ME 33, Fluid Flow
Chapter 7: Dimensional Analysis and Modeling

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Dimensions and Units

Blackboard notes

Dimensional Homogeneity

Blackboard notes

- Nondimensionalization of an equation is useful only when the equation is known!
- In many real-world flows, the equations are either unknown or too difficult to solve.
  - *Experimentation* is the only method of obtaining reliable information
  - In most experiments, geometrically–scaled models are used (time and money).
  - Experimental conditions and results must be properly scaled so that results are meaningful for the full-scale prototype.
  - **Dimensional Analysis**

Dimensional Analysis and Similarity

Primary purposes of dimensional analysis

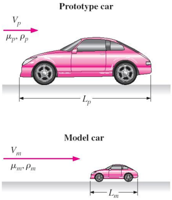
- To generate nondimensional parameters that help in the design of experiments (physical andor numerical) and in reporting of results.
- To obtain scaling laws so that prototype performance can be predicted from model performance.
- To predict trends in the relationship between parameters.

Principle of Similarity

- **Geometric Similarity** – the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.
- **Kinematic Similarity** – velocity as any point in the model must be proportional
- **Dynamic Similarity** – *all forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow.
- **Complete Similarity** is achieved only if all 3 conditions are met. This is not always possible, e.g., river hydraulics models.

Dimensional Analysis and Similarity

- Complete similarity is ensured if all independent Π groups are the same between model and prototype.
- What is Π?
  - We let uppercase Greek letter Π denote a nondimensional parameter, e.g., Reynolds number *Re*, Froude number *Fr*, Drag coefficient, C_D, etc.



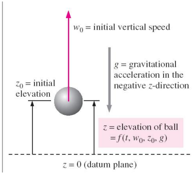
- Consider automobile experiment
- Drag force is $F = f(V, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to $\Pi_1 = f(\Pi_2) \rightarrow C_D = f(Re)$

Method of Repeating Variables

- Nondimensional parameters Π can be generated by several methods.
- We will use the **Method of Repeating Variables**
- Six steps
  1. List the parameters in the problem and count their total number n.
  2. List the primary dimensions of each of the n parameters
  3. Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's, $k = n - j$.
  4. Choose j *repeating parameters*.
  5. Construct the k Π's, and manipulate as necessary.
  6. Write the final functional relationship and check algebra.

Example

Ball falling in a vacuum



- Step 1: List relevant parameters. $z = f(t, w_0, z_0, g) \rightarrow n = 5$
- Step 2: Primary dimensions of each parameter

z	t	w_0	z_0	g
$\{L^1\}$	$\{t^1\}$	$\{L^1 t^{-1}\}$	$\{L^1\}$	$\{L^1 t^{-2}\}$

- Step 3: As a first guess, reduction j is set to 2 which is the number of primary dimensions (L and t). Number of expected Π's is $k = n - j = 5 - 2 = 3$
- Step 4: Choose repeating variables w_0 and z_0

Guidelines for choosing *Repeating parameters*

1. Never pick the dependent variable. Otherwise, it may appear in all the Π's.
2. Chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
3. Chosen repeating parameters must represent *all* the primary dimensions.
4. Never pick parameters that are already dimensionless.
5. Never pick two parameters with the same dimensions or with dimensions that differ by only an exponent.
6. Choose dimensional constants over dimensional variables so that only one Π contains the dimensional variable.
7. Pick common parameters since they may appear in each of the Π's.
8. Pick simple parameters over complex parameters.

Example, continued

- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's.
  - $\Pi_1 = zw_0^{a_1}z_0^{b_1}$
    - a_1 and b_1 are constant exponents which must be determined.
    - Use the primary dimensions identified in Step 2 and solve for a_1 and b_1.
      $\{\Pi_1\} = \{L^0 t^0\} = \{zw_0^{a_1}z_0^{b_1}\} = \{L^1(L^1t^{-1})^{a_1}L^{b_1}\}$
      Time: $\{t^0\} = \{t^{-a_1}\} \rightarrow 0 = -a_1 \rightarrow a_1 = 0$
      Length: $\{L^0\} = \{L^1L^{a_1}L^{b_1}\} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1$
    - This results in $\Pi_1 = zw_0^0 z_0^{-1} = \frac{z}{z_0}$

Example, continued

- Step 5, continued
  - Repeat process for Π_2 by combining repeating parameters with t.
    - $\Pi_2 = t w_0^{a_2} z_0^{b_2}$
    - $\{\Pi_2\} = \{L^0 t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t^1 (L^1 t^{-1})^{a_2} L^{b_2}\}$
      Time: $\{t^0\} = \{t^1 t^{-a_2}\} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1$
      Length:
      $\{L^0\} = \{L^{a_2} L^{b_2}\} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1$
    - This results in $\Pi_2 = t w_0^1 z_0^{-1} = \frac{w_0 t}{z_0}$

Example, continued

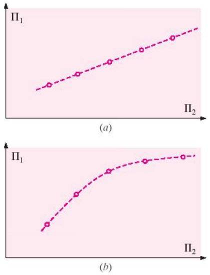
- Step 5, continued
  - Repeat process for Π_3 by combining repeating parameters with g.
    - $\Pi_3 = g w_0^{a_3} z_0^{b_3}$
    - $\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$
      Time: $\{t^0\} = \{t^{-2} t^{-a_3}\} \rightarrow 0 = -2 - a_3 \rightarrow a_3 = -2$
      Length: $\{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \rightarrow 0 = 1 + a_3 + b_3 \rightarrow b_3 = -1 - a_3 \rightarrow b_3 = 1$
    - This results in $\Pi_3 = g w_0^{-2} z_0^1 = \frac{g z_0}{w_0^2}$
    - From experience, it would be recognized that Π_3 can be modified to give $\Pi_{3,modified} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = Fr$

Example, continued

- Step 6:
  - Double check that the Π's are dimensionless.
  - Write the functional relationship between Π's.
    $$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{sqrt g z_0}\right)$$
  - Or, in terms of nondimensional variables, $z^* = f(t^*, Fr)$
- Overall conclusion: Method of repeating variables properly predicts the functional relationship between dimensionless groups.
- However, the method cannon predict the exact mathematical form of the equation.

Experimental Testing and Incomplete Similarity

- On of the most useful applications of dimensional analysis is in designing physical andor numerical experiments, and in reporting the results.
- Setup of an experiment and correlation of data



- Consider a problem with 5 parameters: one dependent and 4 independent.
- Full test matrix with 5 data points for each independent parameter would require $5^4 = 625$ experiments!!
- If we can reduce to 2 Π's, the number of independent parameters is reduced from 4 to 1, which results in $5^1 = 5$ experiments vs. 625!!

Experimental Testing and Incomplete Similarity

Wanapum Dam on Columbia River

Physical model at IIHR

- Flows with free surfaces present unique challenges in achieving complete dynamic similarity.
- For hydraulics applications, depth is very small in comparison to horizontal dimensions. If geometric similarity is used, the model depth would be so small that other issues would arise
  - Surface tension effects (Weber number) would become important.
  - Data collection becomes difficult.
- Distorted models are therefore employed, which requires empirical corrections/correlations to extrapolate model data to full scale.

Experimental Testing and Incomplete Similarity

DDG-51 Destroyer

DTMB Model 5415

- For ship hydrodynamics, Fr similarity is maintained while Re is allowed to be different.
- Why?

$$Re_p = \frac{V_p L_p}{\nu_p} = Re_m = \frac{V_m L_m}{\nu_m}$$
$$\rightarrow \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m}$$
$$Fr_p = \frac{V_p}{\sqrt{gL_p}} = Fr_m = \frac{V_m}{\sqrt{gL_m}}$$
$$\rightarrow \frac{L_m}{L_p} = \left(\frac{V_p}{V_m}\right)^2$$

- This means to match both Re and Fr, viscosity must be changed between model and prototype

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$