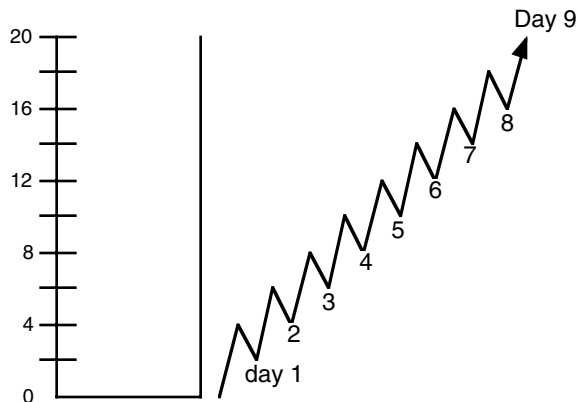


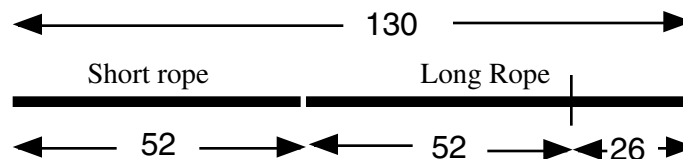
1 Problem Solving

Section 1.1

1. During the first 24 hours the snail climbs up 4 feet and slips back down 2 feet, so it reaches a maximum height of 4 feet and there is a net gain of 2 feet. It seems as if the snail should take 10 days to get out because it makes a net gain of 2 feet each day. This is not correct because on the 8th day it ends up at 16 feet and then on the 9th day it climbs up the remaining 4 feet to get out. It doesn't slide back down on the 9th day because it is already out of the well. The figure below shows one possible solution that uses a drawing.



3. Here is a sketch of the two ropes laid end to end next to each other. The longer rope has an extra 26 feet in length, so if the 26 feet is marked off separately on the longer rope then the remaining section is the same length as the first rope. Together this is $130 - 26 = 104$ feet. So the shorter rope is $104/2 = 52$ feet long and the longer one is $52 + 26 = 78$ feet long.



5. a. Under the old plan writing 10 checks will cost \$2 plus 10 times .15, or $2 + 1.50 = \$3.50$.
Under the new plan it is $\$3 + (10 \times .08) = 3.00 + .80 = \3.80 .

b,c. Here is a partial table for this problem.

| | <u>Checks</u> | <u>Old Plan</u> | <u>New Plan</u> |
|---|---------------|-----------------|-----------------|
| Note that for a small amount of checks | 12 | \$3.80 | \$3.96 |
| the old plan is cheaper and as the number | 13 | \$3.95 | \$4.04 |
| of checks increases, the cost for the old | 14 | \$4.10 | \$4.12 |
| plan goes up faster than the cost for the | 15 | \$4.25 | \$4.20 |
| new plan. At 15 checks the new plan | 17 | \$4.55 | \$4.36 |
| becomes cheaper than the old plan. | 19 | \$4.85 | \$4.52 |

- d. Each time the number of checks increases by one, the new plan gains $15 - 8 = 7$ cents in value against the old plan. For example, in the table above we can see that at 12 checks the new plan is .16 more than the old, but at 13 checks it is only .09 higher. By the time 19 checks have been written, the new plan is 33 cents cheaper than the old one.

Chapter 1

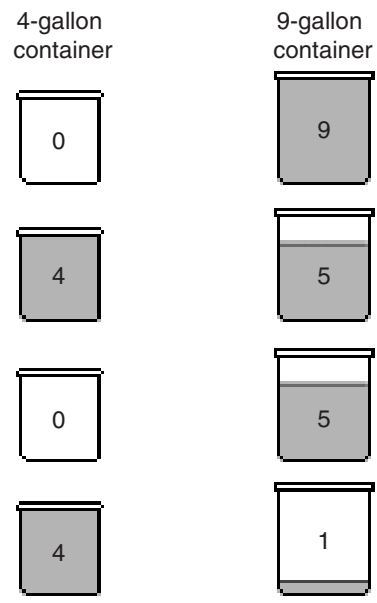
7. One approach to this problem is to create a table similar to the one below. Another good way to create a table for this problem would be to include two additional columns, one for the cost of the postcards and one for the cost of the letters. This table was started with a guess at the solution of 10 postcards and 5 letters. If one started by guessing 3 postcards and 12 letters, then the cost would be too high and the number of letters would need to be reduced. Note that the total of the number of postcards and number of letters must always be 15.

| <u>No. of Postcards</u> | <u>No. of Letters</u> | <u>Cost</u> |
|-------------------------|-----------------------|------------------------|
| 10 | 5 | $2.00 + 1.60 = \$3.60$ |
| 9 | 6 | $1.80 + 1.92 = \$3.72$ |
| 7 | 8 | $1.40 + 2.56 = \$3.96$ |
| 6 | 9 | $1.20 + 2.88 = \$4.08$ |

An alternative approach to this problem would be to notice that each time there is one additional letter and one less postcard the cost increases by 12 cents. This line of thinking could be used in conjunction with the table or in a solution that did not use a table.

9. a,b,c. The most difficult part of this problem may be in reading and understanding. The sum of the digits of the two-digit number 29 is $2 + 9 = 11$. We are looking for pairs of two-digit numbers which are different, but both have the same two digits. This means they will be numbers with their digits reversed, such as 29 and 92, 17 and 71, 53 and 35, etc. But we only want numbers whose digits add to 10. So the only possible pairs of two-digit numbers that could work are 19 and 91, 28 and 82, 37 and 73, 46 and 64, and maybe 55 and 55. We want the pair with a difference of 54, so it must be 28 and 82.
- d. Pairs of two-digit numbers with sums of 12 are: 39 and 93, 48 and 84, 57 and 75, 66 and 66. The difference between 39 and 93 is 54.

11. a. The sketch shows that we start with the 9 gallon container filled and the 4 gallon container empty. Then the 9 gallon container is dumped into the 4 gallon container twice, so that exactly one gallon is left in the 9 gallon container.

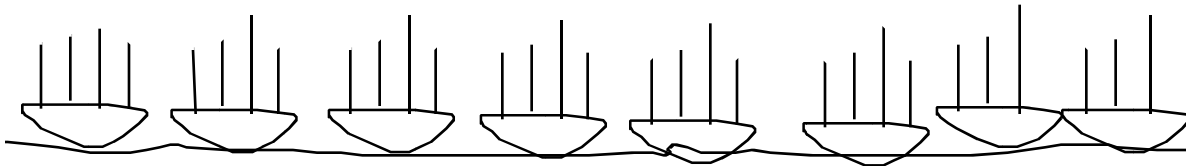


- b. To measure 6 gallons, use the procedure from part a to measure one gallon into the 9 gallon container. Then dump this one gallon into the 4 gallon container. Next fill the 9 gallon container and pour this into the 4 gallon container until it is just full. Since it already had 1 gallon in it, 3 more gallons will be poured out, leaving 6 gallons in the large container.

13. a. The yellow tiles are touching at the corners, and the directions say that no tile should touch the same color at any point.
- b. The center tile touches all of the other tiles, so if it were the same color as any of them it would violate that condition.
- c. With a blue tile in the center and reds in the corners, pairs of the four remaining openings touch each other at their corners. So one additional color will not work. The square can be formed with four colors, but not with three. One reason three will not work is that there are several places in the 3×3 square where four different squares meet in a single point. This forces at least four different colors.
- d. If the same color can touch at a corner, but not on an edge, then a checkerboard pattern will work. Two colors are sufficient for this problem.
15. a. The single square on top will end up opposite the square marked Bottom, so it will be the Top. In the row of four squares in the middle, Front and Back need to alternate with Left and Right. So the second square in this row needs to be labeled Right. Imagine the cube formed, with this square on the Right, then the first square on the left will be facing you and should be labeled Front. That leaves the third square for the Back.
- b. Working from right to left in the middle row, the square to the left of Bottom should be Front, since it will be opposite Back. Next to Front will be Top, since it is opposite Bottom. Now imagine folding the pieces up and placing it with Bottom down, etc. The square above Bottom must be Right and the square below Top must be Left.
[Note: If these problems are difficult to visualize, you might try building the model.]
17. a. Girl A needs to give 30 chips to girl B and 20 to girl C, so she must give up 50 of her 70 chips. At the end of this round, girl A has 20 chips, girl B has 60, and girl C has 40.
- b. We will construct the scores at the end of the second round by working backward from the scores at the end of the third round, assuming that girl C lost the third round. Since both girl A and B doubled their scores to 40 in the third round, they must have had 20 each at the end of the second round. Girl C had to give them each 20 of her chips, so she must have had $40 + 20 + 20 = 80$ chips at the end of the second round.
 A had 20; B had 20; C had 80.
- c. Continuing to work backward, if girl B lost the second round, then she doubled A's score from 10 to 20 and doubled C's from 40 to 80. So B gave away $10 + 40 = 50$ chips in the second round. So the score at the end of the first round was: A had 10; B had 70; C had 40. If girl A lost the first round, she gave 35 chips to B and 20 chips to C. So girl A started the game with $10 + 35 + 20 = 65$ chips. B started with $70 - 35 = 35$ chips and C started with 20 chips. The original distribution was 65 for A, 35 for B, and 20 for C.
- d. Suppose girl C had lost the first round. She would need to give 65 chips to A and 35 to B. But she only had 20 so she couldn't do it. *[What if they had started the game even?]*
19. Amelia took tiles from Ramon's collection first, and then Keiko took half of the remaining tiles. Working backward we will start with the fact that Ramon had 11 left in the end. Just before this Keiko took half of the tiles, so Keiko took 11 from the 22 tiles that Ramon had at that time. Before this, Amelia took 13, so Ramon started with $22 + 13 = 35$ tiles.

21. The student created the diagram by letting a single square represent the second number. Since the first number is twice the second number, the first number is represented by two squares. The third number is twice the first number, so it needs to be four squares. Each square represents the same number. There are a total of seven squares with a sum of 112, so each square represents $112 \div 7 = 16$. The first number is 32, the second one is 16, and the third one is 64.
23. The student is using a drawing to represent the information. There are various ways the student may have used this diagram to help solve the problem. One way is to notice that if we replace a doughnut with a cup of coffee then the cost goes up by 10 cents. So a cup of coffee costs .10 more than a doughnut. Four doughnuts would cost .10 less than one cup of coffee and three doughnuts, so four doughnuts cost .80, and one doughnut costs 20 cents. This means that one cup of coffee costs 30 cents. These prices can be checked with the diagram of the original information.

25. *Solution 1: by making a drawing*



In this diagram, eight boats were drawn. First, three masts were placed on each boat, for a total of 24 masts. Then a fourth mast was added to boats until there were a total of 30 masts. So 6 boats had 4 masts and 2 boats had 3 masts.

Solution 2: by making a table

| | <u>boats with 4 masts</u> | <u>boats with 3 masts</u> | <u>total masts</u> |
|-----------------------------------|---------------------------|---------------------------|--------------------|
| This table could be created by | 4 | 4 | $16 + 12 = 28$ |
| making sure that there are always | 5 | 3 | $20 + 9 = 29$ |
| a total of 8 boats. This is also | 6 | 2 | $24 + 6 = 30$ |
| how a guess and check strategy | | | |
| might work. | | | |

27. *Solution 1: by guessing and checking*

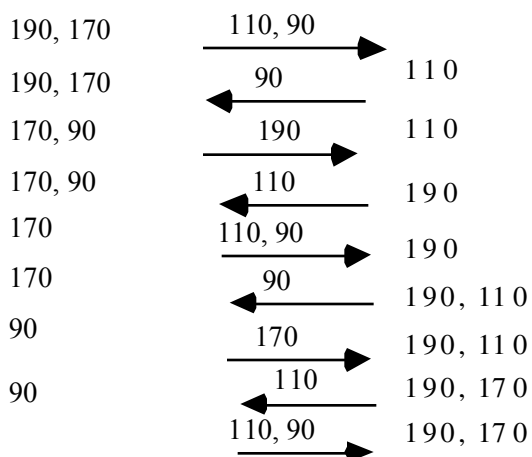
Suppose Claire got 10 free videos. That would mean that she paid for 30 videos. At \$3 each this means she paid \$90 for videos. But we were told that she paid \$132, so the guess of 10 was too low. How much higher should we guess? 90 is about $\frac{2}{3}$ of 132, so 15 might be a good guess. If Claire got 15 free videos that means she paid for 45. At \$3 each that would cost her \$135. This is very close, but she only spent \$132. So, she paid for 44 videos. This is not quite enough to get 15 free, but it is enough for 14 free videos.

Solution 2: by working backwards

Since Claire paid \$132 for her videos, she paid full price for $132 \div 3 = 44$ videos. She gets a free video for every three that she pays for, so she has received $42 \div 3 = 14$ free videos. If she pays for one more, she will get her 15th free video.

[Note: A different interpretation could be made for this problem. Depending on the rules used by the video club, it may be that Claire needs to rent 3 videos all at once in order to receive the free one. If that were the case, we would not have enough information to solve the problem.]

29. Solution 1: by making a drawing



The drawing shows that the four people can cross the river in nine crossings. The 110 and 90 pound people are the only ones who can cross together, so they need to make the first trip and the last trip in order to accomplish the task in the minimum number of crossings.

Solution 2: by using a model

A similar method could be used to solve this problem by using four pieces of paper with the numbers 90, 110, 170, and 190 written on them. One could move them across a “river” following the rules given, experimenting until being convinced of a solution.

31. By a combination of working backward and guess and check:

The numbers 6, 7, and 8 total to 21, so if one disk had a number that is one larger on the back we would get a total of 22. Suppose that the 6 has a 7 on the back. Then we could get 7, 7, and 8 for a total of 22. The smallest possible total needs to be 15. One way to get 15 is with a 6, 5, and 4. Let’s try putting a 5 on the back of the 7 and a 4 on the back of the 8. Then our discs look like this: **6 or 7; 7 or 5; 8 or 4.**

Now check to see if we can get the totals we want. $6 + 5 + 4 = 15$ $7 + 5 + 4 = 16$
 $6 + 7 + 4 = 17$ $7 + 7 + 4 = 18$ $6 + 5 + 8 = 19$ $7 + 5 + 8 = 20$ $6 + 7 + 8 = 21$
 $7 + 7 + 8 = 22$ It works!

[Note: The answer in the back of the text gives a different solution for this problem.

This problem solver is not at all sure that these are the only two possible correct solutions. Can you find another one . . . or prove that these are the only ones?]

33. Start both timers at the same time. The vegetables should be put in the steamer after the seven minute timer has just run out. At that point the 11 minute timer still has four minutes to go, so when it is finished the vegetables will have steamed for four minutes. Then turn over the 11 minute timer again and when it is finished the vegetables will have been steamed for 15 minutes.

35. These problems require careful reading and common sense thinking.
- If **you take** two apples, then **you** will have two apples. (Unless you already had some more.)
 - All **but** nine died, so he still had nine left.
 - “One is not a nickel” can be interpreted to mean that one of the coins is a nickel and one is not. So the coins are a quarter and a nickel.
 - The cider costs 60 cents **more** than the bottle, so if the bottle was 10 cents, the cider would be 70 cents for a total of 80 cents. We need 6 cents more, so make the bottle 13 cents and the cider 73 cents.
 - If it is a hole, it has no dirt in it. The amount of dirt removed is another question.
 - If the hen weighs 3 pounds plus half its weight, we can think of the hen’s weight as being divided into two equal parts. One half is 3 pounds and the other half is half it’s weight. But the two parts have equal weight, so the other half must also be 3 pounds. The hen weighs 6 pounds.
 - As long as there were no **men** playing baseball not a man would cross the plate. The players could have been women or children.
 - Neither phrase is correct as long as the yolk didn’t break. If uncooked the whites are clear and if cooked the whites are white.

Section 1.2

- This pattern has a core of three repeating characters.

▲+○
 - This pattern repeats the core of the first four characters.

××+×
 - One interpretation of this pattern is that it has a core which grows as follows: the first core is circle, star, 2 circles; the second core is circle, star, 3 circles; the third core is circle, star, four circles, etc.

○*○○○○○ ...
- This is an arithmetic sequence with common difference three. So the next three numbers are 26, 29, and 32.
 - The method of finite differences is helpful in this pattern. The sequence of differences between the numbers is 3, 3, 4, 4, 5, 5, 6. So to continue this pattern we want the next three differences to be 6, 7, and 7. That means that the next three numbers are 49, 56, and 63.
 - Again looking at the differences between consecutive numbers in the sequence, we see 5, -2, 5, -2, 5, -2, 5 as the sequence of differences. Continuing this pattern gives 29, 34, and 32 as the next three numbers in the sequence.

5. The sequence of numbers of cannon balls in the first four figures shown is 1, 5, 14, 30. Using the method of finite differences gives us the sequence 4, 9, 16. These numbers may or may not look significant. If not, take finite differences again and get 5, 7. This means that the next number in the sequence 4, 9, 16 should be $16 + 9 = 25$. The numbers 4, 9, 16, and 25 are the squares of the numbers 2, 3, 4, and 5. This makes sense with the cannon ball pattern because each new figure adds a larger base to the previous figure. Each base is a square layer of cannon balls. In the third figure the base is a 3 by 3 layer of 9 cannon balls. So the third figure contains $9 + 4 + 1 = 14$ cannon balls. The sixth figure contains $36 + 25 + 16 + 9 + 4 + 1 = 91$ cannon balls. The tenth pyramid consists of ten layers of cannon balls. The bottom layer contains 10×10 or 100 cannon balls. The next layer has 9×9 , etc. until the top layer is a single cannon ball. So the tenth pyramid contains $100 + 81 + 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = 385$ cannon balls.
7. a. The sequence is arithmetic with a common difference of 3. Three additional cubes are added to each figure.
- b. Here is one description: the 20th figure has 39 cubes on the bottom and 19 cubes stacked on top of the first cube on the left. Another description: the 20th figure has 20 cubes stacked up and 38 cubes lined up next to the stack. Either way, there are 58 cubes. (Other descriptions are also possible.)
9. Multiplying the middle number by three will always give the sum of three consecutive numbers. If we call the middle number n , then the one to its left will be $n - 1$ and the one to its right will be $n + 1$. Adding the three numbers together gives $n - 1 + n + n + 1$. This sum is equal to $n + n + n$, which is the same as the product of 3 times n . 18 times 3 is 54, so $17 + 18 + 19 = 54$.
11. The sum of the 9 numbers in a 3 by 3 array on a calendar will always be equal to 9 times the middle number. This can be seen using algebraic expressions as in #9, or we can look at it another way. In the array shown in the text with 11 in the middle there are four other pairs of numbers, each with sum 22 (or average 11). These four pairs are 3 and 19, 5 and 17, 4 and 18, 10 and 12. Note that these pairs consist of opposite corners, top and bottom, and the two sides. Why does it work this way? For example, the upper left corner of the array will always have a number that is 8 less than the middle number and the bottom right corner will always be 8 more than the middle. Similar relationships hold for the other pairs of numbers. To find a 3 by 3 array whose sum is 198 we would need the middle number to be $198 \div 9 = 22$. *[Could we always find such an array on any month's calendar?]*
13. $1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 1 + 1 + 4 + 9 + 25 = 40 = 5 \times 8$
 $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 1 + 1 + 4 + 9 + 25 + 64 = 104 = 8 \times 13$
 $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 = 1 + 1 + 4 + 9 + 25 + 64 + 169 = 173 = 13 \times 21$

This pattern shows that the sum of the squares of the first n Fibonacci numbers is equal to the product of the n th Fibonacci number times the next (i.e. the $(n + 1)$ st) Fibonacci number. For example the sum of the squares of the first seven Fibonacci numbers is equal to the product of the seventh Fibonacci number times the eighth Fibonacci number. *[It might be an interesting exploration to try to figure out why this works.]*

15. If these are to be Fibonacci-type sequences, then any number in the sequence is found by adding the previous two numbers. So, in part a the two blanks before 11 might be 4 and 7, 3 and 8, 5 and 6, or any two numbers adding to 11. But the only pair of numbers that will work here and that also go with the 1 at the beginning of the sequence are 5 and 6. In part b, the blank between 14 and 20 has to be a 6, because the first two numbers must add to 20. This also confirms that the fourth number should be 26.

Here are the sequences for parts a and b.

- a. 1, 5, 6, 11, 17, 28, 45, 73, 118, 191 b. 14, 6, 20, 26, 46, 72, 118, 190, 308, 498
- c. Exactly ten numbers of the sequence are given here, so we need to add them together.
 $14 + 6 + 20 + 26 + 46 + 72 + 118 + 190 + 308 + 498 = 1298$. The seventh number is 118. $11 \times 118 = 1298$, so it is true that the sum equals 11 times the seventh number.
- d. The first 10 terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Their sum is 143 which is equal to 11×13 .
- e. One reasonable conjecture would be that the sum of the first 10 numbers of any Fibonacci-type sequence is equal to 11 times the seventh number in that sequence. *[What would you do if you wanted to prove or disprove your conjecture?]*

17. a. Here are the three given equations:

$$\begin{aligned} 1 + 2 &= 3 \\ 4 + 5 + 6 &= 7 + 8 \\ 9 + 10 + 11 + 12 &= 13 + 14 + 15 \end{aligned}$$

To continue this pattern, we write $16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$ as the next equation so that it continues counting off the natural numbers and also adds one term in length to each side of the equation.

[Do you notice anything about the first number on the left of each equation?]

- b. The first number in the first row is 1, which is the square of 1. The first number in the second row is 4, which is the square of 2. The third equation starts with 9, which is the square of 3, etc. So the 20th equation will start the sum with 20 squared, which is $20 \times 20 = 400$. Notice also how the number of terms grows on each side of each equation. The 1st equation has 2 terms equal to 1 term, the 2nd one has 3 terms equal to 2, the 3rd has 4 terms on the left and 3 on the right, etc. The 20th equation will have 21 terms on the left and 20 on the right. (We also could have found its length another way. The 21st equation needs to start with $21 \times 21 = 441$.) In any case, the 20th equation is:
 $400 + 401 + 402 + \dots + 420 + 421 = 422 + 423 + \dots + 439 + 440$.

[Why do you think this pattern works? You might try taking the first number on the left and distributing it among the other numbers on the left.]

19. a. The third diagonal in Pascal's triangle contains the Triangular numbers, so the numbers in this diagonal are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, etc. where the difference between consecutive numbers keeps increasing by one. The tenth number in this sequence is 55. *[It is an interesting coincidence that the 10th Fibonacci number is also 55.]*
- b. The first few numbers in the fourth diagonal are 1, 4, 10, 20, 35. Look at the differences between these numbers and also look back at part a. What do you notice? The *differences* between consecutive numbers in the fourth diagonal are consecutive triangular numbers. So the sequence continues with $35 + 21 = 56$ and then $56 + 28 = 84$, etc. The first ten numbers in the fourth diagonal are 1, 4, 10, 20, 35, 56, 84, 120, 165, 220.

21. The sum of each row of Pascal's triangle is a power of 2. The sum of the numbers in the 12th row is 2 to the 12th power, which is 4096.

23. a. This is an arithmetic sequence because there is a common difference. To obtain any number in the sequence, add 5 to the previous number.

The sequence is 4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, etc. so the 12th number is 59.

[Could this have been found without writing out all the terms?]

b. This is a geometric sequence because there is a common ratio. To obtain the next number, multiply the number by 2. The sequence is 15, 30, 60, 120, 240, 480, 960, 1920, 3840, 7680, 15360, 30720 so the 12th number is 30720.

c. This is an arithmetic sequence because it has a common difference. The numbers are decreasing, so the difference is negative. (Or think of subtracting instead of adding.) To obtain any number in the sequence, subtract 4 from (or add -4 to) the previous number. The sequence is 24, 20, 16, 12, 8, 4, 0, -4 , -8 , -12 , -16 , -20 so the 12th number is -20 .

d. This is a geometric sequence because there is a common ratio. To obtain the next number, multiply the number by 3. The sequence is 4, 12, 36, 108, 324, 972, 2916, 8748, 26244, 78732, 236196, 708588 so the 12th number is 708588.

25. a. We can use the method of finite differences to find the next number in a diagonal of Pascal's triangle. For the first and second diagonals finite differences are not even needed (although they could be used). The first diagonal is all 1's and the second diagonal is the natural numbers (with constant finite difference 1). The third diagonal is the triangular numbers: 1, 3, 6, 10, 15, etc. with finite differences 2, 3, 4, 5, etc. so the next number after 15 is $15 + 6 = 21$. The fourth diagonal starts with 1, 4, 10, 20, 35. Finite differences here are 3, 6, 10, 15. Notice that these are the numbers in the third diagonal. In the fourth diagonal the next number after 35 is $35 + 21 = 56$.

[Can you see why the pattern works based on how Pascal's triangle is constructed?]

b. What happens when we use finite differences on the sequence 1, 2, 4, 8, ...?

| | | | | | | |
|---|---|---|---|----|----|----|
| 1 | 2 | 4 | 8 | 16 | 32 | 64 |
| | 1 | 2 | 4 | 8 | 16 | 32 |
| | | 1 | 2 | 4 | 8 | 16 |

No matter how many times we take differences we will always just have the same sequence again. So the method of finite differences is not helpful here.

27. a. Taking finite differences 3 times gives us the following:

| | | | | | |
|---|---|---|----|----|-----|
| 1 | 2 | 7 | 22 | 53 | 106 |
| | 1 | 5 | 15 | 31 | 53 |
| | | 4 | 10 | 16 | 22 |
| | | | 6 | 6 | 6 |

When we get down to this row where the difference is constant at 6, then we know that the next number in the second row up is $22 + 6 = 28$. This means that the number after 53 is $53 + 28 = 81$. Then this gives us the next number in the sequence. It is $106 + 81 = 187$.

Chapter 1

- b. This sequence only requires two rows of finite differences before reaching a constant difference.

| | | | | | |
|---|---|----|----|----|----|
| 1 | 3 | 11 | 25 | 45 | 71 |
| 2 | 8 | 14 | 20 | 26 | |
| 6 | 6 | 6 | 6 | 6 | |

So the next number in the middle row is 32, and the next number in the sequence is $71 + 32 = 103$.

29. A square number can be found by multiplying a natural number by itself. The sequence 1, 4, 9, 16, . . . can also be written as 1×1 , 2×2 , 3×3 , 4×4 , . . . So the next three numbers are 5×5 , 6×6 , and 7×7 ; or, 25, 36, 49. The 100th number in the sequence is 100×100 , which is 10,000. This is why taking the second power is often called squaring.

31. a. Looking at the dimensions of the arrays in the sequence of oblong numbers helps to give us a clue. The first one is 1×2 , the second is 2×3 , the third is 3×4 , and the fourth is 4×5 . The fifth oblong number will be 5×6 , or 30.

- b. The 20th oblong number will be 20×21 , or 420.

[Do you see a connection between the oblong numbers and the triangular numbers? You might want to look at the arrays of dots.]

33. a. The pentagonal numbers are 1, 5, 12, 22, 35, 51, . . . The first sequence of finite differences starts with 4, 7, 10, 13, 16. This is an arithmetic sequence because it has a common difference of 3.

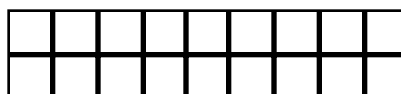
- b. Yes, we can use this method to continue the sequence of pentagonal numbers. The difference between the 5th and 6th numbers was 16, so the difference between the 6th and 7th will be 19, the difference between the 7th and 8th will be 22, etc. The next few pentagonal numbers after 51 will be 70, 92, $92 + 25 = 117$, $117 + 28 = 145$, etc.

35. The reasoning used by the researchers in this article was inductive reasoning because the conclusions were based on observing a pattern that appeared in a sample population. These results were then extrapolated to the general population.

37. a. Here is a sketch of the 9th even number. It consists of two rows of nine;

or nine sets of two.

The 9th even number is 18.



- b. The 45th even number is 45 sets of two (or two 45s), so it is $45 \times 2 = 90$.

39. If we try the procedure on 8, we double 8 to get 16, then place them side by side to get 168. The number 168 is divisible by 7 because $7 \times 24 = 168$. So 8 is not a counterexample. If we try 12, we double 12 to get 24, then write the number 2412. This number is not divisible by 7. $344 \times 7 = 2408$ and $345 \times 7 = 2415$. So the number 12 is a counterexample, which shows that this procedure does not always give a number that is divisible by 7. The procedure does work for all one digit numbers.

41. a. To find a counterexample we just need to find one example where the suggested property does not work. So we are looking for two numbers that we can multiply so that the product is not divisible by 2. $7 \times 4 = 28$ is not a counterexample. $7 \times 5 = 35$ is a counterexample. *[What kinds of numbers do we need to use here to find counterexamples?]*
- b. We want to find a whole number greater than five which can not be written as the sum of two consecutive numbers and also can not be written as the sum of three consecutive numbers. Every odd number is the sum of two consecutive numbers. Look at the diagrams in #38 if you are not convinced of this. So we need to look in the even numbers. Can any even numbers be the sum of two consecutive numbers? Why not?
So for our counterexample we need to find an even number which can not be written as the sum of three consecutive numbers. If we add any three consecutive numbers the result is always a multiple of three. To see this, draw a staircase pattern showing the sum of three consecutive numbers, like $6 + 7 + 8$. This staircase can always be leveled off to a rectangle with width 3 simply by moving one square. For example, $6 + 7 + 8 = 7 + 7 + 7$. So, if we can find an even number (greater than five) that is not a multiple of three then it will be our counterexample. Eight and ten are the smallest counterexamples. Fourteen is the next one.
43. a. The statement “the sum of any four consecutive whole numbers is divisible by 4” is false. A counterexample can be found in the first four whole numbers. $1 + 2 + 3 + 4 = 10$. Ten is not divisible by 4. If we change the word “sum” to “product”, then the statement will be true. We could prove this by noticing that any four consecutive numbers will contain one number that is a multiple of four. A product containing a factor which is a multiple of 4 will also be divisible by 4.
- b. This statement is true. Since we are only looking at 14 numbers, we could check the statement with each of the numbers.
45. a. By the end of the 2nd month the three first month members will have each brought in two new members, so that there will be 9 members. During the 3rd month each of the 9 members brings in two more members. Counting each old member with their two friends as a group of three, we see that there are $9 \times 3 = 27$ members. In fact, the system results in tripling the membership each month. So, after 6 months there will be $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$ members. This product can also be written using an exponent. $3^6 = 729$.
- b. After 1 year (12 months) there will be $3^{12} = 531,441$ members.
47. a. On the 12th day there were $12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 78$ gifts given. This is the 12th triangular number. It can also be found by using a staircase as in Sec. 1.1.
- b. The total number of gifts received during all 12 days is the sum of the first 12 triangular numbers, which is $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 = 364$.
49. This problem will be easier to solve if we first look at a simpler problem. Suppose that the refrigeration car was the 3rd from the front and the 2nd from the end. Then the train would look like this: xxRx. The R is 3rd from the front and 2nd from the back. This train has 4 cars, which is $2 + 1 + 1$ (or, $3 + 2 - 1$). If the refrigeration car is the 147th from the front, then there are 146 others in front of it. And if it is 198th from the back, then there are 197 others behind it. So there are $146 + 1 + 197 = 344$ cars total. (Or, $147 + 198 - 1 = 344$ cars).

51. First look at the cards numbered from 1 to 100. Between cards number 1 and number 59, there are 6 cards containing a 6 (6, 16, 26, 36, 46, and 56). Each of cards 60 through 69 contains at least one 6, so this is 10 more cards. Also cards 76, 86, and 96 each contain a 6. There are a total of 19 cards between 1 and 100 containing at least one 6. The counts will be the same for the 100's, 200's, 300's, and 400's. So there are a total of $19 \times 5 = 95$ cards with a 6.

Section 1.3

1. a. The variable x represents the depth. By replacing x with 10 in the expression $.43x + 14.7$, we find that the pressure at a depth of 10 feet is $.43(10) + 14.7 = 4.3 + 14.7 = 19$ pounds per square inch. If x is 100 we get a pressure of $43(100) + 147 = 4.3 + 147 = 151.3$ pounds per square inch. To find the pressure at the surface we substitute 0 for x and get $43(0) + 14.7 = 0 + 14.7 = 14.7$ pounds per square inch.
- b. Here we replace x with the number of chirps. For 20 chirps per minute, the temperature is $20/4 + 40 = 5 + 40 = 45$ degrees. For 100 chirps per minute, the temperature is $100/4 + 40 = 25 + 40 = 65$ degrees.
3. a. To find the cost of all of the main floor seats we would multiply the number of main floor seats by \$28. An expression for the cost is $28m$.
- b. To find the cost of all of the main floor seats we would multiply the number of main floor seats by \$19. An expression for the cost is $19b$.
- c. If the total for the main floor seats was greater than the total for the balcony seats then subtracting the total cost for the balcony seats from the total cost for the main floor seats will give us the positive difference. Using the answers from parts a and b, we get the expression $28m - 19b$.
5. a. The correct equation is $s = 6p$. If there are 6 students for each professor, multiplying the number of professors by 6 will give the number of students. For example, if there are 10 professors, then the equation $s = 6p$ says that there are 60 students.
- b. If one were to translate the verbal statement "6 times as many students as professors" into symbols without thinking too hard about what the symbols say, it might be easy to translate 6 times as many students to the symbols $6s$. The thinking might be "6 times the students equals the professors" but that is not what the original statement said.
7. a. There are four more chips on the right side than on the left side, so we need to have two chips in each box for the scale to balance. If x represents the number of chips in one box, then the balance scale can be represented by the equation $2x + 5 = 9$.
- b. Removing one box from each side of the scale we have 2 boxes and 4 chips on the left and 11 chips on the right side. Now remove 4 chips from each side. The scale still balances and there are 2 boxes on the left balancing 7 chips on the right. Each box must contain 3.5 chips. The balance scale can be represented by the equation $3x + 4 = x + 11$.

9. a. Step 1: Simplification of the left side using the distributive property.
 Step 2: Addition Property of Equality. (30 was added to both sides)
 Step 3: Simplification on both sides.
 Step 4: Subtraction Property of Equality. ($7x$ was subtracted from both sides)
 Step 5: Simplification on both sides.
 Step 6: Division Property of Equality. (Both sides divided by 5)
 Step 7: Simplification on both sides.
- b. Step 1: Simplification of the left side.
 Step 2: Subtraction Property of Equality. (2 was subtracted from both sides)
 Step 3: Simplification on both sides.
 Step 4: Division Property of Equality. (Both sides divided by 33)
 Step 5: Simplification on both sides.
11. The solutions of these equations given below show only one possible track. There are other correct ways to do each of these. For example, in part a we could start by subtracting $17x$ from both sides first.

| | | |
|----|---|---|
| a. | $43x - 281 = 17x + 8117$ $43x - 281 + 281 = 17x + 8117 + 281$ $43x = 17x + 8398$ $43x - 17x = 17x + 8398 - 17x$ $26x = 8398$ $26x/26 = 8398/26$ $x = 323$ | Addition Property Simplification Subtraction Property Simplification Division Property Simplification |
| b. | $17(3x - 4) = 25x + 218$ $51x - 68 = 25x + 218$ $51x - 68 + 68 = 25x + 218 + 68$ $51x = 25x + 286$ $51x - 25x = 25x + 286 - 25x$ $26x = 286$ $26x/26 = 286/26$ $x = 11$ | Simplification Addition Property Simplification Subtraction Property Simplification Division Property Simplification |
| c. | $56(x + 1) + 7x = 45,353$ $56x + 56 + 7x = 45,353$ $63x + 56 = 45,353$ $63x + 56 - 56 = 45,353 - 56$ $63x = 45,297$ $63x/63 = 45,297/63$ $x = 719$ | Simplification (Distributive Property) Simplification Subtraction Property Simplification Division Property Simplification |
| d. | $3x + 5 = 2(2x - 7)$ $3x + 5 = 4x - 14$ $3x + 5 - 3x = 4x - 14 - 3x$ $5 = x - 14$ $5 + 14 = x - 14 + 14$ $19 = x$ | Simplification Subtraction Property Simplification Addition Property Simplification |

Chapter 1

13. a. If we remove 5 chips from each side, then the right side will still have the same amount of excess weight. Then there are 2 boxes on the left and 7 chips on the right. The right side will stay heavier as long as there are less than 3.5 chips in each box.
An inequality to represent the original situation is $2x + 5 < 12$.
An inequality representing the solution is $x < 3.5$

b. If we remove 2 chips from each side, then the left side will still have the same amount of excess weight. Then there are 3 boxes on the left and 9 chips on the right. The left side will stay heavier as long as there are more than 3 chips in each box.
An inequality to represent the original situation is $3x + 2 > 11$.
An inequality representing the solution is $x > 3$.

15. a. Step 1: Subtraction Property of Inequality. ($2x$ was subtracted from both sides)
Step 2: Simplification on both sides.
Step 3: Subtraction Property of Inequality. (11 was subtracted from both sides)
Step 4: Simplification on both sides.
Step 5: Division Property of Inequality. (Both sides divided by 4)
Step 6: Simplification on both sides.

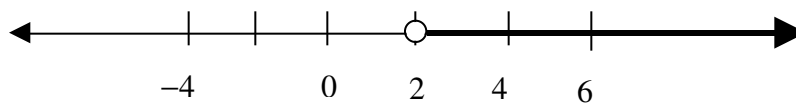
b. Step 1: Subtraction Property of Inequality.
Step 2: Simplification on both sides. (Note that $2x - 3x = -1x$, or $-x$)
Step 3: Multiplication Property of Inequality. (Both sides Multiplied by -1)
Note that the inequality sign changes direction here.
Step 4: Simplification on both sides.

17. a.

| | |
|---------------------------|--------------------------------------|
| $6(x + 5) > 11x$ | |
| $6x + 30 > 11x$ | Simplify using distributive property |
| $6x + 30 - 6x > 11x - 6x$ | Subtraction Property |
| $30 > 5x$ | Simplification |
| $30/5 > 5x/5$ | Division Property of Inequality |
| $6 > x$ (or, $x < 6$) | Simplification |

b.

| | |
|--------------------------|----------------------|
| $5(x + 8) - 6 > 44$ | |
| $5x + 40 - 6 > 44$ | Simplification |
| $5x + 34 > 44$ | Simplification |
| $5x + 34 - 34 > 44 - 34$ | Subtraction Property |
| $5x > 10$ | Simplification |
| $5x/5 > 10/5$ | Division Property |
| $x > 2$ | Simplification |



19. a. Each post card costs \$.20. So if the number of post cards that Marci wrote is x , then the total cost to mail the post cards is $.20x$.
- b. Marci sent a total of 18 post cards and letters. If we know the number of post cards, then we can find the number of letters by subtracting this amount from 18. For example, if she sent 10 post cards then she sent $18 - 10 = 8$ letters. Since x is the number of post cards she sent, $18 - x$ is an algebraic expression for the number of letters she sent.
- c. The cost for the letters is $.33(18 - x)$ and the cost for the post cards is $.20x$ so the total cost is $.33(18 - x) + .20x$.
- d. Since the total cost is \$4.38, we can write the equation $.33(18 - x) + .20x = 4.38$. Simplifying the left side gives $5.94 - .13x = 4.38$. Subtracting 5.94 from both sides gives $-.13x = -1.56$. Dividing both sides of this equation by $-.13$ gives us $x = 12$. So Marci mailed 12 postcards.
21. a. If x is the number of compact discs, then at \$10.50 each their cost is $10.50x$.
- b. There were three more tapes than discs, so there were $x + 3$ tapes.
- c. At \$8 each, the total cost of the tapes was $8(x + 3)$. Note that it is important to put the $x + 3$ in parentheses here.
- d. The sum of the costs can be represented by $10.50x + 8(x + 3)$. We want this sum to be less than \$120, so we want the solutions to the inequality $10.50x + 8(x+3) < 120$.
- | | |
|-------------------------------|-----------------------|
| Simplify the left side to get | $18.50x + 24 < 120$. |
| Subtract 24 on both sides | $18.50x < 96$. |
| Divide both sides by 18.5 | $x < 5.189189 \dots$ |
- To spend less than \$120 Merle had to buy 5 or fewer compact discs.
Check: If he bought 5 CD's, then he bought 8 tapes.
This would cost $10.50(5) + 8(8) = \$116.50$.
23. Since we want to find the length of a side of the square region it makes sense to try letting x represent the length of a side of the square. Marcia put a fence around the four sides of the square, so she used $4x$ feet of fencing. She had 110 feet of fence left over, so she started with $4x + 110$ feet of fencing. But we also know that she started with 350 feet of fence.
- So we know that $4x + 110 = 350$.
- | | |
|---|------------|
| Solve by subtracting 110 from both sides to get | $4x = 240$ |
| And divide both sides by 4 to get | $x = 60$. |
- The length of a side of the square region was 60 feet.
Note: Other lines of reasoning might lead to starting with the equations
 $4x = 350 - 110$ or $4x = 240$.
25. Another way to say the statement given is “The sum of a number and 14 is less than 3 times the number”. Translating this phrase into symbols gives us $x + 14 < 3x$.
- | | |
|--------------------------------------|-------------|
| Subtract $1x$ from both sides to get | $14 < 2x$. |
| Divide both sides by 2 | $7 < x$. |
- The solution says that the statement is true whenever $x > 7$.
(Try this with some numbers.)

27. a. If n is the first number, then the next number is one more than n , or $n + 1$. The number following $n + 1$ is $n + 2$, and the next number is $n + 3$. So $n, n + 1, n + 2$, and $n + 3$ are four consecutive numbers.
- b. The sum of four consecutive numbers in which the first number is n could be written as $n + (n + 1) + (n + 2) + (n + 3)$. The parentheses are optional in this expression.
- c. Since this sum is to have a value of 350, we can solve the equation

$$n + n + 1 + n + 2 + n + 3 = 350.$$
 The four n 's added together equal $4n$ and $1 + 2 + 3 = 6$, so an equivalent equation is

$$4n + 6 = 350.$$
 Subtracting 6 from each side we get, $4n = 344$. Dividing by 4 on each side gives $n = 86$.
- d. If 350 could be written as the sum of three consecutive numbers then we could solve the equation $n + n + 1 + n + 2 = 350$ and obtain a whole number for n .
 But this gives $3n + 3 = 350$, or $3n = 347$. Since 347 is not evenly divisible by 3, there are no three consecutive numbers with sum 350. On the other hand, the equation
 $n + n + 1 + n + 2 + n + 3 + n + 4 = 350$ does have a whole number solution for n .
 Find it! So, there are 5 consecutive numbers that sum to 350. [*How about 6? or 7?*]
29. For example, start with the number 17. Adding 221 gives 238. Multiplying 238 by 2652 gives 631,176. Subtract 1326 to get 629,850. Divide by 663 to get 950. Subtract 870 to get 80. Dividing by 4 gives us 20. Then if we subtract our original number of 17 we have 3.
 Try starting with a different number and see if you also end up with 3.
 Why does this work? Suppose we call our original number x . Then after the first step we have $x + 221$. Next is $2652(x + 221)$; then $2652(x + 221) - 1326$;
 then $[2652(x + 221) - 1326] \div 663$; then $[(2652(x + 221) - 1326) \div 663] - 870$;
 then $[(2652(x + 221) - 1326) \div 663] - 870$;
 then $\{[(2652(x + 221) - 1326) \div 663] - 870\} \div 4$;
 Finally we subtract x to get $(\{[(2652(x + 221) - 1326) \div 663] - 870\} \div 4) - x$.
 Simplifying this expression starting with the innermost parentheses gives us:

$$\begin{aligned} & (\{[(2652x + 586092 - 1326) \div 663] - 870\} \div 4) - x \\ &= (\{[(2652x + 584766) \div 663] - 870\} \div 4) - x \\ &= (\{(4x + 882) - 870\} \div 4) - x \\ &= ((4x + 12) \div 4) - x \\ &= (4x \div 4) + (12 \div 4) - x \\ &= x + 3 - x \\ &= 3 \end{aligned}$$
 To really understand how this works be sure that you can justify each step to yourself.
 [*Try creating your own number trick.*]
31. The first scale shows that two cubes and a bolt balance with 8 nails. The second one shows that one cube balances a bolt and a nail. Since both scales are balanced, if we put the items from the left pan of the second scale onto the left pan of the first scale and put the items from the right pan of the second scale onto the right pan of the first scale, it will still balance. This means that 3 cubes and a bolt balances with 9 nails and a bolt. Remove a bolt from each side to see that 3 cubes balances 9 nails. Then one cube will balance with 3 nails.

33. If x represents the distance from town C to town D, then $10x$ represents the distance from B to C, since it is 10 times farther. And, it is 10 times further yet from A to B, so the distance from A to B is $10(10x)$, or $100x$. The total distance from town A to town D is the sum of these three distances. This sum is $x + 10x + 100x$. We also know that this total distance is 1332 miles. So we can write the equation $x + 10x + 100x = 1332$. To solve this equation we simplify the left side to be $111x$ and then divide both sides by 111 to get $x = 12$. This is the distance from C to D. The distance from town A to town B is $100x$, which is 1200 miles.
35. There are various ways to see this pattern. You might see three sets of the figure number, with some overlap. For example, in the 4th figure there are three sets of 4 tiles, two sets going vertically and one set going horizontally. But there are two tiles that are counted twice using this method. So there are $3(4) - 2$ tiles. The 10th figure would have $3(10) - 2$ tiles and the n th figure would have $3n - 2$ tiles. Or, you might look at the 4th figure and see three sets of 3 tiles, plus one extra tile. Then there are $3(3) + 1$. The 10th figure then would have $3(9) + 1$ tiles and the n th figure would have $3(n - 1) + 1$ tiles. (Notice that this expression can be written as $3n - 3 + 1$, which is also equal to $3n - 2$.)

So we will try to solve the equation $3n - 2 = 8230$. Adding 2 to both sides gives $3n = 8232$. Dividing both sides by 3 gives $n = 2744$. So the 2744th figure in the pattern would have 8230 tiles.

37. There are several different correct ways to view this pattern. For example, in describing the 3rd figure in the pattern, one person might see a 5 by 5 square of yellow tiles surrounded by a border of red tiles. Another person might see 7 reds on top, 7 on the bottom, and 5 additional on each side, with 5 rows of 5 yellow tiles in the middle. A third viewpoint might be a 7 by 7 square of red tiles with the 5 by 5 tiles in the center changed to yellow. Other correct views are possible.

- a. Solution 1: The fourth figure would have a 7×7 square of yellow tiles in the middle, surrounded by red tiles. There would be $4 \times 8 = 32$ red tiles. This method counts the top left corner with the top row, the top right corner with the right side, the bottom right corner with the bottom and the bottom left corner with the left side. Then the fifth figure would have $9 \times 9 = 81$ yellow tiles, surrounded by $4 \times 10 = 40$ red tiles.

Solution 2: The fourth figure would have 9 reds on top, 9 on the bottom, and 7 on the left and 7 on the right, for a total of $9 \times 2 + 7 \times 2 = 32$ reds tiles. It has $7 \times 7 = 49$ yellow tiles. The fifth figure has $11 \times 2 + 9 \times 2 = 40$ red tiles and $9 \times 9 = 81$ yellow tiles.

Solution 3: The fourth figure has $9 \times 9 - 7 \times 7 = 81 - 49 = 32$ red tiles. It has $7 \times 7 = 49$ yellow tiles. The fifth figure has $11 \times 11 - 9 \times 9 = 121 - 81 = 40$ red tiles and $9 \times 9 = 81$ yellow tiles.

The first figure has 8 red tiles and 1 yellow tile. The second has 16 red and 9 yellow. The third has 24 red and 25 yellow.

- b. Here are three descriptions of the 100th figure, using the thinking of the 3 solutions in a.
- 1.) Figure 100 has a 199×199 square of yellow tiles in the middle. (199 is the 100th odd number.) It is surrounded by a border of 4×200 red tiles. There are 800 red tiles and 39,601 yellow tiles. The total number of tiles is $800 + 39,601 = 40,401$.
 - 2.) Figure 100 has 201 reds on top, 201 on the bottom, and 199 on each side. There are $2 \times 201 + 2 \times 199 = 800$ red tiles. It has $199 \times 199 = 39,601$ yellow tiles. The total number of tiles is $800 + 39,601 = 40,401$.
 - 3.) Figure 100 has $201 \times 201 - 199 \times 199 = 40,401 - 39,601 = 800$ red tiles. It has $199 \times 199 = 39,601$ yellow tiles. The total number of tiles is $201 \times 201 = 40,401$.

Chapter 1

- c. Solution 1: The n^{th} figure has $(2n - 1)(2n - 1)$ or $(2n - 1)^2$ yellow tiles.
It has $4 \times 2n$ or $8n$ red tiles. There are a total of $(2n - 1)^2 + 8n$ tiles.

Solution 2: The n^{th} figure has $(2n - 1)(2n - 1)$ or $(2n - 1)^2$ yellow tiles.
It has $2n + 1$ reds on top, $2n + 1$ on the bottom, and $2n - 1$ on each side for a total of $8n$ reds. There are a total of $(2n - 1)^2 + 8n$ tiles.

Solution 3: The n^{th} figure has $(2n - 1)(2n - 1)$ or $(2n - 1)^2$ yellow tiles.
It has $(2n + 1)^2 - (2n - 1)^2$ red tiles.
There are a total of $(2n - 1)^2 + (2n + 1)^2 - (2n - 1)^2 = (2n + 1)^2$ tiles.

[Note: If these algebraic expressions in part c do not make sense to you, try thinking of n as being 100 and compare the part c answers with those in part b.]

Chapter 1 Test

1. Polya's four steps for problem solving are: Understanding the problem, Devising a plan, Carrying out the plan, and Looking back. These are discussed in Section 1.1 of the text.
2. The eight problem solving strategies introduced in this chapter are:

| | |
|-----------------------|---------------------------|
| Making a drawing | Using a model |
| Guessing and checking | Working backward |
| Making a table | Finding a pattern |
| Using a variable | Solving a simpler problem |
3. $1 + 2 = 3$
 $1 + 2 + 5 = 8$
 $1 + 2 + 5 + 13 = 21$
 $1 + 2 + 5 + 13 + 34 = 55$
Each of these sums is itself a Fibonacci number. Each one is the next Fibonacci number after the last number being added. Another observation is that each answer in the pattern is the sum of the previous answer and the last number being added (e.g., $55 = 21 + 34$).
4. Here is row nine of Pascal's Triangle: 1 9 36 84 126 126 84 36 9 1. The sum of these numbers is 512. Another way to find the sum of the ninth row of Pascal's triangle is to recall that each row of Pascal's triangle sums to a power of 2.
The sum of row nine is $2^9 = 512$.
5. a. This is a geometric sequence with a common ratio of 3. The next term is $81 \times 3 = 243$.
b. This is an arithmetic sequence with a common difference of 3. The next term is 18.
c. This is an arithmetic sequence with a common difference of 6. The next term is 30.
d. This is a sequence of consecutive square numbers. The next square number is 36. The pattern for this sequence can also be found by taking finite differences.

- e. Taking finite differences we get:
- | | | | | |
|---|---|----|----|----|
| 3 | 5 | 11 | 21 | 35 |
| | 2 | 6 | 10 | 14 |
| | | 4 | 4 | 4 |

So the next number is $35 + 18 = 53$.

6. In problem 5, sequence a is geometric and sequences b and c are arithmetic. Sequences d and e are neither arithmetic nor geometric because in both of them there is neither a common difference nor a common ratio.

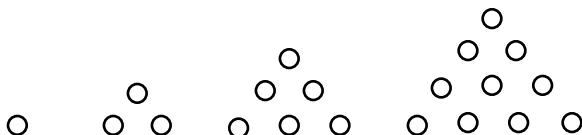
7. a. Taking finite differences we get:
- | | | | | |
|---|---|----|----|----|
| 1 | 5 | 14 | 30 | 55 |
| | 4 | 9 | 16 | 25 |
| | | 5 | 7 | 9 |
| | | | 2 | 2 |

Continuing the pattern to the right, the third row will be 5, 7, 9, 11, 13, 15; and the second row will be 4, 9, 16, 25, 36, 49, 64. To get the next numbers after 1, 5, 14, 30, 55, we can add using the row two numbers: $55 + 36 = 91$, $91 + 49 = 140$, $140 + 64 = 204$.

- b. Taking finite differences we get:
- | | | | |
|---|---|----|----|
| 2 | 9 | 20 | 35 |
| | 7 | 11 | 15 |
| | | 4 | 4 |

Continuing the pattern to the right, the second row will be 7, 11, 15, 19, 23, 27 and the sequence will be 2, 9, 20, 35, 54, 77, 104.

8. a. The first four triangular numbers are shown in the figure below. The difference between successive triangular numbers increases by one with each new number in the sequence. The sequence of triangular numbers is 1, 3, 6, 10, 15, 21, 28, ...
The fifth triangular number is 15.



- b. The square numbers can be drawn similarly, with the figures being squares instead of triangles. The sequence of square numbers is 1, 4, 9, 16, 25, 36, 49, ...
The fifth square number is 25.

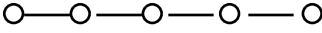
- c. The sequence of pentagonal numbers is 1, 5, 12, 22, 35, 51, 70, ...
The fifth pentagonal number is 35.

[Do you see a pattern in these three questions? Can it be generalized?]

9. If we add the first seven consecutive whole numbers, $1 + 2 + 3 + 4 + 5 + 6 + 7$ we get 28. This number is divisible by 4, so it is not a counterexample. But, $2 + 3 + 4 + 5 + 6 + 7 + 8$ gives us 35, which is not evenly divisible by 4. So these seven consecutive whole numbers provide a counterexample. There are many other possible counterexamples too.

Chapter 1

10. a. Remove two boxes from each side and remove five chips from each side. The beam will remain balanced and we will see that one box balances with seven chips.
- b. Remove two boxes from each side and remove seven chips from each side. The beam will remain in the same position and we will see that there are six chips on the left side and three boxes on the right. The three boxes will remain heavier than the six chips if each box contains more than two chips.
11. a.
- | | |
|-----------------------------|----------------------|
| $3(x - 40) = x + 16$ | Simplification |
| $3x - 120 = x + 16$ | Subtraction Property |
| $3x - 120 - x = x + 16 - x$ | Simplification |
| $2x - 120 = 16$ | Addition Property |
| $2x - 120 + 120 = 16 + 120$ | Simplification |
| $2x = 136$ | Division Property |
| $2x/2 = 136/2$ | Simplification |
| $x = 68$ | |
- b.
- | | |
|-----------------------------------|----------------|
| $4x + 18 = 2(441 - 34x)$ | Simplification |
| $4x + 18 = 882 - 68x$ | Addition |
| $4x + 18 + 68x = 882 - 68x + 68x$ | Simplification |
| $72x + 18 = 882$ | Subtraction |
| $72x + 18 - 18 = 882 - 18$ | Simplification |
| $72x = 864$ | Division |
| $72x/72 = 864/72$ | Simplification |
| $x = 12$ | |
12. a.
- | | |
|------------------------------|----------------|
| $7x - 3 < 52 + 2x$ | Subtraction |
| $7x - 3 - 2x < 52 + 2x - 2x$ | Simplification |
| $5x - 3 < 52$ | Addition |
| $5x - 3 + 3 < 52 + 3$ | Simplification |
| $5x < 55$ | Division |
| $5x/5 < 55/5$ | Simplification |
| $x < 11$ | |
- b.
- | | |
|-------------------------------|----------------|
| $6x - 46 > 79 - 4x$ | Addition |
| $6x - 46 + 4x > 79 - 4x + 4x$ | Simplification |
| $10x - 46 > 79$ | Addition |
| $10x - 46 + 46 > 79 + 46$ | Simplification |
| $10x > 125$ | Division |
| $10x/10 > 125/10$ | Simplification |
| $x > 12.5$ | |
13. a. Step 1: Subtraction Property of Inequality. (2x was subtracted from both sides)
Step 2: Simplification on both sides.
Step 3: Addition Property of Inequality. (217 was added on both sides)
Step 4: Simplification on both sides.
Step 5: Division Property of Inequality. (Both sides divided by 13)
Step 6: Simplification on both sides.
- b. Step 1: Simplification using the distributive property.
Step 2: Subtraction Property of Inequality. (15x was subtracted from both sides)
Step 3: Simplification on both sides.
Step 4: Division Property of Inequality. (Both sides Divided by 23)
Step 5: Simplification on both sides.

14. Suppose that the fence was only 40 feet long. 
- This diagram shows that a 40 foot long straight fence that begins and ends with a post and has posts that are 10 feet on center requires 5 posts. A 30 foot fence would need 4 posts. A 100 foot fence would need 11 posts. A 2000 foot fence would need 201 posts, one every 10 feet and one extra so that both ends have a post. This solution used the strategies of looking at a simpler problem and making a drawing.

Other strategies could also be used to solve this problem.

15. Right before finishing with 170 chips, Pauli won 80 chips, so before winning these 80 chips she had $170 - 80 = 90$ chips. The step that took her to 90 was losing half of her total chips, so she must have had double 90, or 180 chips before losing half of them. To get to 180, she won 50 chips, so before winning those 50, she had $180 - 50 = 130$. In the first round she lost half her chips, which took her to the 130 position. So at the beginning of the game she had $130 \times 2 = 260$ chips. This solution uses the strategy of working backward. That seems to be the most natural approach for this problem, but other strategies could be used, such as a combination of guessing and checking and using a table.
16. One approach to solving this problem is to notice that the diagram of the tower is a representation of triangular numbers. The number of tiles required to build a tower like this with 25 tiles along the base and 25 rows of tiles will be the 25th triangular number. The sequence of the first 25 triangular numbers is: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325. So it would take 325 tiles. If we have done enough explorations with the triangular numbers, we might recall that the 25th triangular number is equal to $(25 \times 26) \div 2$ which is also equal to 325. The n^{th} triangular number is $n(n + 1) \div 2$. The strategies used here included finding a pattern and solving a simpler problem. Other useful strategies might include using a model, making a drawing, or making a table.
17. a. This problem seems to be a good one in which to use the finding a pattern strategy, since we are shown a pattern. One might look for a visual pattern or a pattern in the numbers of dots. The numbers are consecutive multiples of 4, so the fourth square should have $4 \times 4 = 16$ dots. There are a number of different ways to view the visual pattern. Counting the dots on the top, bottom, and sides of each square, one might count the dots in the fourth square as 5 on top, 5 on the bottom, and 3 between on each side. Or, one might count 3 on each of the four sides, plus the 4 corners. Another way to look at the pattern is to see a square array of dots with an inner square array removed. In this case one might count the dots in the fourth square as $5^2 - 3^2$. In any case, the fourth square has 16 dots.
- b. The method for determining the number of dots in the 50th square will depend on the way one views the pattern. Four ways were mentioned in part a. Performing the calculations using each of the four methods in the same order as they are described above gives:

$$4 \times 50 = 200$$

$$51 + 51 + 49 + 49 = 200$$

$$4 \times 49 + 4 = 200$$

$$51^2 - 49^2 = 2601 - 2401 = 200$$

- c. Each of the following is a correct expression for the number of dots in the n^{th} square:

$$4n$$

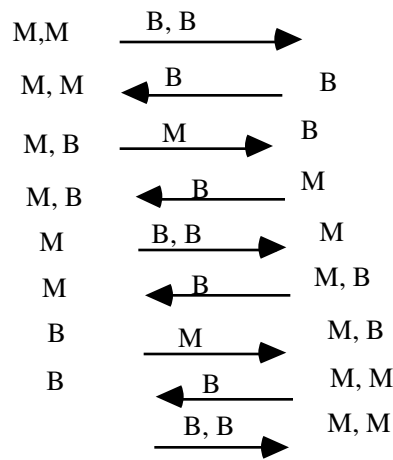
$$2(n + 1) + 2(n - 1)$$

$$4(n - 1) + 4$$

$$(n + 1)^2 - (n - 1)^2$$

18. Let's look at a simpler problem first. Suppose that there are only four people around the table. Call the people A, B, C, and D. A shakes hands with B and with D. B shakes hands with A and with C. C shakes hands with B and with D. D shakes hands with C and with A. This looks like 8 handshakes, but each handshake has been listed twice here, so there are actually four handshakes. Each person shakes the hand of two other people, but if we count each handshake separately this way, each shake gets counted twice. So as long as there are at least three people, there will be exactly as many handshakes as there are people. For 78 people there will be 78 handshakes. For n people there are n handshakes. This solution used the strategies of solving a simpler problem and of finding a pattern. Another solution would be to represent the people around the table as vertices of a polygon, and to represent the handshakes as the edges of the polygon. This might be done with a drawing or with a model.

19. The drawing shows that it would take nine times across the river for the small canoe to get everyone to the other side. A similar procedure could be done using a model.



20. There are several ways to look at this pattern. Here are two of them:

- I. For example, in the 3rd figure notice that there are $3 + 2$ tiles across the top and two sets of three tiles hanging below them for a total of $3 + 3 + (3 + 2) = 11$ tiles.
- II. Notice that the 3rd figure consists of a 4 by 5 rectangle with a 3 by 3 square removed. So it contains $4(5) - 3(3) = 20 - 9 = 11$ tiles.
- III. You may have another way to look at the pattern.

- a. Using method I: The 5th figure has $5 + 5 + (5 + 2) = 17$ tiles.
The 150th figure has $150 + 150 + (150 + 2) = 452$ tiles.
Using method II: The 5th figure has $6(7) - 5(5) = 42 - 25 = 17$ tiles.
The 150th figure has $151(152) - 150(150) = 22952 - 22500 = 452$ tiles.

- b. Using method I: The n^{th} figure has $n + n + (n + 2) = 3n + 2$ tiles.
Using method II: The n^{th} figure has $(n + 1)(n + 2) - n(n)$ tiles.
[With some algebra techniques, we can see that these two expressions are equivalent.]