to speeds of 18 knots. In subsequent chapters, towing tanks will be referred to on a number of occasions.

CHAPTER 3

Foundations of Flow Analysis

3.1 THE VELOCITY FIELD

In particle and rigid-body dynamics we are able to describe the motion of each body in a separate and discrete manner. For instance, the velocity of the *n*th particle of an aggregate of particles moving in space can be specified by the scalar equations

$$\begin{aligned}(V_x)_n &= f_n(t)\\ (V_y)_n &= g_n(t)\\ (V_z)_n &= h_n(t)\end{aligned} \qquad [3.1]$$

Note that the identification of a particle is easily facilitated with the use of a subscript. However, in a deformable continuum such as a fluid, there are, for practical purposes, an infinite number of particles whose motions are to be described, which makes this approach unmanageable; so we employ spatial coordinates to help identify particles in a flow. The velocity of all particles in a flow can therefore be expressed in the following manner:

$$\begin{aligned}V_x &= f(x, y, z, t)\\ V_y &= g(x, y, z, t)\\ V_z &= h(x, y, z, t)\end{aligned} \qquad [3.2]$$

Specifying coordinates xyz and the time t and using these values in functions f, g, and h in Eq. 3.2, we can directly determine the velocity components of a fluid element at the particular position and time specified. The spatial coordinates thus take the place of the subscript n of the discrete systems studied in mechanics. This is called the *field approach*. If properties and flow characteristics at each position in space remain constant with time, the flow is called *steady flow*. A time-dependent flow, on the other hand, is designated as an *unsteady flow*. The steady-flow velocity field would then be given as

$$\begin{aligned}V_x &= f(x, y, z)\\ V_y &= g(x, y, z)\\ V_z &= h(x, y, z)\end{aligned} \qquad [3.3]$$

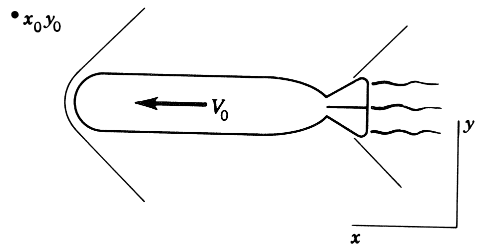

Figure 3.1
Unsteady-flow field relative to *xy*.

Often, a steady flow may be derived from an unsteady-flow field by simply changing the space reference. To illustrate this, examine the flow pattern created by a torpedo moving near the free surface through initially undisturbed water at constant speed V_0 relative to the stationary reference xy, as shown in Fig. 3.1. It can be seen that this is an unsteady-flow field, as viewed from xyz. Thus, the velocity at position x_0, y_0 in the field, for instance, will at one instant be zero and later, owing to the oncoming waves and wake of the torpedo, will be subjected to a complicated time variation. To establish a steady-flow field, we now consider a reference $\xi\eta$ *fixed* to the torpedo. The flow field relative to such a moving reference is shown in Fig. 3.2. The velocity at fixed position $\xi_0\eta_0$, as seen from the torpedo, clearly does not change with respect to time. This must be true since this position is fixed in an unchanging flow pattern as seen from the torpedo. Note from Fig. 3.2 that the water far upstream of the torpedo has a velocity relative to the torpedo and thus relative to the axes $\xi\eta$, which is equal to $-V_0$. You can now see that the transition from an unsteady flow, relative to reference xy depicted in Fig. 3.1, to a steady flow, relative to reference xy, could have been accomplished by superimposing a velocity $-V_0$ on the entire flow field of Fig. 3.1 to arrive at the steady field of Fig. 3.2. *This may be done any time a body is moving with constant speed through an initially undisturbed fluid.*

Flows are usually depicted graphically with the aid of *streamlines*. These lines are drawn so as to be always tangent to the velocity vectors of the fluid particles in a flow. This is illustrated in Fig. 3.3. For a steady flow the orientation of the streamlines will be fixed. Fluid particles, in this case, will proceed along paths coincident with the streamlines. In unsteady flow, however, an indicated streamline pattern

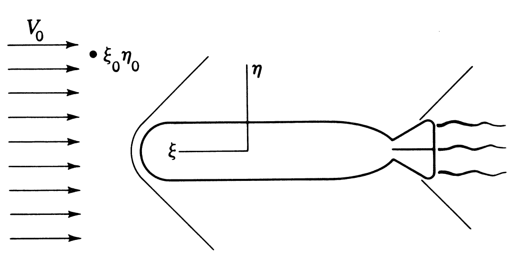

Figure 3.2
Steady-flow field relative to $\xi\eta$.

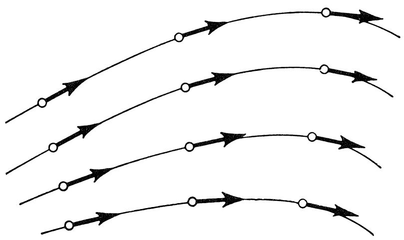

Figure 3.3
Streamlines.

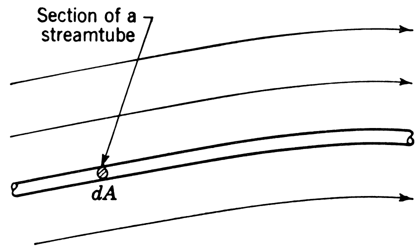

Figure 3.4
Streamtube.

yields only an instantaneous flow representation, and for such flow there will no longer be a simple correspondence between path lines and streamlines.

Streamlines proceeding through the periphery of an infinitesimal area at some time t will form a tube, which is useful in discussions of fluid phenomena. This is called the *streamtube,* which is illustrated in Fig. 3.4. From considerations of the definition of the streamline, it is obvious that there can be no flow through the lateral surface of the streamtube. In short, the streamtube acts like an impervious container of zero wall thickness and infinitesimal cross section. A continuum of adjacent streamtubes arranged to form a finite cross section is often called a *bundle of streamtubes*.

3.2 TWO VIEWPOINTS

In Sec. 3.1 we discussed various general aspects of the velocity field $\mathbf{V}(x, y, z, t)$. Two procedures will now be set forth by which the field may be utilized in computations involving the motion of fluid particles making up the flow. For instance, by stipulating fixed coordinates x_1, y_1, z_1 in the velocity-field functions and letting time pass, we can express the velocity of particles moving by this position at any time. Mathematically, this may be given by the formulation $\mathbf{V}(x_1, y_1, z_1, t)$. Hence, by this technique we express, at a fixed position in space, the velocities of a continuous "string" of fluid particles moving by this position. This viewpoint is sometimes called the *Eulerian* viewpoint.

On the other hand, to study "any one" particle in the flow one must "follow the particle." This means that x, y, z in the expression $\mathbf{V}(x, y, z, t)$ must not be fixed but must vary continuously in such a way as always to locate the particle. This approach is called the *Lagrangian* viewpoint. For any *particular* particle, $x(t)$, $y(t)$, and $z(t)$ become specific time functions which are different, in general, from corresponding time functions for other particles in the flow. Furthermore, the functions $x(t)$, $y(t)$, and $z(t)$ for a particular particle must have particular values $x(0)$, $y(0)$, and $z(0)$ at time $t = 0$ for that particular particle. In most cases, however, we do *not* identify a particular particle in our work, so for any one particle, $x(t)$, $y(t)$, and $z(t)$ are *unspecified* time functions which have the capability, nevertheless, when the form of the time functions and initial positions are chosen, of focusing on any particular particle. Thus we say in this case that

$$\begin{aligned} V_x &= f[x(t), y(t), z(t), t] \\ V_y &= g[x(t), y(t), z(t), t] \\ V_z &= h[x(t), y(t), z(t), t] \end{aligned} \tag{3.4}$$

In fluid dynamics there is ample occasion to employ both techniques.[1]

These considerations do not depend on whether the field is steady or unsteady and should not be confused with the conclusions of Sec. 3.1. You may note that the Eulerian viewpoint was utilized in that section in the discussion of both the steady and unsteady flows about the torpedo.

3.3 ACCELERATION OF A FLOW PARTICLE

We will soon use Newton's law for any one particle in a flow, and we will need the time rate of change of velocity of any one particle in a flow. In using the velocity field we will then have to use the Lagrangian viewpoint. Thus, noting that x, y, z are functions of time, we may establish the acceleration field by employing the chain rule of differentiation in the following way:

$$\mathbf{a} = \frac{d}{dt}\mathbf{V}(x, y, z, t) = \left(\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{\partial z}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right) \tag{3.5}$$

Since x, y, z are coordinates of any one particle, it is clear that dx/dt, dy/dt, and dz/dt must then be the scalar velocity components of any one particle and can be denoted as V_x, V_y, and V_z, respectively. Hence

$$\mathbf{a} = \left(V_x\frac{\partial \mathbf{V}}{\partial x} + V_y\frac{\partial \mathbf{V}}{\partial y} + V_z\frac{\partial \mathbf{V}}{\partial z}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right) \tag{3.6}$$

The three scalar equations corresponding to Eq. 3.6 in the three cartesian-coordinate directions are, taking components of the vector $\mathbf{V}$,

$$\begin{aligned} a_x &= \left(V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right) + \left(\frac{\partial V_x}{\partial t}\right) \\ a_y &= \left(V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z}\right) + \left(\frac{\partial V_y}{\partial t}\right) \\ a_z &= \left(V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right) + \left(\frac{\partial V_z}{\partial t}\right) \end{aligned} \tag{3.7}$$

[1]A simple-minded way of thinking of the two viewpoints is to consider a golf tournament where the players are the "particles." If you station yourself as the observer at any particular tee in order to observe the various players coming by this location, you are using the Eulerian viewpoint. On the other hand, if you select your favorite player and move around the course with him/her for purposes of observation, you are using the Lagrangian viewpoint.

Now the acceleration $\mathbf{a}$ of any one particle is given in terms of the velocity field, the partial spatial derivatives, and the partial time derivative of $\mathbf{V}$. But $\mathbf{V}$ is a function of x, y, z, and t. Hence the acceleration $\mathbf{a}$ is then given in terms of x, y, z and t and is thus also a field variable.

The acceleration of fluid particles in a flow field may be imagined as the superposition of two effects:

1. In expressions in the first parentheses on the right-hand sides of Eqs. 3.6 and 3.7, the *explicit* time variable t is held constant. Hence, in these expressions at a given time t, the field is assumed to become and remain steady. The particle, under such circumstances, is in the process of changing position in this steady field. It is as a result, undergoing a change in velocity because the velocity at various positions in this field will, in general, be different at any time t. This time rate of change of velocity due to changing position in the field is aptly called the *acceleration of transport,* or *convective acceleration*.
2. The term within the second parentheses in the acceleration equations does not arise from the change of particle position, but rather from the rate of change of the velocity field itself at the position occupied by the particle at time t. It is sometimes called the *local acceleration*.

The differentiation carried out in Eq. 3.6 is called the *substantial,* or *total, derivative*. To emphasize the fact that the time derivative is carried out as one follows the particle, the notation D/Dt is often used in place of d/dt. Hence, the substantial derivative of the velocity is given by $D\mathbf{V}/Dt$. The increased complexity over that which we experienced in mechanics of discrete particles is the price we pay for having, by necessity, brought in spatial coordinates to identify particles in a deformable continuous medium. It should be understood that the substantial derivative is by no means restricted to the velocity field vector. Thus for any vector field $\mathbf{H}$ associated with a flow we can say:

$$\frac{D\mathbf{H}}{Dt} = \left(V_x \frac{\partial \mathbf{H}}{\partial x} + V_y \frac{\partial \mathbf{H}}{\partial y} + V_z \frac{\partial \mathbf{H}}{\partial z} \right) + \frac{\partial \mathbf{H}}{\partial t}$$

Note that we have, in effect, two vector fields involved in this equation. There is first the field $\mathbf{H}$ undergoing the substantial derivative, and for any such vector field $\mathbf{H}$ associated with the flow there is always the fluid velocity field $\mathbf{V}$ whose components in the above equation facilitate *following any one particle* as one computes the rate of change of $\mathbf{H}$ for the particle. We have offered several problems with different $\mathbf{H}$ fields at the end of the chapter.

In many analyses, it is useful to think of a set of streamlines as part of a coordinate system. In such cases the letter s indicates the position of the particle along a particular streamline, and accordingly $\mathbf{V} = \mathbf{V}(s, t)$. Hence, for the acceleration of transport we have $(\partial \mathbf{V}/\partial s)(ds/dt)$, which gives the acceleration that results from the action of the particle's changing position along a streamline. The complete acceleration is then given as

$$\mathbf{a} = V \frac{\partial \mathbf{V}}{\partial s} + \frac{\partial \mathbf{V}}{\partial t} \qquad [3.8]$$

Let us consider the special case of steady flow, where, as we pointed out earlier, there is a fixed streamline pattern and where streamlines are the same as path lines. We can decompose the acceleration of transport vector for such flow into two scalar components by choosing one component a_T tangent to the path and the other component a_N normal to the path in the osculating plane.[2] You will recall from earlier mechanics courses that the acceleration component a_T can be given as

$$a_T = V\frac{dV}{ds} = \frac{1}{2}\frac{dV^2}{ds} \tag{3.9}$$

and, taking the direction *toward* the center of curvature in the osculating plane as positive, that the other component of acceleration a_N can be given as

$$a_N = \frac{V^2}{R} \tag{3.10}$$

where R is the radius of curvature. We will have occasion to use acceleration components in the ensuing chapters, particularly Chap. 11.

EXAMPLE 3.1

■ Problem Statement

To illustrate some of the definitions and ideas of Sec. 3.3, we examine a simple two-dimensional flow (see Fig. 3.5) with the upper boundary that of a rectangular hyperbola, given by the equation $xy = K$. Assume that the scalar components of the velocity field are known to be

$$\begin{aligned} V_x &= -Ax \\ V_y &= Ay \qquad A = \text{const} \\ V_z &= 0 \end{aligned} \tag{a}$$

(Note that the flow is steady.) Determine the streamline pattern and the acceleration field.

■ Strategy

We will first determine the streamline equations, by considering the relation between the streamline slope and the velocity components. This will give us means of getting the acceleration components via the definition of the substantial derivative of the velocity vector field.

[2]The osculating plane at a particular point on a path is the limiting plane formed by the point and two additional points, on the path, ahead and behind, as they are brought ever closer to the particular point. See I. H. Shames, *Engineering Mechanics: Statics and Dynamics,* 4th ed., Prentice-Hall, Englewood Cliffs, NJ, Chap. 11.

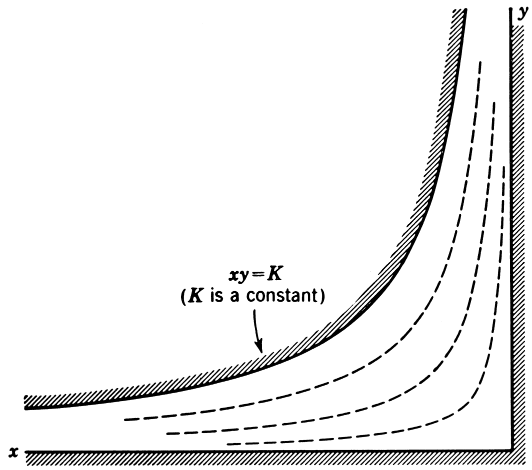

Figure 3.5
Two-dimensional flow showing streamlines.

■ Execution

By definition, the streamlines must have the same slope as the velocity vectors at all points. Equating these slopes, we get

$$\left(\frac{dy}{dx}\right)_{\text{str}} = \frac{V_y}{V_x} = -\frac{y}{x} \qquad \text{[b]}$$

Separating the variables and integrating, we have

$$\ln y = -\ln x + \ln C$$

Hence,

$$xy = C$$

Note that the streamlines form a family of rectangular hyperbolas. The wetted boundaries are part of the family, as is to be expected.

The components of acceleration may now easily be determined. Since this is steady flow, there will be only the acceleration of transport. Employing Eq. 3.7 under these conditions, we get

$$\begin{aligned} a_x &= (-Ax)(-A) + (Ay)(0) + (0)(0) = A^2x \\ a_y &= (-Ax)(0) + (Ay)(A) + (0)(0) = A^2y \\ a_z &= 0 \end{aligned} \qquad \text{[c]}$$

Hence

$$\mathbf{a} = A^2x\mathbf{i} + A^2y\mathbf{j} \qquad \text{[d]}$$

To give the acceleration of a particle at position $x'y'$ at any time, merely substitute x', y' into Eq. d.

■ Debriefing

This example should give us a clear picture of a streamline and the fact that a boundary will always form one of the streamlines. If the flow is inviscid there generally will be nonzero fluid velocity along such a streamline. If the flow is viscous, remember that the fluid velocity along this boundary will have the same velocity as the boundary itself (in this case it would be zero velocity).

EXAMPLE 3.2

■ Problem Statement

Given the velocity field

$$\mathbf{V}(x, y, z, t) = 10x^2\mathbf{i} - 20yx\mathbf{j} + 100t\mathbf{k} \text{ m/s}$$

determine the velocity and acceleration of a particle at position $x = 1$ m, $y = 2$ m, $z = 5$ m, and $t = 0.1$ s.

■ Strategy

We will use the definition of a *field* to get the desired velocity. We will use the definition of the *substantial derivative* to get the desired acceleration.

■ Execution

The velocity is determined by inserting the proper spatial coordinates and time into the vector velocity field to get a specific velocity as follows:

$$\mathbf{V} = (10)(1)\mathbf{i} - (20)(2)(1)\mathbf{j} + (100)(0.1)\mathbf{k} = 10\mathbf{i} - 40\mathbf{j} + 10\mathbf{k} \text{ m/s}$$

To get the acceleration of any one particle, we must use the *Lagrange* viewpoint to establish the acceleration field. Thus

$$\begin{aligned}\mathbf{a}(x, y, z, t) &= \left(V_x\frac{\partial \mathbf{V}}{\partial x} + V_y\frac{\partial \mathbf{V}}{\partial y} + V_z\frac{\partial \mathbf{V}}{\partial z}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right)\\ &= [(10x^2)(20x\mathbf{i} - 20y\mathbf{j}) + (-20yx)(-20x\mathbf{j})] + 100\mathbf{k}\\ &= 200x^3\mathbf{i} + (-200x^2y + 400yx^2)\mathbf{j} + 100\mathbf{k} \text{ m/s}^2\end{aligned}$$

For the particle of interest, the acceleration is

$$\begin{aligned}\mathbf{a} &= (200)(1^3)\mathbf{i} + [-200(1^2)(2) + 400(2)(1^2)]\mathbf{j} + 100\mathbf{k}\\ &= 200\mathbf{i} + 400\mathbf{j} + 100\mathbf{k} \quad \text{m/s}^2\end{aligned}$$

■ Debriefing

Note that we used the field approach for both the velocity field and the acceleration field emerging from use of the substantial derivative.

3.4 IRROTATIONAL FLOW

Earlier we presented the velocity field $\mathbf{V}(x, y, z, t)$, permitting us to give the velocity of a particle of fluid anywhere in the flow field. We learned in physics that it is the *relative motion* between *adjacent* atoms and molecules that is related to bonding forces between atoms and molecules. Similarly in fluid flow, it is the *relative motion* between *adjacent* flow particles that is related most simply to stresses. We now examine this relative movement.

We wish to point out first that the word "adjacent" will connote for us particles infinitesimally apart. Accordingly, we have shown in Fig. 3.6 two adjacent particles A and B a distance $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ apart at time t. To aid in the consideration of the relative movement between A and B, we have shown in Fig. 3.7 a rectangular parallelepiped for which $\overline{AB}$ is the diagonal. Now if we can effectively describe the deformation and rotation rates of this rectangular parallelepiped, we can in some way give the relative motion between A and B in terms of these rates. To accomplish

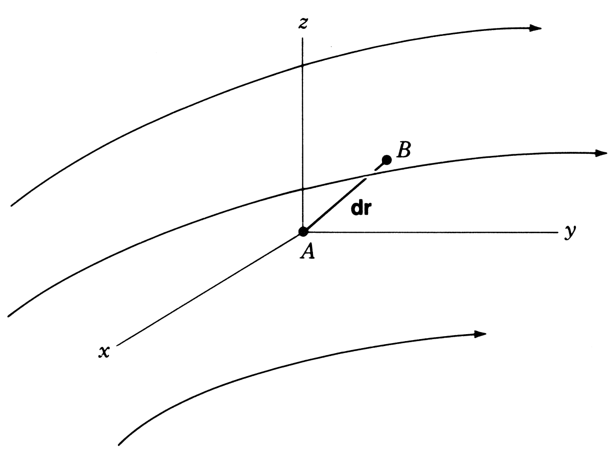

Figure 3.6
Adjacent particles A and B.

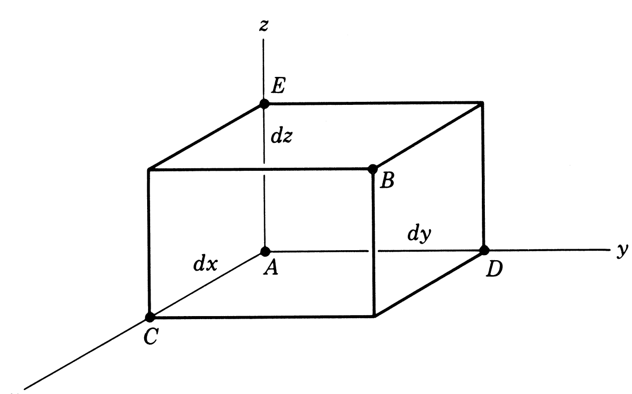

Figure 3.7
Adjacent particles along reference axes.

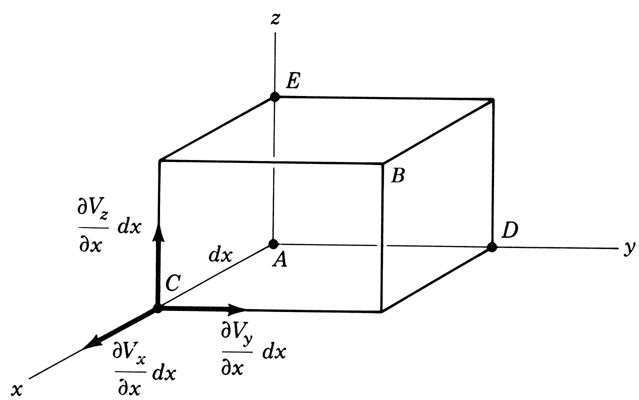

Figure 3.8
Components of $(\mathbf{V}_C - \mathbf{V}_A)$.

this, we have shown three additional particles C, D, and E at corners of the rectangular parallelepiped along axes xyz. If we know the relative motion between C and A, between D and A, and between E and A, we then know the deformation and rotation rates of the rectangular parallelepiped, and we can then express the relative motion between B and A in terms of the aforementioned relative motions.

Hence, we start with particle C. The velocity $\mathbf{V}_C$ of this particle can be given in terms of the velocity of particle A, namely $\mathbf{V}_A$, plus an infinitesimal increment, since C is a distance dx apart from A. Thus we have

$$\mathbf{V}_C = \mathbf{V}_A + \left(\frac{\partial \mathbf{V}}{\partial x}\right) dx$$
$$\therefore\ (\mathbf{V}_C - \mathbf{V}_A) = \left(\frac{\partial V_x}{\partial x}\right) dx\,\mathbf{i} + \left(\frac{\partial V_y}{\partial x}\right) dx\,\mathbf{j} + \left(\frac{\partial V_z}{\partial x}\right) dx\,\mathbf{k} \qquad [3.11]$$

The relative motion between C and A is $(\mathbf{V}_C - \mathbf{V}_A)$. It will be simplest to consider A as stationary and C as moving. The resulting conclusions will still be general. The components of $(\mathbf{V}_C - \mathbf{V}_A)$ as given by Eq. 3.11 are then shown in Fig. 3.8. We can set forth motion, respectively, of particles D and E relative to A in the same manner. In Fig. 3.9 we have shown the velocity components for particles C, D, and E relative to particle A. Consider now particle C. It is clear in Fig. 3.9 that $(\partial V_x/\partial x)\,dx$ is the rate of elongation of line segment AC. And if we express this elongation rate per *unit original length,* we have simply $\partial V_x/\partial x$.

But from your course in strength of materials you will recall that the elongation of an infinitesimal line segment in the x direction per unit original length is the normal strain ϵ_{xx}.[3] Thus we may conclude that

$$\frac{\partial V_x}{\partial x} = \dot{\epsilon}_{xx}$$

[3]See I. H. Shames and J. Pitarresi *Introduction to Solid Mechanics,* 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, Chap. 3.

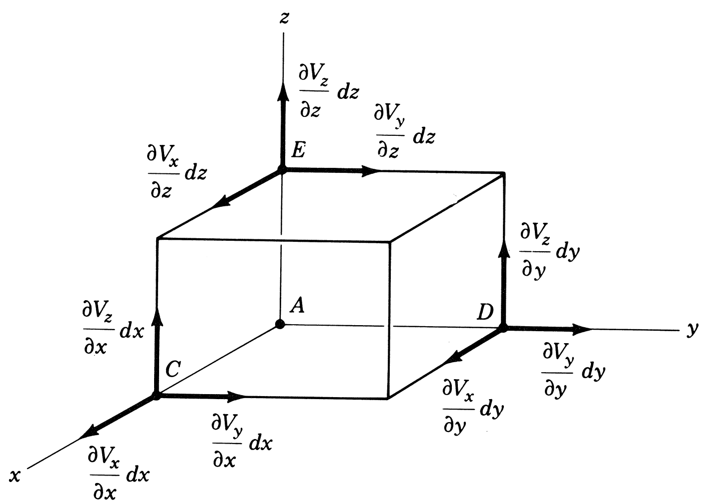

Figure 3.9
Relative velocity components for adjacent particles C, D, and E.

where the dot represents a time rate of change. Similarly we can say that

$$\frac{\partial V_y}{\partial y} = \dot{\epsilon}_{yy}$$

$$\frac{\partial V_z}{\partial z} = \dot{\epsilon}_{zz}$$

Thus we have depicted the time rates of *elongation* per unit original length (normal strain rates) of the sides of the rectangular parallelepiped. Next, we investigate the rate of *angular* change of the sides of the rectangular parallelepiped. Note in examining Fig. 3.9 that the velocity $(\partial V_y/\partial x)\,dx$ divided by dx is the angular velocity of AC about the z axis. Similarly at D, $(-\partial V_x/\partial y)\,dy$ divided by dy is the angular velocity of AD about the z axis. We can make two conclusions at this juncture:

1. The *average* rate of angular rotation about the z axis of the orthogonal line segments AC and AD is

$$\frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \qquad [3.12]$$

2. *The rate of change of the angle CAD* (a right angle at time t) becomes

$$\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \qquad [3.13]$$

The second result, you may recall from strength of materials, where we had $\gamma_{xy} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$, is the time *rate of change* of the *shear angle* γ_{xy} so that

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)$$

Similarly,

$$\dot{\gamma}_{xz} = \dot{\gamma}_{zx} = \left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)$$

$$\dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$

Accordingly, we have available to describe the deformation rate of the rectangular parallelepiped the strain rate terms which we now set forth as follows:[4]

$$\begin{bmatrix} \dot{\epsilon}_{xx} & \frac{\dot{\gamma}_{xy}}{2} & \frac{\dot{\gamma}_{xz}}{2} \\ \frac{\dot{\gamma}_{yx}}{2} & \dot{\epsilon}_{yy} & \frac{\dot{\gamma}_{yz}}{2} \\ \frac{\dot{\gamma}_{zx}}{2} & \frac{\dot{\gamma}_{zy}}{2} & \dot{\epsilon}_{zz} \end{bmatrix} = \text{strain rate tensor} \qquad [3.14]$$

Now experience from solid mechanics and intuition indicates that it is the strain rate tensor part of relative motion that is most simply related to the stress tensor.

We have thus far described two kinds of relative movement between the adjacent particles along coordinate axes. The normal strain rates give the rate of stretching or shrinking of the sides of the associated rectangular parallelepiped, while the shear-strain rates give rate of change of angularity of the edges of the rectangular parallelepiped. What's left of the relative movement must then be rigid-body *rotation*. Thus, the expression

$$\frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

is actually more than just the average rotation of line segments dx and dy about the z axis—it represents for a deformable medium what may be considered as the

[4]A note to the advanced reader: By using $\gamma/2$ instead of γ, you may have learned in strength of materials that the nine strain terms without dots form a symmetric second-order tensor. Taking time derivatives of each quantity and thereby forming an array of strain rates does not in any way alter the tensor character of the terms.

rigid-body angular velocity ω_z about the z axis.[5] This is,

$$\omega_z = \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \qquad [3.15]$$

Similarly for the other axes we have, by permuting indices,

$$\omega_x = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) \qquad [3.16]$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) \qquad [3.17]$$

$$\therefore\ \boldsymbol{\omega} = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k} \qquad [3.18]$$

Had we used a different coordinate system, we would have arrived at formulations which have a different form from Eqs. 3.15 to 3.18, but they would all pertain to the angular motion of fluid elements. Since the angular motion of fluid elements is a physical action not dependent on man-made coordinate systems, we have devised a vector operator called the *curl*[6] which when operating on a vector field **V** portrays twice the angular velocity. Thus Eq. 3.18 becomes

$$\boldsymbol{\omega} = \tfrac{1}{2}(\mathbf{curl\ V}) \equiv \tfrac{1}{2}\boldsymbol{\nabla} \times \mathbf{V} \qquad [3.19]$$

Note that Eq. 3.19 alludes to no particular coordinate system. Like the divergence operator and the gradient operator, the curl operator takes on a particular form when carried out in a particular coordinate system.[7] For instance, for cartesian coordinates we see from Eq. 3.18 that

$$\mathbf{curl\ A} \equiv \boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \qquad [3.20]$$

[5] The expression given by Eq. 3.15 is the *average* angular velocity of two orthogonal vanishingly small line segments dx and dy about the z axis. One can show that it is *also* the average angular velocity about the z axis of *all line segments* in the vanishingly small region dv. The "rigid body" interpretation obtains from the conclusion that if the fluid element in dv were imagined to become frozen at time t with the surrounding fluid made to simultaneously disappear, the frozen element would have the above angular velocity ω_z about the z axis at time t.

[6] The mathematical definition of the curl operator is given as

$$\mathbf{curl\ B} = -\lim_{\Delta V \to 0}\left[\frac{1}{\Delta V}\iint_S \mathbf{B} \times \mathbf{dA}\right]$$

where ΔV is any volume in space and S is the surface enclosing the volume.

[7] It is to be pointed out that there are straightforward general methods for forming the various vector operators for orthogonal coordinate systems. These may be found in mathematics books dealing with vector analysis.

We will not at this time evaluate the curl operator on other coordinate systems. It should be pointed out the curl can be used on any continuous vector field, and the physical interpretation of the resulting curl vector so formed will depend on the particular field operated on. The physical picture of rotation of an element is thus restricted to the curl of the velocity field, but understanding this particular case will help you interpret the curl of other fields.

At this time, we define *irrotational* flows as those for which $\boldsymbol{\omega} = \mathbf{0}$ at each point in the flow. *Rotational* flows are those where $\boldsymbol{\omega} \neq \mathbf{0}$ at points in the flow. For irrotational flow, we require that

$$\boxed{\begin{aligned} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} &= 0 \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} &= 0 \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} &= 0 \end{aligned}} \qquad [3.21]$$

From Eq. 3.19 it becomes clear that another criterion for irrotationality, and the one we will use, is

$$\boxed{\textbf{curl V} = \mathbf{0}} \qquad [3.22]$$

Finally we point out that $2\boldsymbol{\omega}$ is often called the *vorticity* vector.

3.5 RELATION BETWEEN IRROTATIONAL FLOW AND VISCOSITY

We now discuss some conditions under which we can expect rotational and irrotational types of flows. A development of rotation in a fluid particle in an initially irrotational flow would require shear stress to be present on the particle surface. It will be recalled that shear stress on a surface may be evaluated for parallel flows by the relation $\tau = \mu(\partial V/\partial n)$. Thus the shear stress in such flows and in more general flows will depend on the viscosity of the fluid and the manner of spatial variation of velocity (or the so-called velocity gradient) in the region. For fluids of small viscosity, such as air, irrotational flow will then persist in regions where large velocity gradients are not encountered. This may very often be over a great part of the flow. For instance, for an airfoil section moving through initially undisturbed air (Fig. 3.10), the fluid motion relative to the airfoil is that of an irrotational flow over most of the field. However, it is known that no matter how small the viscosity, real fluids "stick" to the surface of a solid body. Thus at point *A* on the airfoil the fluid velocity must be zero relative to the

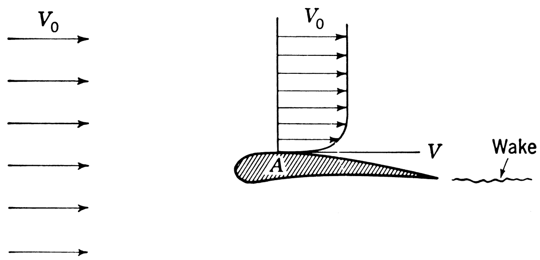

Figure 3.10
Velocity profile shows large velocity gradients near airfoil.

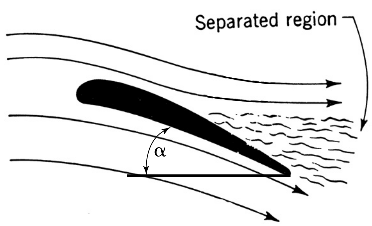

Figure 3.11
Flow separation for airfoil. The angle of attack is α.

airfoil, and at a comparatively short distance away it is almost equal to the free-stream velocity V_0. This is illustrated in the velocity profile of the diagram. Thus one sees that there is a thin region adjacent to the boundary where sizable velocity gradients must be present. Here, despite low viscosity, shear stresses of consequential magnitude are present, and the flow becomes rotational. This region adjacent to the solid boundary is called the *boundary layer*. It is fortunate, however, that much of the main flow is very often little affected by the flow conditions in the boundary layer, so that irrotational analysis may be carried out over a large part of the problem.

Another rotational-flow region may be found behind the trailing edge of the airfoil, where flows of different velocities from the upper and lower surfaces come into contact. Here again, large velocity gradients are present and consequently a rotational flow is present over a region behind the airfoil. This region is often called the *wake*.

Finally, we examine a condition called *separation,*[8] where the fluid flow cannot follow the boundary smoothly, as illustrated in Fig. 3.11 in the case of the airfoil at high angle of attack. Inside the separated regions we can again expect rotational flow.

In the flow shown in Fig. 3.11, it may be that the flow downstream of the separation point has regions of relatively small velocity gradients (hence small shear stress), where the flow is rotational. In the complete absence of further viscous action this rotation would persist indefinitely, so one may admit with good reason the theoretical possibility of frictionless rotational flow.

[8]The boundary layer and the separation process will be discussed at length in Chap. 12.

3.6 BASIC AND SUBSIDIARY LAWS FOR CONTINUOUS MEDIA

Now that means for describing fluid properties and flow characteristics have been established, we turn to the considerations of the interrelations among scalar, vector, and tensor quantities that we have set forth. Experience dictates that in the range of engineering interest four *basic laws* must be satisfied for any continuous medium. These are:

1. Conservation of matter (continuity equation).
2. Newton's second law (momentum and moment-of-momentum equations).
3. Conservation of energy (first law of thermodynamics).
4. Second law of thermodynamics.

In addition to these general laws, there are numerous *subsidiary laws,* sometimes called *constitutive* relations, that apply to specific types of media. We have already discussed two subsidiary laws, namely, the equation of state for the perfect gas and Newton's viscosity law for certain viscous fluids. Furthermore, for elastic solids there is the well-known Hooke's law, which you studied in strength of materials.

3.7 SYSTEMS AND CONTROL VOLUMES

In employing the basic and subsidiary laws, either one of the following modes of application may be adopted:

1. The activities of each and every given mass must be such as to satisfy the basic laws and the pertinent subsidiary laws.
2. The activities in each and every volume in space must be such that the basic laws and the pertinent subsidiary laws are satisfied.

In the first instance the laws are applied to an identified quantity of matter called the *system*. A system may change shape, position, and thermal condition but must *always entail the same matter*. For example, one may choose the steam in an engine cylinder (Fig. 3.12) after the cutoff [9] to be the system. As the piston moves, the volume of the system changes but there is no change in the quantity and identity of mass.

For the second case, a definite volume, called the *control volume,* is designated in space, and the boundary of this volume is known as the *control surface*.[10] The amount and identity of the matter in the control volume may change with time, but the shape of the control volume to be used in this text is fixed.[11] For instance, to

[9] No further addition of steam takes place after cutoff during the expansion stroke of the steam engine.

[10] In some thermodynamics texts the term *closed system* corresponds to our *system* and *open system* corresponds to our *control volume.*

[11] Some problems can be solved by employing a control volume of variable shape. However, in this text the control volume will always have a fixed shape.

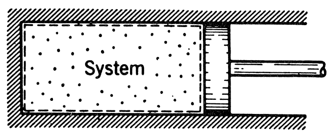

Figure 3.12
A system.

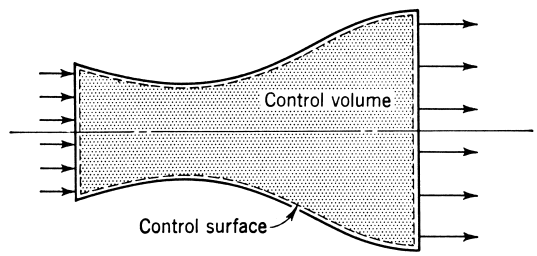

Figure 3.13
Control volume for the inside of a nozzle.

study flow through a nozzle, one could choose, as a control volume, the interior of the nozzle as shown in Fig. 3.13. We note that the control volume and the system can be infinitesimal.

In rigid-body mechanics it was the system approach (at that time called the *free-body diagram*) that was invariably used, since it was easy and direct to identify the rigid body, or portions thereof, in the problem and to work with each body as a discrete entity. However, since infinite numbers of particles having complicated motion relative to each other must be dealt with in fluid mechanics, it will often be advantageous to use control volumes in certain computations.

3.8 A RELATION BETWEEN THE SYSTEM APPROACH AND THE CONTROL-VOLUME APPROACH

In Sec. 3.2 we presented two viewpoints involving vector fields associated with a velocity field. These viewpoints allow us either to observe particles moving by a fixed position in space or to follow any one particle. We will now consider these viewpoints for *aggregates* of fluid elements constituting a finite mass where, in following the aggregate as per the Lagrange viewpoint, we are using the system approach. On the other hand, in stationing ourselves and observing in a finite region of space as per the Eulerian viewpoint, we are adopting the control-volume approach. We will now be able to relate the system approach and the control-volume approach for certain fluid and flow properties which we next describe.

In thermodynamics one usually makes a distinction between those properties of a substance whose measure depends on the amount of mass of the substance present and those properties whose measure is independent of the amount of mass of the substance present. The former are called *extensive* properties; the latter are called *intensive* properties. Examples of extensive properties are weight, momentum, volume, and energy. Clearly, changing the amount of mass directly changes the measure of these properties, and it is for this reason that we think of extensive properties as directly associated with the material itself. For each extensive variable such as

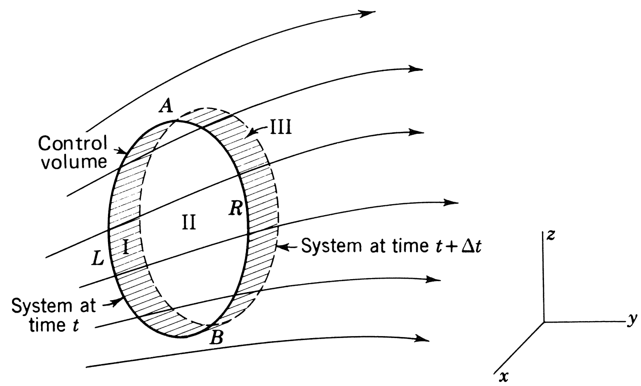

Figure 3.14
Simplified view of a moving system.

volume V and energy E, one can introduce by *distributive measurements* the corresponding intensive properties, namely, volume per unit mass v and energy per unit mass e, respectively. Thus we have $V = \iiint v\rho\, dv$[12] and $E = \iiint e\rho\, dv$. Clearly, v and e do not depend on the amount of matter present and are hence the intensive quantities related to the extensive properties V and E by distributive measure. Also, such quantities are termed *specific,* i.e., specific volume and specific energy, and are generally denoted by lowercase letters. Furthermore, such properties as temperature and pressure are by their *mass-independent nature* already in the category of the intensive property. Thus any portion of a metal bar at uniform temperature T_0 also has the same temperature T_0. Nor does the pressure of 1 ft^3 of air in a 10-ft^3 tank at uniform pressure p_0 differ from the pressure of 3 ft^3 of air in the tank. It is with *extensive* properties that we will now relate the system approach with the control-volume approach.

Consider next an arbitrary flow field $\mathbf{V}(x, y, z, t)$ as seen from some reference xyz wherein we observe a *system* of fluid of finite mass at times t and $t + \Delta t$, as shown in a highly idealized manner in Fig. 3.14 by the full line curve and the dashed line curve, respectively. The streamlines correspond to those at time t. In addition to this system, we will consider that the volume in space occupied by the system at time t is a *control volume fixed* in position and shape in xyz. Hence, at time t our system is identical to the fluid inside our control volume, shown by the full line curve. Let us now consider some arbitrary extensive property N of the fluid for the purpose of relating the rate of change of this property for the system with the variations of this property associated with the control volume. The distribution of N per unit mass will be given as η, such that $N = \iiint \eta\rho\, dv$ with dv representing an element of volume.

To do this, we have divided up the overlapping systems at time $t + \Delta t$ and at time t into three regions, as you will note in Fig. 3.14, where region II

[12]In this text we use v to represent specific volume and dv to represent the volume of a fluid element. Although the same letter is used in both terms, there should be no confusion if the terms are taken in context.

is common to the system at both times t and $t + \Delta t$. Let us compute the rate of change of N with respect to time for the system by the following limiting process:

$$\left(\frac{dN}{dt}\right)_{\text{system}} = \frac{DN}{Dt}$$
$$= \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{III}} \eta\rho\, dv + \iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{I}} \eta\rho\, dv + \iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] \qquad [3.23]$$

We may use the rule that the sum of the limits equals the limit of the sums to rearrange the equation above in the following manner:

$$\frac{DN}{Dt} = \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right]$$
$$+ \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{III}} \eta\rho\, dv\right)_{t+\Delta t}}{\Delta t}\right] - \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{I}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] \qquad [3.24]$$

Each one of the limiting processes above will now be considered separately. In the first one, we see on noting that $(\iiint_{\text{II}} \eta\rho\, dv)$ is a function of time that we have here by definition a partial time derivative of this function of time. And as $\Delta t \to 0$, the volume II becomes that of the control volume and the subscript II is replaced by the subscript CV. Also, as $\Delta t \to 0$, the time derivative is taken at time t. Accordingly, we can say that

$$\lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] = \frac{\partial}{\partial t}\iiint\limits_{\text{CV}} \eta\rho\, dv \qquad [3.25]$$

In the next limiting process of Eq. 3.24, we can consider the integral $(\iiint_{\text{III}} \eta\rho\, dv)_{t+\Delta t}$ to approximate the amount of property N that has crossed part of the control surface, which we have shown diagrammatically as ARB in Fig. 3.14 during the time Δt, so the ratio $(\iiint_{\text{III}} \eta\rho\, dv)_{t+\Delta t}/\Delta t$ approximates the average rate of efflux of N across ARB during the interval Δt. In the limit as $\Delta t \to 0$, this ratio becomes the *exact* rate of *efflux* of N through the control surface. Similarly, in considering the last limiting process of Eq. 3.24, we can consider for flows with continuous-flow characteristics and properties that the integral $(\iiint_{\text{I}} \eta\rho\, dv)_t$ approximates the amount of N that has passed *into* the control volume during Δt through the remaining portion of the control surface, which we have shown diagrammatically in Fig. 3.14 as ALB. In the limit, the ratio $(\iiint_{\text{I}} \eta\rho\, dv)_t/\Delta t$ then becomes the *exact* rate of *influx* of N into the control volume

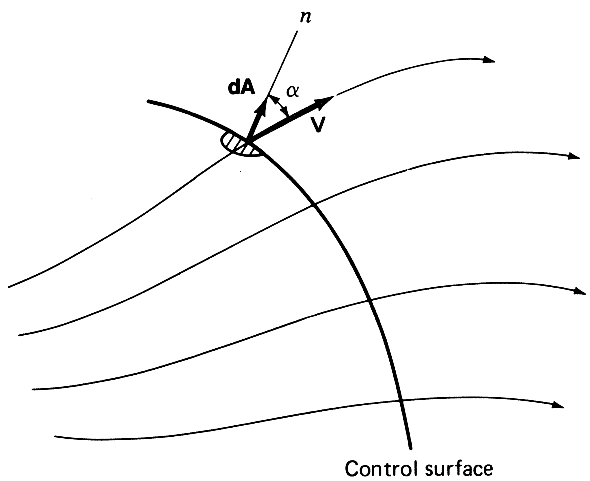

Figure 3.15
Interface dA at control surface at time t.

at time[13] t. Hence, the last two integrals of Eq. 3.24 give the *net* rate of *efflux* of N from the control volume at time t as

$$\lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\mathrm{III}} \eta\rho\, dv\right)_{t+\Delta t}}{\Delta t}\right] - \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\mathrm{I}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] = \text{Net efflux rate of } N \text{ from } CV \tag{3.26}$$

We thus see that by these limiting processes, we have equated the rate of change of N for a *system* at time t with the sum of two things:

1. The rate of change of N inside the control volume having the shape of the system at time t (Eq. 3.25).
2. The rate of efflux of N through the control surface at time t (Eq. 3.26).

We will now express Eq. 3.26 in a more compact, useful form. In this regard, consider Fig. 3.15, where we have a steady-flow velocity field and a portion of a control surface. An area **dA** on this surface has been shown. Now this area is also the interface of fluid that is just touching the control surface at the time t shown in the diagram. In Fig. 3.16 we have shown that interface of fluid at time $t + dt$. Note that the interface has moved a distance $V\,dt$ along a direction tangent to the streamline at that point. The volume of fluid dv that occupies the region swept out by dA in time dt thus forming a streamtube is

$$dv = (V\,dt)(dA\cos\alpha)$$

[13]Hence it is *minus* the *efflux* of N through ALB.

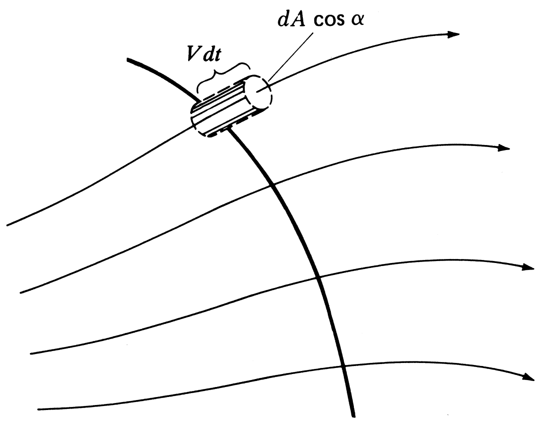

Figure 3.16
Interface dA at control surface at time $t + dt$.

Using the definition of the dot product, this becomes

$$dv = \mathbf{V} \cdot \mathbf{dA}\, dt$$

It should be apparent that dv is the volume of fluid that has crossed dA of the control surface in time dt. Multiplying by ρ and dividing by dt then gives the instantaneous rate of mass flow of fluid, $\rho\mathbf{V} \cdot \mathbf{dA}$, leaving the control volume through the indicated area dA.

The *efflux rate* of N through the control surface can be given approximately as[14]

$$\text{Efflux rate through CS} \approx \iint_{ARB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})$$

Note next that for fluid *entering* the control volume (see Fig. 3.17) the expression $\rho\mathbf{V} \cdot \mathbf{dA}$ must be negative because of the dot product. Hence, the *influx rate* expression of N through the control surface requires a negative sign to make the result the positive value that we know must exist. Hence, we have

$$\text{Influx rate through CS} \approx -\iint_{ALB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})$$

The approximate *net efflux* rate of N is then

$$\begin{aligned}\text{Net efflux rate} &\approx \text{efflux rate on } ARB - \text{influx rate on } ALB \\ &= \iint_{ARB} \eta(\rho\mathbf{V} \cdot \mathbf{dA}) - \left[-\iint_{ALB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})\right]\end{aligned}$$

[14]Considering the units of the expression $\eta(\rho\mathbf{V} \cdot \mathbf{dA})$, we get

$$\eta\left(\frac{\text{units of } N}{\text{mass}}\right)\rho\mathbf{V} \cdot \mathbf{dA}\left(\frac{\text{mass}}{\text{unit time}}\right)$$

which is the efflux of N per unit time through $\mathbf{dA}$.

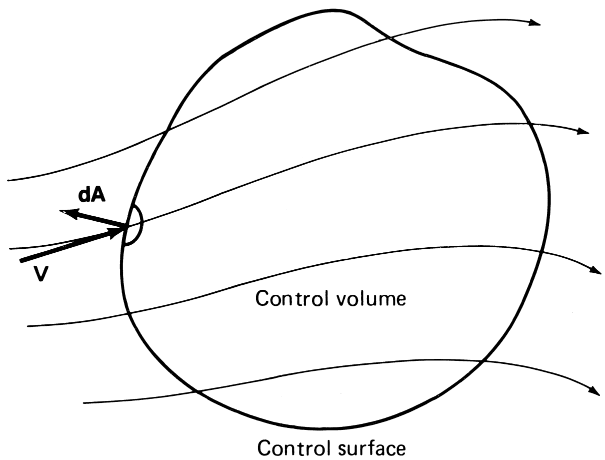

Figure 3.17
Control surface showing influx of mass.

In the limit as $\Delta t \to 0$, the approximations become exact, so we can express the right side of the equation above as $\oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA})$, where the integral is a closed surface integral over the entire control surface. Thus Eq. 3.26 can now be given as

$$\text{Net efflux rate of } N \text{ from CV} = \oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) \qquad [3.27]$$

It is to be pointed out that the development of Eq. 3.27 was made for simplicity for a steady-flow velocity field. However, it also holds for unsteady flow, since unsteady effects are of second order for this development. Now using Eqs. 3.27 and 3.25 for the various limiting processes, we can go back to Eq. 3.23 and state that

$$\boxed{\frac{DN}{Dt} = \oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} \eta \rho \, dv} \qquad [3.28]$$

This is called the *Reynolds transport* equation.[15] This equation permits us to change from a system approach to a control-volume approach.

You will note in the development that the velocity field was measured relative to some reference xyz and the control volume was *fixed* in this reference. This makes it clear that the fluid velocity $\mathbf{V}$ in the equation above *is in effect measured relative to the control volume*. Furthermore, you will recall from mechanics that the time

[15]Although the Reynolds transport equation has been carefully developed from a mathematical point of view, it does have a rather straightforward physical interpretation. We can illustrate this most simply by considering your classroom as the control volume and the system consisting of all the students in the classroom at any time t. Let N be the mass of the system. After the bell has rung for the end of the class period, there will be, at time t, an efflux rate of mass through the doorways (part of the control surface) with a resulting rate of change of mass inside the classroom. The Reynolds transport equation requires that $dN/dt = 0$ at any time t since we are not destroying students nor are we creating students. Thus, the efflux rate of mass plus the rate of change of mass inside at this time t clearly should be zero. (Would you have it any other way?)

rate of change of a vector quantity depends on the reference from which the change is observed. This is an important consideration for us here, since N (and η) can be a vector quantity (as, for example, momentum). Since the system moves in accordance with the velocity field given relative to xyz in our development, we see that the time rate of change of N is observed also from the xyz reference. Or, a more important conclusion, *the time rate of change of N is in effect observed from the control volume. Thus all velocities and time rates of change of Eq. 3.28 are those seen from the control volume*. Since we could have used a reference xyz having any arbitrary motion in the development above, it means that our control volume can have any motion whatever. Equation 3.28 will then instantaneously be correct if we measure the time derivatives and velocities relative to the control volume, no matter what the motion of the control volume may be. Finally, it can be shown that for an *infinitesimal control volume,* and an *infinitesimal system,* Eq. 3.28 reduces to an identity. This will explain why the system and control-volume equations as developed in subsequent chapters become redundant for infinitesimal considerations.

In Chaps. 4 and 5 we formulate the control-volume approach for the basic laws mentioned earlier by starting in each case with the familiar system formulation and extending it with the aid of the Reynolds transport equation to the control-volume formulation. As you do this several times in the next Chap. 4, you will develop a greater physical feel for the Reynolds transport equation, which may seem at this time "artificial." Perhaps the realization that all human efforts to explain nature analytically are artificial may be of some comfort. Two additional "artificialities" will now be presented to permit us to use the basic laws, soon to be developed, with greater effect.

3.9 ONE- AND TWO-DIMENSIONAL FLOWS

In every analysis a hypothetical substance or process is set forth which lends itself to mathematical treatment while still yielding results of practical value. We have already discussed the continuum concept. Now, simplified flows are set forth, which, when used with discretion, will permit the use of highly developed theory on problems of engineering interest.

One-dimensional flow is a simplification where all properties and flow characteristics are assumed to be expressible as functions of *one space coordinate* and *time*. The position is usually the location along some path or conduit. For instance, a one-dimensional flow in the pipe shown in Fig. 3.18 would require that the velocity, pressure, and so forth be constant over any given cross section at any given time, and vary only with s at this time t.

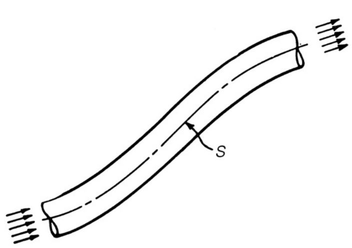

Figure 3.18
One-dimensional (1-D) flow.

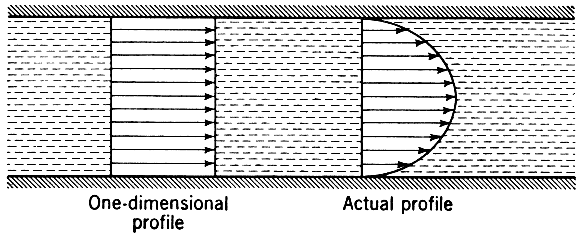

Figure 3.19
Comparison of 1-D flow and actual flow.

In reality, flow in pipes and conduits is never truly one dimensional, since the velocity will vary over the cross section. Shown in Fig. 3.19 are the respective velocity profiles of a truly one-dimensional flow and that of an actual case. Nevertheless, if the departure is not too great or if average effects at a cross section are of interest, one-dimensional flow may be assumed to exist. For instance, in pipes and ducts this assumption is often acceptable where

1. Variation of cross section of the container is not too excessive.
2. Curvature of the streamlines is not excessive.
3. Velocity profile is known not to change appreciably along the duct.

Two-dimensional flow is distinguished by the condition that all properties and flow characteristics are functions of two cartesian coordinates, say, x, y, and time, and hence do not change along the z direction at a given instant. All planes normal to the z direction will, at the given instant, have the same streamline pattern. The flow past an airfoil of infinite aspect ratio[16] or the flow over a dam of infinite length and uniform cross section are mathematical examples of two-dimensional flows. Actually, in a real problem a two-dimensional flow is often assumed over most of the airfoil or dam, and "end corrections" are made to modify the results properly.

EXAMPLE 3.3

■ Problem Statement

Consider a viscous, steady flow through a pipe (Fig. 3.20). We will learn in Chap. 11 that the velocity profile forms a paraboloid about the pipe centerline, given as

$$V = -C\left(r^2 - \frac{D^2}{4}\right) \quad \text{m/s} \qquad \text{[a]}$$

where C is a constant.

a. What is the flow rate of mass $\frac{DM}{Dt}$ through the left end of the control surface, shown dashed?

b. What is the flow rate of kinetic energy $\frac{DKE}{Dt}$ through the left end of the control surface? Assume that the velocity profile does not change along the pipe.

[16]A wing of constant cross section and infinite length.

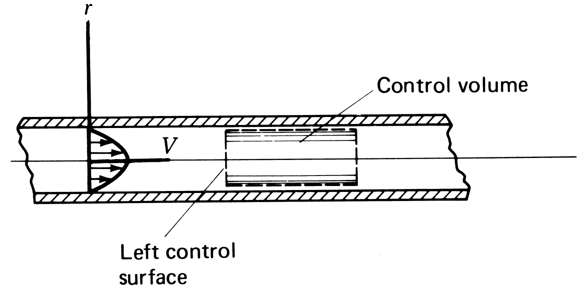

Figure 3.20
Steady viscous flow in a pipe.

■ Strategy

We shall consider in the cross section of the pipe a concentric ring of infinitesimal thickness. Because the velocity depends only on the variable r, the velocity will be constant through the ring. The integration of the mass flow through the rings covering the cross section will now be simple.

Next, using the intensive property for kinetic energy, and then including it in the integral for the mass flow, we will get on integration the kinetic energy flow rate in the pipe.

■ Execution

In Fig. 3.21, we have shown a cross section of the pipe. Using infinitesimal circular rings, we can say, noting that **V** and **dA** are colinear but of opposite sense,

$$\frac{DM}{Dt} = \iint \rho \mathbf{V} \cdot \mathbf{dA} = \rho \int_0^{D/2} C\left(r^2 - \frac{D^2}{4}\right) 2\pi r\, dr$$

$$\frac{DM}{Dt} = 2\pi\rho C\left[\frac{r^4}{4} - \frac{D^2}{4}\frac{r^2}{2}\right]_0^{D/2}$$

$$\frac{DM}{Dt} = -\frac{\rho C \pi D^4}{32} \quad \text{kg/s} \qquad \text{[b]}$$

We now turn to the flow of kinetic energy through the left end of the control surface. The kinetic energy for an element of fluid is $\frac{1}{2} dm V^2$. This corresponds to an infinitesimal amount of an extensive property. To get η, the corresponding intensive property, we divide by dm and use v for V to get

$$\eta = \tfrac{1}{2} v^2 \qquad \text{[c]}$$

We accordingly wish to compute

$$\iint \eta(\rho \mathbf{V} \cdot \mathbf{dA}) = \iint \left(\tfrac{1}{2} v^2\right)\{\rho \mathbf{V} \cdot \mathbf{dA}\}$$

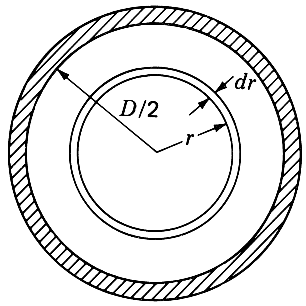

Figure 3.21
Cross section of pipe with infinitesimal ring of fluid.

Employing Eq. a for V, and noting again that $\mathbf{V}$ and $\mathbf{dA}$ are collinear but of opposite sense, we get

$$\frac{DKE}{Dt} = \iint \eta(\rho \mathbf{V} \cdot \mathbf{dA}) = \int_0^{D/2} \frac{1}{2} C^2 \left(r^2 - \frac{D^2}{4}\right)^2 \left\{\rho \left[C\left(r^2 - \frac{D^2}{4}\right) 2\pi r\, dr\right]\right\}$$

$$\frac{DKE}{Dt} = \rho C^3 \pi \int_0^{D/2} \left(r^2 - \frac{D^2}{4}\right)^3 r\, dr$$

$$\frac{DKE}{Dt} = \frac{\rho C^3 \pi D^8}{2048} \quad \text{N} \cdot \text{m/s} \tag{d}$$

where we could have facilitated the integration by making a change of variable for $\left(r^2 - \frac{D^2}{4}\right)$ to a single variable—say ξ.

■ Debriefing

We have demonstrated the setting up of integrals for computing two flow rates of extensive properties to get DM/Dt and $D(KE)/Dt$ through what could be part of a control surface. We will be making similar calculations starting with Chap. 4 and continuing through the book for integrals stemming from the Reynolds transport equation.

EXAMPLE 3.4

■ Problem Statement

For Example 3.3, assume a *one-dimensional* model with the same mass flow. Compute the kinetic energy flow through a section of the pipe for this flow.

■ Strategy

Using a constant axial velocity component V_{av} times the cross section area, we will get, on including the mass density, the mass flow rate for this axial velocity for a one-dimensional flow. Setting the mass flow rate developed in Example 3.3 for an actual velocity profile, equal to that of the one-dimensional case, we will determine the proper value of the aforementioned constant velocity required for the one-dimensional simplification. Using this constant velocity, we will determine by ordinary multiplication the flow rate of kinetic energy for the one-dimensional model.

■ Execution

We first proceed to compute the constant velocity needed to achieve a mass flow rate in a one-dimensional flow in the pipe equal to the actual mass flow rate in the pipe as developed in Example 3.3 (see Eq. b). Thus, equating these mass flow rates,

$$-(V_{\text{av}})\left(\frac{\rho\pi D^2}{4}\right) = -\frac{\rho C D^4 \pi}{32}$$
$$\therefore\ V_{\text{av}} = \frac{CD^2}{8} \quad \text{m/s} \qquad \text{[a]}$$

The kinetic energy flow for the one-dimensional model is then

$$\iint \frac{V^2}{2}(\rho \mathbf{V} \cdot \mathbf{dA}) = -\frac{\rho}{2}\left(\frac{CD^2}{8}\right)^3\left(\frac{\pi D^2}{4}\right)$$
$$= -\frac{\rho C^3 D^8 \pi}{4096} \quad \text{N} \cdot \text{m/s} \qquad \text{[b]}$$

We now define the *kinetic-energy correction* factor α as the ratio of the actual flow of kinetic energy through a cross section to the flow of kinetic energy for a one-dimensional model for the same mass flow. That is

$$\alpha = \frac{KE \text{ flow for section}}{KE \text{ flow for 1-D model}} \qquad \text{[c]}$$

For the case at hand, we have from Eq. b of this example and Eq. d of Example 3.3

$$\alpha = \frac{-\rho C^3 \pi D^8/2048}{-\rho C^3 \pi D^8/4096} = 2 \qquad \text{[d]}$$

The factor α exceeds unity, so there is an underestimation of kinetic energy flow for a one-dimensional model. We will have more to say about this point later in the text.

■ Debriefing

This example gives us the opportunity to assess the degree of error incurred using the one-dimensional model for the single case of kinetic energy flow. Clearly, this kind of error must at times be taken into account for other variables when modeling flows for the purpose of simplifying calculations of problems.

HIGHLIGHTS

When we are dealing with a finite set of particles, we can identify any one particle by using a subscript. Then, incorporating this into a time function, we can easily describe the velocity of any one particle. This is exactly what we did in your dynamics course where we used such notation as, for example, $(V(t))_n$. In the case of a fluid, where countless particles are involved, it is clear that a different approach must be used. Here, instead of using a subscript to identify any one particle, we use the spatial coordinates and the time to identify any one particle. And we incorporate these coordinates into a function to give the velocity for any one particle. Thus, we use the notation $\mathbf{V}(x, y, z, t)$ where the position of any one particle as well as the velocity of this particle can be specified. This is called the **field approach.**

We demonstrated that the field approach can be used very creatively. First, in $\mathbf{V}(x, y, z, t)$ we can pick fixed coordinates which we denote as (x_0, y_0, z_0, t) and allow t to progress. The resulting velocity formulation $\mathbf{V}(x_0, y_0, z_0, t)$ then conveys the velocity as time progresses of a string of particles as they pass by the chosen fixed point. This very useful approach is called the **Eulerian viewpoint.** On the other hand, we can imagine following any one particle in the flow as time progresses. For this use, the spatial coordinates must vary with time in such a way as to always locate the chosen particle at any time t. This is called the **Lagrangian viewpoint.** Why is this useful? In the dynamics of a single particle, Newton's law applies to this particle as it is followed. We are doing the same thing here for a fluid wherein we are in the presence of countless particles requiring the use of a field approach to manage this.

Let us then straightway go to Newton's law for a fluid. We must use the Lagrangian viewpoint to focus on any one particle in the flow. We treat the spatial coordinates as certain time functions varying in such a manner as to follow any one particle. However, we shall not specify the time functions but realize that at any later time they could be specified for any particular particle. We are thus keeping the discussion open-ended at this point. We will need the acceleration of any one particle. We thus take the time derivative of $\mathbf{V}$ using the familiar **chain rule** of differential calculus. We get

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \left(\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{at} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt}\right) + \frac{\partial \mathbf{V}}{\partial t}$$

It should be clear that to follow any one particle we require $\dfrac{dx}{dt} = V_x, \dfrac{dy}{dt} = V_y, \dfrac{dz}{dt} = V_z$, namely the velocity components of the particle.

The resulting formulation, called the **substantial derivative** or the **total derivative,** has the notation D replacing d,

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \left(V_x \frac{\partial \mathbf{V}}{\partial x} + V_y \frac{\partial \mathbf{V}}{\partial y} + V_z \frac{\partial \mathbf{V}}{\partial z}\right) + \frac{\partial \mathbf{V}}{\partial t}$$

The first bracketed expression gives the acceleration resulting from the particle being in the process of changing position in a velocity field. This velocity field is mathematically held constant by virtue of the fact that we are holding t constant. During the computation, we are allowing the particle to be in the process of moving. We are thus in the process of changing the position of the particle in this steady flow field. Because the velocity is a varied function of position (albeit a steady velocity field), the particle is hence in the process of accelerating. This acceleration is aptly called the **acceleration of transport.** For the last expression, we are mathematically holding the spatial coordinates stationary and are getting the acceleration contribution by virtue of our allowing the velocity field to be in the process of varying with time.

To apply the preceding concept, we will remind you of two simple definitions. The **system** is an identified aggregate of matter whose mass is constant but whose shape may be changing arbitrarily. A **control volume** is an identified fixed volume in space wherein there may be flow through the boundary, called the **control surface,** and where the amount of mass inside can be changing. Also, we define an extensive property N for a body as one which depends for its value on the amount of mass of the body. We then presented an extremely useful equation in this chapter called the **Reynolds transport equation** that we will use in Chapters 4 and 5. Noting that η is N per unit volume, we have

$$\frac{DN}{Dt} = \oiint_{\mathrm{CS}} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{\mathrm{CV}} \eta(\rho \, dv)$$

In essence, this theorem relates the time derivative of N of a system at time t, as one follows it (much like you did in dynamics), with the time rate of change of N determined by focusing on a control volume corresponding to the volume occupied by the system at time t. That is, we are relating a Lagrangian viewpoint for a system with an Eulerian viewpoint using the control volume, which is the boundary of the system at time t. How is the latter step, which perhaps is the least familiar to you, accomplished? At time t we have the rate of flow of N passing through the control surface, and we add to this the rate of change of N inside. In this way, we account for the total rate of change of N as we look at the control volume. Since the system has the identical volume as the control volume at time t and since the control volume entails identically the same matter as the system at time t, one would intuitively expect the same rates of change of N from both viewpoints at time t. However, since we proved the relation earlier we do not have to depend on intuition, although it is nice when it can be applied.

3.10 CLOSURE

In this chapter we have laid the foundation for the handling of fluid flow. Specifically, we have presented (1) kinematical procedures and concepts which enable us to describe the motion of fluids including the concept of irrotationality; (2) the four basic laws which will form the basis for our calculating the motion and flow characteristics of fluids; (3) the system and control-volume viewpoints by which we can apply these laws effectively to physical problems, and (4) the Reynolds transport equation relating the system approach to the control-volume approach or, in other words, relating the Eulerian and the Lagrangian viewpoints. In Chapters 4 and 5, we will develop these basic laws for both finite systems and finite-control volumes in a very general form. And, in solving problems in those chapters, we will make liberal usage of the one-and two-dimensional flow models.

*3.11 COMPUTER EXAMPLE

COMPUTER EXAMPLE 3.1

■ Computer Problem Statement

You are making plans for a sailing regatta. The starting point is at A (see Fig. C3.1). You plan to make only one tack to get to the buoy at B as shown in the diagram. The speed of the boat depends on the orientation of the sail and the wind velocity, the latter as seen from the boat. We shall assume that the skipper has set the sail to achieve the maximum speed for any velocity direction of the sailboat as measured by the angle α shown in the diagram. We will estimate that the velocity of the boat is given as

$$V_i = 6.3 - 0.055\alpha_i \quad \text{knots with } \alpha_i \text{ in degrees}$$

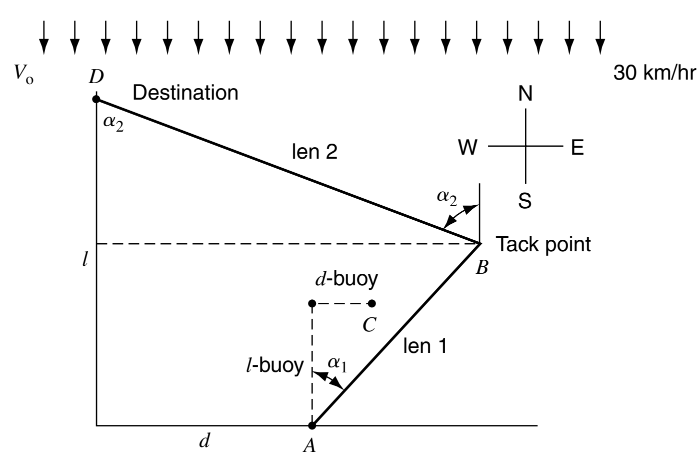

Figure C3.1

The range of each α_i is from 10 degrees to 45 degrees. Write an interactive program for the skipper for the time of passage from *A* to *B* asking for

What distance in feet is the starting point south of the destination?
What distance in feet is the starting point east of the destination?
How far north of the starting point is the buoy you have to go around?

Using the command `[a, b] = min` find the minimum value of "`a`" from each column, and "`b`" will then be the corresponding row for the matrix. This gives minimum time. Do the same for the "`c,d`" matrix. This will get the minimum time, the angle α_i, and the distance before tacking to win the race.

Use the program for the following data:
Starting point is 2900 ft south of destination.
Starting point is 800 ft east of destination.
Buoy is 1000 ft north of starting point.
Buoy is 500 ft east of starting point.

■ Strategy

We will use matrices to store values of every possible combination of distance before tacking (`len1`) and angle of departure (`a1`). Since we have every possible combination, we can use these to determine which combination of initial angle and distance to tack makes for the fastest time to the finish, based on the starting and ending point locations and the location of the buoy, `b` (which we must go around). Once the minimum time is pinpointed in the total time matrix, its location can be used to determine which "`len1`" and "`a1`" were used in determining it. These two values will tell us all we need to know to win the race!

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

con=pi./180;
%This is the constant for converting from degrees to
%radians.

len=input('What distance in feet is the starting
point south of the destination?\n');
%This is the total distance the starting point (a) is
%south of the ending point (d).

dist=input('What distance in feet is the starting
point east of the destination?\n');
%This is the total distance the starting point (a) is
%east of the ending point (d).
```

```
1_buoy=input('How far north of the starting point is
the buoy you have to go around?\n');
d_buoy=input('How far east of the starting point is
the buoy you have to go around?\n');

11_vector=linspace(sqrt(1_buoy.^2+d_buoy.^2), 2.*(sqrt
(1_buoy.^2+d_buoy.^2)));
%We choose our tack to be after the buoy (obviously)
%but before twice the distance, ab, to the buoy.

al_vector=linspace(atan(d_buoy./1_buoy).*(180./pi), 89
,length(11_vector));
%The initial angle must be chosen to at least clear
%the buoy. We also want the size of this array to be
%the same size as the vector "11_vector".

for i=1:length(11_vector);

for j=1:length(a1_vector);

len1 (i, j)=11_vector (i);
%This makes a matrix out of a vector by making rows
%of each value in the vector "11_vector".

a1 (i,j)=a1_vector(j);
%This makes a matrix out of a vector by making
%columns of each value in the vector "al_vector".

end

end

a2=(atan((dist+len1.*sin(a1.*con))./(len-
len1.*cos(a1.*con)))).*(180./pi);
%This is the equation solving for alpha2 (in
%degrees).

len2=sqrt((dist+len1.*sin(a1.*con)).^2+(len-
len1.*cos(a1.*con)).^2);
%This is the equation for the distance between the
%tack and the ending point.

v1=6.3-.055.*a1;
%This is the equation for the velocity in knots
%between the beginning point and the tack.

t1=len1./(v1.*1.6878);
%Since we know the velocity and the distance of
%travel we can determine the time it takes. The
```

```
%1.6878 is to convert the velocity from knots to
%ft/sec.

v2=6.3-.055.*a2;
%This is the equation for the velocity in knots
%between the tack and the ending point.

t2=len2./(v2.*1.6878);
%This is the time between the tack and the ending
%point.

time=t1+t2;
%Once we add "t1" and "t2" then we can find the
%minimum value for the total time and find the "a1"
%and "11" that correspond to this minimum time.

[a,b]=min(time);
%When executed, "a" will be the minimum value from
%each column of matrix "time" and "b" will be the
%corresponding row of the matrix that the value was
%found in.

[c,d]=min(a);
%When executed, "c" will be the minimum value of the
%row vector created immediately above and "d" will be
%the corresponding column that it was found in.

fprintf('\nThe minimum amount of travel time is:
%4.2f minutes.\n\n',c./60);
fprintf('The value of a1 that will give us the
minimum time is: %4.2f\n\n', a1(b(d),d));
fprintf('The value of len1 that will give us the
minimum time is: %4.2f\n\n',len1(b(d),d));
fprintf('Therefore, continue a course of %4.2f
degrees for %4.2f feet before
tacking. \n\n',a1(b(d),d), len1 (b(d),d));

x1=linspace(dist,dist+len1(b(d),d).*sin(a1(b(d),d).*c
on));
y1=tan((90-a1(b(d),d)).*con).*x1-tan((90-
a1(b(d),d)).*con).*dist;
x2=linspace(0,len2(b(d),d).*sin(a2(b(d),d).*con));
y2=-tan((90-a2(b(d),d)).*con).*x2+len;
axis([0(dist+len1(b(d),d).*sin(a1(b(d),d).*con))
+1000 0 len]);
plot(x1,y1,'b',x2,y2,'g');
hold on;
plot(x1(1)+d_buoy,y1(1)+1_buoy,'o');
```

```
grid;
title ('The Best Course To Take To Win The Race!!');
text(x1(1)+100,y1(1)+100,`Point A: The starting
point');
text(x1(100),y1(100),`"Tacking!"');
text(x2(1),y2(1)-100,`Point C: The ending point');
text(x1(1)+d_buoy-500,y1(1)+1_buoy,`Buoy');
%All this just gives us a plot of the course we must
take to win the race and labels things appropriately
(see Fig. C3.2).
```

■ Debriefing

In MATLAB, matrices of values can be easily manipulated without the need of loop iteration. The only reason we even used loops in this problem was to generate the matrices. Once you have matrices they can be added, subtracted, and multiplied very easily without loops. If you want random values, MATLAB has an intrinsic function `"rand"` and `"randn"` which will generate a `n x n` matrix as easily as `"rand (n, n)"` and there is no limit on `"n"`!

In world class 30 m racing more careful computations are made in deciding the racing strategy. The initial direction that the yacht takes is very important for determining the route to take and where to take the tacks.

■ Computer Output

```
EDU>> mp14a
What distance in feet is the starting point south of the
destination?
2900
What distance in feet is the starting point east of the
destination?
800
How far north of the starting point is the buoy you have
to go around?
1000
How far east of the starting point is the buoy you have
to go around?
500
The minimum amount of travel time is: 7.44 minutes.
The value of a1 that will give us the minimum time is:
26.57
The value of len1 that will give us the minimum time is:
1118.03
Therefore, continue a course of 26.57 degrees for
1118.03 feet before tacking.
EDU>>
```

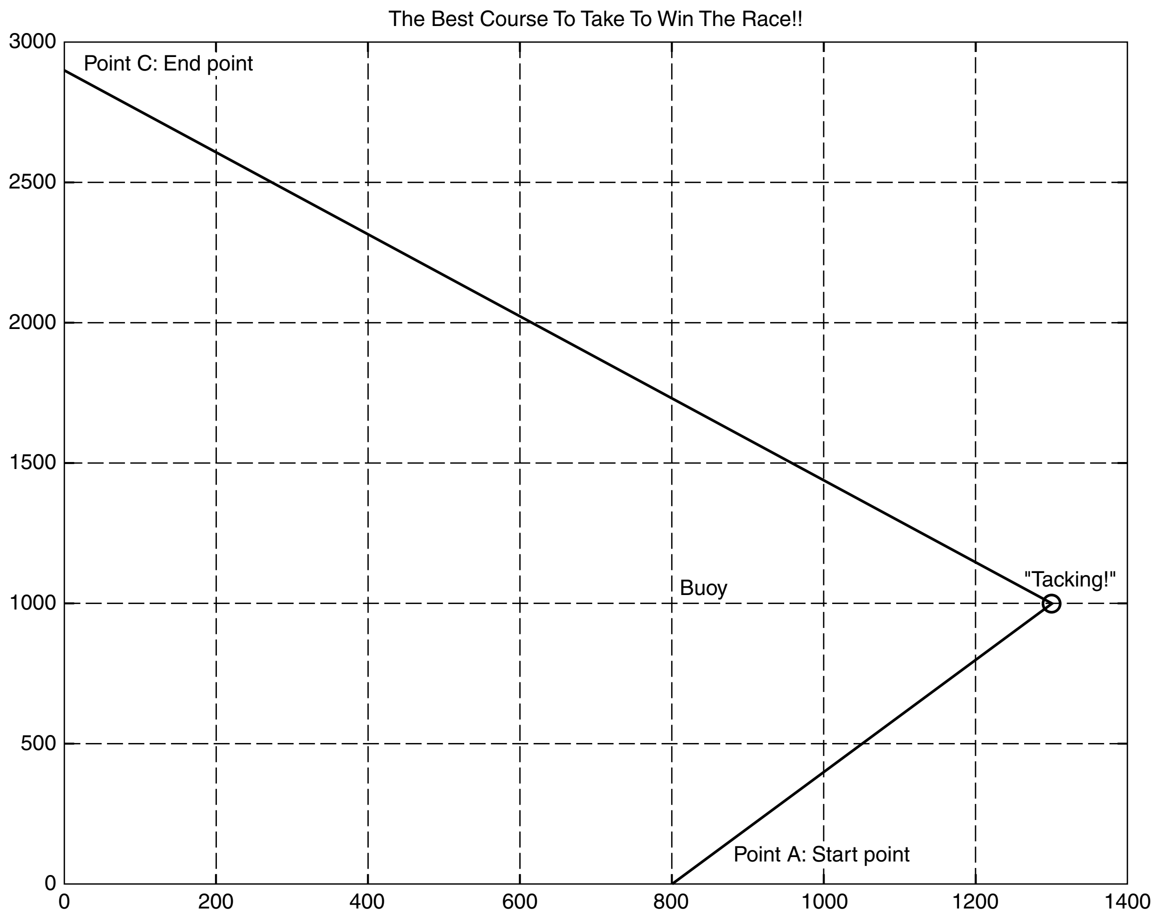
The Best Course To Take To Win The Race!!
Point C: End point
Buoy
"Tacking!"
Point A: Start point
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200
1400

Figure C3.2

PROBLEMS

Problem Categories

Velocity fields 3.1–3.3
Substantial derivatives 3.4–3.8
Streamlines 3.9–3.13
Noninertial references 3.14–3.15
Velocity field with cylindrical coordinates 3.16
Rotation and strain rates 3.17–3.20
Gradients 3.21–3.22
Rotationality and irrotationality 3.23–3.24
Basic laws 3.25–3.27
One-dimensional flows 3.28–3.30
Kinetic energy in flows 3.31–3.32
Computer problems 3.33–3.34

3.1 A flow field is given as

$$\mathbf{V} = 6x\mathbf{i} + 6y\mathbf{j} - 7t\mathbf{k} \quad \text{m/s}$$

What is the velocity at position $x = 10$ m and $y = 6$ m when $t = 10$ s? What is the slope of the streamlines for this flow at $t = 0$ s? What is the equation of the streamlines at $t = 0$ s up to an arbitrary constant? Finally, sketch streamlines at $t = 0$ s.

3.2 We will later learn that the two-dimensional flow around an infinite stationary cylinder is given as follows, using cylindrical coordinates:

$$V_r = V_0 \cos\theta - \frac{\chi \cos\theta}{r^2}$$

$$V_\theta = -V_0 \sin\theta - \frac{\chi \sin\theta}{r^2}$$

where V_0 and x are constants. (Note that there is no flow in the z direction.) What is the slope (dy/dx) of a streamline at $r = 2$ m and $\theta = 30°$? Take $V_0 = 5$ m/s and $\chi = \frac{5}{4}\,\text{m}^3/\text{s}$. Show that at $r = \sqrt{\chi/V_0}$ (i.e., on the boundary of the cylinder) the streamline must be tangent to the cylinder wall. *Hint:* What does this imply about normal component V_N at the boundary?

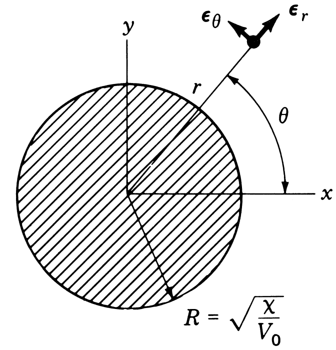

Figure P3.2

3.3 Given the following *unsteady*-flow field,

$$\mathbf{V} = 3(x - 2t)(y - 3t)^2\mathbf{i} + (6 + z + 4t)\mathbf{j} + 25\mathbf{k} \quad \text{ft/s}$$

can you specify by inspection a reference $x'y'z'$ moving at constant speed relative to xyz so that $\mathbf{V}$ relative to $x'y'z'$ is *steady*? What is $\mathbf{V}$ for this reference? What is the speed of translation of $x'y'z'$ relative to xyz? *Hint:* For the last step, imagine a point fixed in $x'y'z'$. How must $x'y'z'$ then move relative to xyz to get correct relations between x' and x, y' and y, and z' and z?

3.4 Using data from Prob. 3.1, determine the acceleration field for the flow. What is the acceleration of the particle at the position and time designated in Prob. 3.1?

3.5 Given the velocity field

$$\mathbf{V} = 10\mathbf{i} + (x^2 + y^2)\mathbf{j} - 2yx\mathbf{k} \quad \text{ft/s}$$

what is the acceleration of a particle at position (3, 1, 0) ft?

3.6 Given the velocity field

$$\mathbf{V} = (6 + 2xy + t^2)\mathbf{i} - (xy^2 + 10t)\mathbf{j} + 25\mathbf{k} \quad \text{m/s}$$

what is the acceleration of a particle at (3, 0, 2) m at time $t = 1$ s?

3.7 A flow of charged particles (a plasma) is moving through an electric field $\mathbf{E}$ given as

$$\mathbf{E} = (x^2 + 3t)\mathbf{i} + yz^2\mathbf{j} + (x^2 + z^2)\mathbf{k} \quad \text{N/C}$$

The velocity field of the particles is given as

$$\mathbf{V} = 10x^2\mathbf{i} + (5t + \sqrt{y})\mathbf{j} + t^3\mathbf{k} \quad \text{m/s}$$

If the charge per particle is 10^{-5} C, what is the time rate of change of force on any one particle as it moves through the field?

3.8 The force $\mathbf{F}$ on a particle with electric charge q moving through a magnetic field $\mathbf{B}$ is given as

$$\mathbf{F} = q\mathbf{V} \times \mathbf{B}$$

Consider a flow of charged particles moving through a magnetic field $\mathbf{B}$ given as

$$\mathbf{B} = (10 + t^2)\mathbf{i} + (z^2 + y^2)\mathbf{k} \quad \text{W/m}^2$$

where the velocity field is given as

$$\mathbf{V} = (20x + t^2)\mathbf{i} + (18 + zy)\mathbf{j} \quad \text{m/s}$$

What is the time rate of change of $\mathbf{F}$ for a flow particle with charge 10^{-5} C? Do not take time to multiply out terms in final computation.

3.9 The equation for streamlines corresponding to a two-dimensional doublet (to be studied in Chap. 11) is given in meters as

$$x^2 + y^2 - \frac{\chi}{C}y = 0 \qquad \text{[a]}$$

where χ is a constant for the flow and C is a constant for a streamline. What is the direction of the velocity of a particle at position $x = 5$ m and $y = 10$ m? If $V_x = 5$ m/s, what is V_y at the point of interest?

3.10 In Prob. 3.9, it should be apparent from analytic geometry that the streamlines represent circles. For a given value of χ and for different values of C, along what axis do the centers of the aforementioned circles lie? Show that all circles go through the origin. Sketch a system of streamlines.

3.11 In Example 3.1, what is the equation of the streamline passing through position $x = 2$, $y = 4$? Remembering that the radius of curvature of a curve is

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

determine the acceleration of a particle in a direction normal to the streamline and toward the center of curvature at the aforementioned position.

3.12 We are given the following family of curves representing streamlines for a two-dimensional source (Chap. 11):

$$y = Cx \qquad (1)$$

where C is a constant for each streamline. Also we know that

$$|\mathbf{V}| = \frac{K}{\sqrt{x^2 + y^2}} \qquad (2)$$

where K is a constant for the flow. What is the velocity field $\mathbf{V}(x, y, z)$ for the flow? That is, show that

$$V_x = \frac{Kx}{x^2 + y^2} \qquad V_y = \frac{Ky}{x^2 + y^2}$$

Suggestion: Start by showing that

$$|\mathbf{V}| = V_x\sqrt{1 + \left(\frac{V_y}{V_x}\right)^2} \quad \text{and} \quad \frac{V_y}{V_x} = C = \frac{y}{x}$$

3.13 A *path line* is the curve traversed by any one particle in the flow and corresponds to the *trajectory* as employed in your earlier course in particle mechanics. Given the velocity field

$$\mathbf{V} = (6x)\mathbf{i} + (16y + 10)\mathbf{j} + (20t^2)\mathbf{k} \quad \text{m/s}$$

what is the path line of a particle which is at (2, 4, 6) m at time $t = 2$ s? *Suggestion:* Form dx/dt, dy/dt, and dz/dt. Integrate: solve for constants of integration; then eliminate the time t to relate xyz in a single equation.

3.14 Consider a velocity field $\mathbf{V}(x, y, z, t)$ as measured from reference xyz. The reference xyz is moving relative to another reference XYZ with an angular velocity $\boldsymbol{\omega}$ and a translational velocity $\dot{\mathbf{R}}$ and has, in addition, an angular acceleration $\dot{\boldsymbol{\omega}}$ and a translational acceleration $\ddot{\mathbf{R}}$. From your earlier dynamics course, you may have learned that the acceleration of a particle relative to XYZ (that is, $\mathbf{a}_{XYZ}$) is given as

$$\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

where $\mathbf{a}_{xyz}$ and $\mathbf{V}_{xyz}$ are taken relative to xyz.

We have the following data at an instant:

$$\mathbf{V} = 10x\mathbf{i} + 30xy\mathbf{j} + (3x^2z + 10)\mathbf{k} \quad \text{m/s}$$
$$\boldsymbol{\omega} = 10\mathbf{i} \quad \text{rad/s}$$
$$\dot{\mathbf{R}} = 0 \quad \text{m/s}$$
$$\ddot{\mathbf{R}} = 16\mathbf{k} \quad \text{m/s}^2$$
$$\dot{\boldsymbol{\omega}} = 5\mathbf{k} \quad \text{rad/s}^2$$

What is the acceleration relative to xyz and XYZ, respectively, of a particle at

$$\boldsymbol{\rho} = 3\mathbf{i} + 3\mathbf{k} \quad \text{m}$$

at the instant of interest?

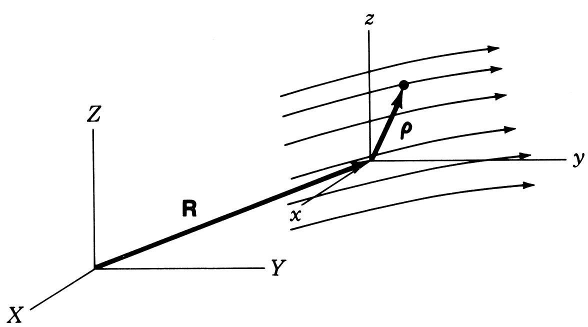

Figure P3.14

3.15 Think up and discuss a few situations where the formulations developed in Prob. 3.14 would be of use.

3.16 Consider a steady two-dimensional inviscid flow about a cylinder of radius a. Using cylindrical coordinates, we can express the velocity field of a nonviscous incompressible flow in the following manner,

$$\mathbf{V}(r, \theta) = -\left(V_0 \cos\theta - \frac{a^2V_0}{r^2}\cos\theta\right)\boldsymbol{\epsilon}_r + \left(V_0 \sin\theta + \frac{a^2V_0}{r^2}\sin\theta\right)\boldsymbol{\epsilon}_\theta$$

where V_0 is a constant and $\boldsymbol{\epsilon}_r$ and $\boldsymbol{\epsilon}_\theta$ are unit vectors in the radial and transverse directions, respectively, as shown in the diagram. What is the acceleration of a fluid particle at $\theta = \theta_0$ at the *boundary* of the cylinder whose radius is a? *Suggestion:* Use path coordinates. *Hint:* What must V_r be at the boundary?

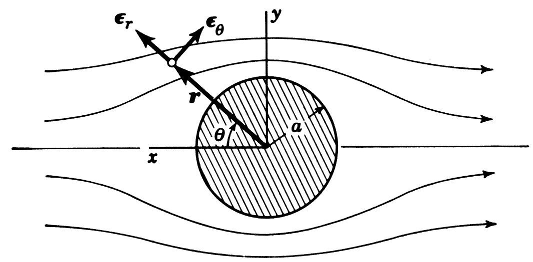

Figure P3.16

3.17 Given the following velocity field

$$\mathbf{V} = 10x^2y\mathbf{i} + 20(yz + x)\mathbf{j} + 13\mathbf{k} \quad \text{m/s}$$

what is the strain rate tensor at (6, 1, 2) m?

3.18 In Prob. 3.17, what is the total angular velocity of a fluid particle at (1, 4, 3) m?

3.19 Given the velocity field

$$\mathbf{V} = 5x^2y\mathbf{i} - (3x - 3z)\mathbf{j} + 10z^2\mathbf{k} \quad \text{m/s}$$

compute the angular velocity field $\boldsymbol{\omega}(x, y, z)$.

3.20 A flow has the following velocity field:

$$\mathbf{V} = (10t + x)\mathbf{i} + yz\mathbf{j} + 5t^2\mathbf{k} \quad \text{ft/s}$$

What is the angular velocity of a fluid element at $x = 10$ ft, $y = 3$ ft, and $z = 5$ ft? Along what surface is the flow always irrotational?

3.21 Show that any velocity field $\mathbf{V}$ expressible as the gradient of a scalar ϕ must be an irrotational field.

3.22 If $\mathbf{V} = \textbf{grad}\,\phi$, what irrotational flow is associated with the function

$$\phi = 3x^2y - 3x + 3y^2 + 16t^3 + 12zt$$

Read Prob. 3.21 before proceeding.

3.23 Is the following flow field irrotational or not?

$$\mathbf{V} = 6x^2y\mathbf{i} + 2x^3\mathbf{j} + 10\mathbf{k} \quad \text{ft/s}$$

3.24 Explain why in a capillary tube the flow is virtually always rotational.

3.25 What were the basic laws and subsidiary laws that you used in your course in strength of materials?

3.26 In the studies of rigid-body mechanics, how was conservation of mass ensured? Also, was conservation of energy a law independent and apart from Newton's laws? Explain the reason for your answer.

3.27 Have we placed any restrictions on the motion of a control volume? Can it have material other than fluid inside or passing through?

3.28 A fluid is moving along a curved circular pipe such that the pressure, velocity, and so forth are uniform at each section of the pipe and are functions of the position s along the centerline of the pipe and time. How would we classify this flow in the light of our discussion in this chapter? If the flow properties were also functions at a section of the radial distance r from the centerline in addition to s and t, would this then be a two-dimensional flow? Why?

3.29 In Example 3.3, compute the linear momentum flow through a cross section of the control volume. Recall that the linear momentum of a particle is $m\mathbf{V}$.

3.30 In Prob. 3.29 find a momentum correction factor which would be the ratio for the actual momentum flow to that of the one-dimensional model of the flow for the same mass flow. In the previous problem, we got the result

$$\iint V(\rho\mathbf{V} \cdot \mathbf{dA}) = -\frac{\rho C^2 \pi D^6}{192}$$

Do not consult Example 3.3 while doing this problem.

3.31 In Example 3.3, compute the kinetic energy flow through one face of the control surface if it is moving to the left at a speed of V_0 relative to the ground.

3.32 In Chap. 11, we discuss the simple *vortex* where in cylindrical coordinates

$$V_r = 0 \qquad V_z = 0$$
$$V_\theta = \frac{\Lambda}{2\pi r}$$

Λ is a constant called the *strength* of the vortex. Draw the streamlines for the simple vortex. What is the mass flow and kinetic energy flow through the surface shown in the diagram?

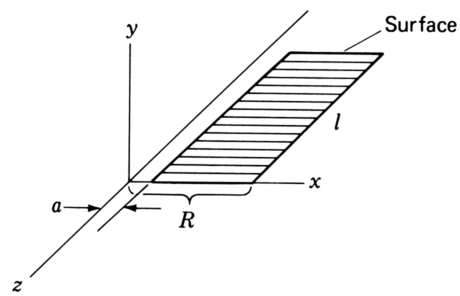

Figure P3.32

3.33 Given the following velocity field parallel to the xy plane

$$\mathbf{V} = (3t^2x^3 + t^{1/2})\mathbf{i} + (\ln y)(t^{3/2})\mathbf{j} \quad \text{ft/sec}$$

with t in seconds, plot the path of a fluid particle starting from (2, 5) ft at time $t = 0$. Observe the particle for 20 seconds, one second at a time.

3.34 In Problem 3.9, plot the streamlines for a two-dimensional doublet for which $\chi = 10$ m^3 for different constant values of C (which identifies the contour lines of the velocity field). Use the values of C equal to 2, 5, 8, and 12.