Chapter 6

# An Introduction to Trigonometric Functions

## Chapter Outline


**6.1** Angle Measure, Special Triangles, and Special Angles 566

**6.2** Unit Circles and the Trigonometry of Real Numbers 582

**6.3** Graphs of the Sine and Cosine Functions 599

**6.4** Graphs of the Tangent and Cotangent Functions 619

**6.5** Transformations and Applications of Trigonometric Graphs 637

**6.6** The Trigonometry of Right Triangles 656

**6.7** Trigonometry and the Coordinate Plane 674


## Preview

In this chapter we'll draw on numerous concepts from previous chapters as we begin a study of trigonometry. The classical approach involves **static trigonometry** and a study of right triangles, the word *static* meaning, "standing or fixed in one place" (Merriam Webster). We'll also look at **dynamic trigonometry**—the trigonometry of objects in motion or in the process of change. Sometimes you'll still be able to see a "right triangle connection" in the applications we consider, but more often the applications will be free from any concept of angle measure as we focus on the trigonometry of real numbers. All in all, we hope to create a framework that unites algebra, geometry, and trigonometry, as it was indeed their union that enabled huge advances in knowledge, technology, and an understanding of things that are.

# 6.1 Angle Measure, Special Angles, and Special Triangles

## LEARNING OBJECTIVES

*In Section 6.1 you will learn how to:*

**A.** Use the vocabulary associated with a study of angles and triangles

**B.** Find fixed ratios of the sides of special triangles using a polygon inscribed in a circle

**C.** Use radians for angle measure and compute circular arc length and area using radians

**D.** Convert between degrees and radians for nonstandard angles

**E.** Solve applications involving angular velocity and linear velocity using radians

## INTRODUCTION

Trigonometry, like its sister science geometry, has its origins deeply rooted in the practical and scientific use of measurement and proportion. In this section we'll look at the fundamental concepts on which trigonometry is based, which we hope will lead to a better understanding and a greater appreciation of the wonderful science that trigonometry has become.

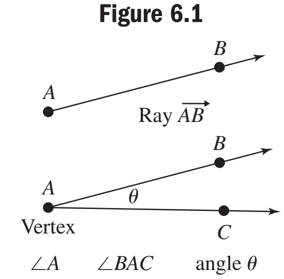
### POINT OF INTEREST

While angle measure based on a 360° circle has been almost universally accepted for centuries, its basic construct is contrived, artificial, and no better or worse than other measures proposed and used over time (stadia, gons, cirs, points, mils, gradients, and so on). In the 1870s, mathematician Thomas Muir and James Thomson (the brother of Lord Kelvin) began advocating the need for a new unit, which stated the measure of an angle in terms of a circle's inherent characteristics, rather than an arbitrarily declared number like 360 (degrees), 400 (gradients), or 1000 (mils). The new measure, which after some deliberation they called *radians,* conveniently expressed the standard angles used since antiquity $\left(30° = \frac{\pi}{6}, 45° = \frac{\pi}{4}, 60° = \frac{\pi}{3}, \text{etc.}\right)$, while contributing to the simplification of many mathematical formulas and procedures. The use of radians also allows a clearer view of the trigonometric functions as *functions of a real number,* rather than merely functions of an angle, thus extending their influence on both pure and applied mathematics.

## A. Angle Measure and Triangles

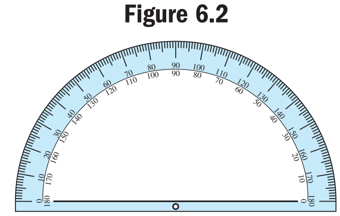
Beginning with the common notion of a straight line, a **ray** is a half line, or all points extending from a single point, in a single direction. An **angle** is the joining of two rays at a common endpoint called the **vertex.** Arrowheads are used to indicate the half lines continue forever and can be extended if necessary. Angles can be named using a single letter at the vertex, the letters from the rays forming the sides, or by a single Greek letter, with the favorites being **alpha** $\alpha$, **beta** $\beta$, **gamma** $\gamma$, and **theta** $\theta$. The symbol $\angle$ is often used to designate an angle (Figure 6.1).

**Figure 6.1**

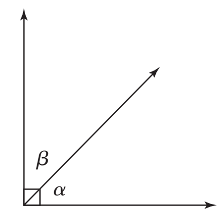



Euclid (325–265 B.C.), who is often thought of as the *father of geometry,* described an angle as "the inclination of one to another of two lines which meet in a plane." This *amount of inclination* gives rise to the common notion of angle measure in degrees, often measured with a semicircular **protractor** like the one shown in Figure 6.2. The notation for degrees is the ° symbol. By

**Figure 6.2**

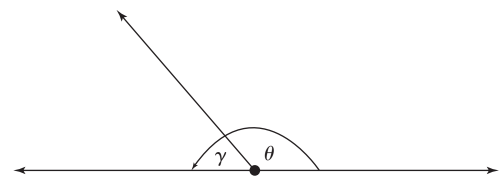


definition 1° is $\frac{1}{360}$ of a full rotation, so this protractor can be used to measure any angle from 0° (where the two rays are on top of each other) to 180° (where they form a straight line). An angle measuring 180° is called a **straight angle,** while an angle that measures 90° is called a **right angle.** Two angles that sum to 90° are said to be **complementary,** while two that sum to 180° are called **supplementary** angles.




$\alpha + \beta = 90°$

$\alpha$ and $\beta$ are complements




$\theta + \gamma = 180°$

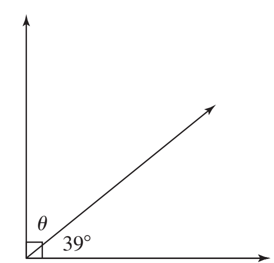
$\theta$ and $\gamma$ are supplements

**EXAMPLE 1** Determine the measure of each angle described.

**a.** the complement of a 57° angle

**b.** the supplement of a 132° angle

**c.** the measure of angle $\theta$ shown in the figure.




**Solution:**

**a.** The complement of 57° is 33° since $90 - 57 = 33°$.

**b.** The supplement of 132° is 48° since $180 - 132 = 48°$.

**c.** Since $\theta$ and 39° are complements, $\theta = 90 - 39 = 51°$.

NOW TRY EXERCISES 7 THROUGH 10

In a study of trigonometry, it helps to further classify the various angles we encounter. An angle greater than 0° but less than 90° is called an **acute** angle. An angle greater than 90° but less than 180° is called an **obtuse** angle. For very fine measurements, each degree is divided into 60 smaller parts called **minutes,** and each minute into 60 smaller parts called **seconds.** This means that a minute is $\frac{1}{60}$ of a degree, while a second is $\frac{1}{3600}$ of a degree. The angle whose measure is "sixty-one degrees, eighteen minutes, and forty-five seconds" is written as $61°\,18'\,45''$ using the notation indicated. The degrees-minutes-seconds (DMS) method of measuring angles is commonly used in aviation and navigation, while in other areas **decimal degrees** such as 61.3125° are preferred. You will sometimes be asked to convert between the two.

**EXAMPLE 2** Convert as indicated:

**a.** $61°\,18'\,45''$ to decimal degrees

**b.** 142.215° to DMS

**Solution:**

**a.** Since $1' = \frac{1}{60}$ of a degree and $1'' = \frac{1}{3600}$ of a degree, we have $(61 + \frac{18}{60} + \frac{45}{3600})° = 61.3125°$.

**b.** For the conversion to DMS we write the whole number part separate from the fractional part to compute the number of minutes it represents: $142° + 0.215° = 142° + 0.215(60)'$ or $142°\ 12.9'$. We then extend the process to find the number of seconds represented by 0.9 minutes. $142°\ 12.9' = 142°\ 12' + 0.9(60)'' = 142°\ 12'\ 54''$.

NOW TRY EXERCISES 11 THROUGH 26 ▶

## B. Triangles and Properties of Triangles

A triangle is a closed plane figure with three straight sides and three angles. Regardless of its shape or orientation, triangles have the following properties.

**PROPERTIES OF TRIANGLES**

Given triangle $ABC$ with sides $a$, $b$, and $c$ respectively,

I. The sum of the angles is 180°:

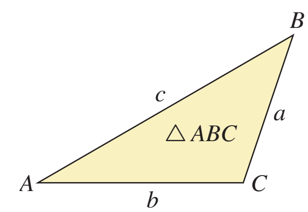
$A + B + C = 180°$

II. The combined length of any two sides exceeds that of the third side:

$a + b > c$, $a + c > b$, and $b + c > a$.

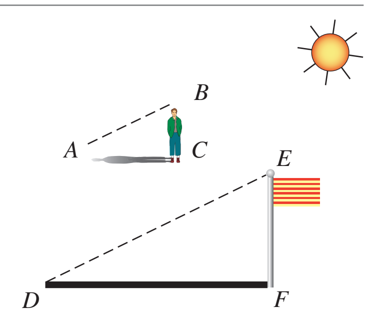
III. Larger angles are opposite larger sides:

If $C > B$, then $c > b$.




Two triangles are **similar triangles** if they have three equal angles. Since antiquity it's been known that *if two triangles are similar, corresponding sides are proportional*. This relationship was used extensively by the engineers of virtually all ancient civilizations and, in fact, undergirds our study of trigonometry. Example 3 illustrates how proportions and similar triangles are often used. Note that *corresponding sides* are those opposite the equal angles from each triangle.

**EXAMPLE 3** ▶ To estimate the height of a flagpole, Laureen reasons that $\triangle ABC$ formed by her height and shadow must be similar to $\triangle DEF$ formed by the flagpole. She is 5 ft 6 in. tall and casting an 8-ft shadow, while the shadow of the flagpole measures 44 ft. How tall is the pole?




**Solution:** ▶ Let $H$ represent the height of the flagpole.

$$\frac{\text{Height}}{\text{Shadow Length}}: \quad \frac{5.5}{8} = \frac{H}{44} \qquad \text{original proportion}$$

$$8H = 242 \qquad \text{cross multiply}$$

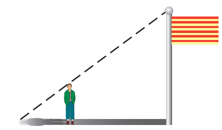
$$H = 30.25 \qquad \text{result}$$

The flagpole is $30\frac{1}{4}$ ft tall.

NOW TRY EXERCISES 27 THROUGH 34 ▶

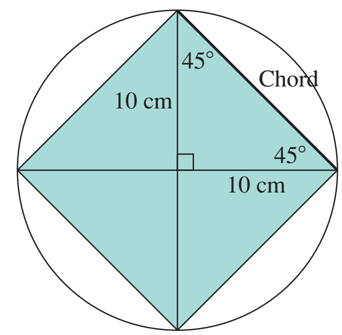
Figure 6.3 shows Laureen standing along the shadow of the flagpole, again illustrating the proportional relationships that exist. Early mathematicians quickly recognized the power of these relationships, realizing if the angle of inclination and the related fixed proportions were known, they had the ability to find mountain heights and the widths of lakes, and even the ability to estimate distances to the Sun and Moon. What was needed was an accurate and systematic method of finding these "fixed proportions" for various angles, so they could be applied more widely. In support of this search, two special triangles were often used. These triangles, commonly called **45-45-90** and **30-60-90** triangles, are *special* because no estimation, interpolation, or formula is needed to find the relationships between their sides.

**Figure 6.3**



Consider a circle with $r = 10$ cm circumscribed about a square. By drawing the diagonals of the square, four right triangles (triangles having a $90°$ angle) are formed, each with a base and height of 10 cm (see Figure 6.4). The hypotenuse of each is a *chord of the circle* (a line segment whose endpoints lie on the circumference). Due to the symmetry, the two nonright angles are equal and measure $45°$, and the Pythagorean theorem shows the length of the chord (hypotenuse) is $\sqrt{200} = 10\sqrt{2}$. This 45-45-90 triangle has sides in the proportion $10:10:10\sqrt{2}$. Actually, the relationship easily generalizes for circles of any radius and the result is stated here.

**Figure 6.4**

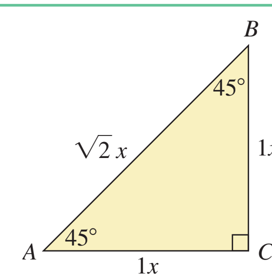



**45-45-90 TRIANGLES**

Given a 45-45-90 triangle with one side of length $x$, the proportional relationship between the sides is:

$$1x:1x:\sqrt{2}\,x.$$

In words, the two legs are equal and the hypotenuse is $\sqrt{2}$ times either leg.



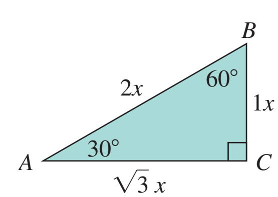

The proportional relationship for a 30-60-90 triangle is likewise developed (see Exercise 106) and the result is stated here.

**30-60-90 TRIANGLES**

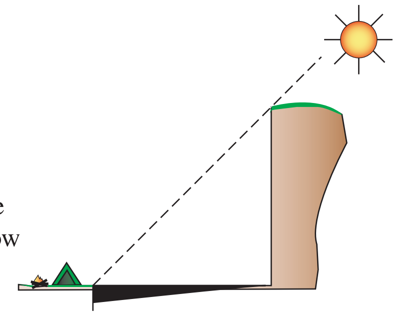
Given a 30-60-90 triangle with the shortest side of length $x$, the proportional relationship between the sides is:

$$1x:\sqrt{3}\,x:2x.$$

In words, the hypotenuse is 2 times the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.



**EXAMPLE 4** ▶ A group of campers has pitched their tent some distance from the base of a tall cliff. The evening's conversation turns to a discussion of the cliff's height and they all lodge an estimate. Then one of them says, "Wait . . . how will we know who's closest?" Describe how a 45-45-90 triangle can help determine a winner.



**Solution:** ▶ In the morning, cut a pole equal in height to any of the campers. Then follow the shadow of the cliff as it moves, with the selected camper occasionally laying the pole at her feet and checking her shadow's length against the length of the pole. At the moment her shadow is equal to the pole's length, the sun is shining at a 45° angle and the campers can use the pole to measure the shadow cast by the cliff, which will be equal to its height since a 45-45-90 triangle is formed.

NOW TRY EXERCISES 35 AND 36 ▶

## C. Angle Measure in Radians; Arc Length and Area

As stated in the *Point of Interest,* there are numerous ways to measure angles and these can actually be used interchangeably. However, where degree measure has its roots in measuring "the amount of inclination" between two rays, angle measure in radians uses a slightly different perspective. For two rays joined at a vertex, angle measure can also be considered as the *amount of rotation* from a fixed ray called the **initial side,** to the rotated ray called the **terminal side.** This allows for the possibility of angles greater than 180°, and for positive or negative angles, depending on the direction of rotation. Angles formed by a counterclockwise rotation are considered **positive angles,** and angles formed by a clockwise rotation are **negative angles.** We can then name an angle of any size, including those greater than 360° where the amount of rotation exceeds one revolution. See Figures 6.5 through 6.10, noting that in Figure 6.10 angles $\theta$ and $\alpha$ share the same initial and terminal sides and are called **coterminal angles.**

**Figure 6.5**

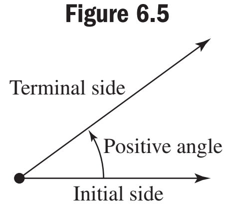



**Figure 6.6**

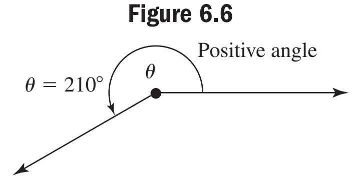



**Figure 6.7**

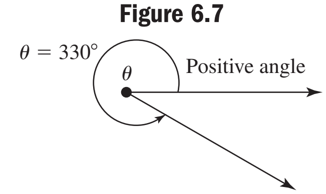



**Figure 6.8**

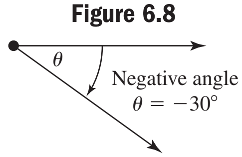



**Figure 6.9**

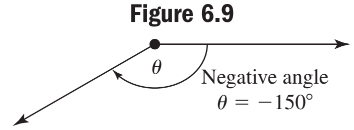
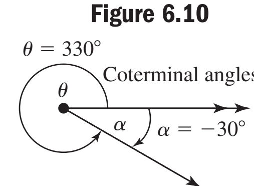
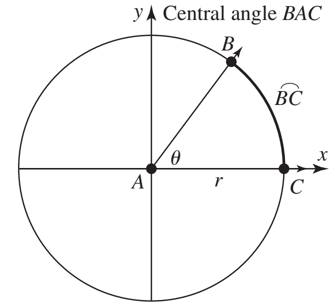



**Figure 6.10**



An angle is said to be in **standard position** in the $xy$-plane if its vertex is at the origin and the initial side is along the $x$-axis. In standard position, the terminal sides of 90°, 180°, 270°, and 360° angles coincide with one of the axes and are called **quadrantal angles.** To help develop these ideas further, we use a **central circle,** or a circle in the $xy$-plane with its center at the origin. A **central angle** is an angle of the circle that's in standard position. For central angle $\theta$ intersecting the circle at points $B$ and $C$, we say $\overset{\frown}{BC}$ (arc $BC$) **subtends** $\angle BAC$, as shown in Figure 6.11. The arc encompassing the 360° of a whole circle is identical to its circumference, and has length $C = 2\pi r$. To find the length of a shorter arc, we simply use a *proportional part* of the 360°: $\frac{\text{amount of rotation}}{360°} = \frac{\text{length of arc}}{2\pi r}$. It is customary to name the arc length $s$, and the proportion becomes $\frac{\theta}{360°} = \frac{s}{2\pi r}$. Solving for $s$ gives $s = \left(\frac{\theta}{360°}\right)2\pi r$, which can be rewritten as $s = \left(\frac{2\pi}{360°}\right)\theta r$ by regrouping. While this formula is very usable, it is unwieldy and has limited value in more advanced applications. Thomas Muir and James Thomson (see the *Point of Interest*) had the idea that if the angle $\theta$ were defined in terms of $s$ and $r$, this formula and many others could be greatly simplified. In particular, if you define **1 radian** (abbreviated rad) to be the measure of an angle subtended by an arc equal in length to the radius (see Figure 6.12), the preceding formula gives $\frac{2\pi}{360°} = 1$ by direct substitution:

**Figure 6.11**

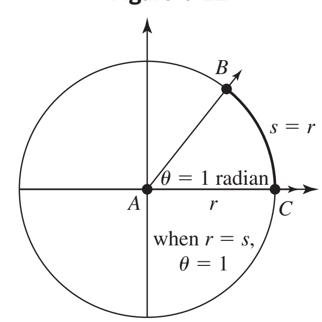



$$s = \left(\frac{2\pi}{360°}\right)\theta r \qquad \text{arc length}$$

$$s = \left(\frac{2\pi}{360°}\right)(1)(s) \qquad \text{substitute } s \text{ for } r \text{ and 1 for } \theta$$

$$1 = \frac{2\pi}{360°} \qquad \text{simplify}$$

**Figure 6.12**



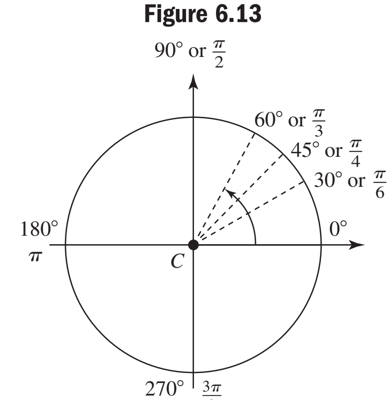

This creates a much simpler **arc length formula,** since we can now substitute 1 for $\frac{2\pi}{360°}$ in the formula $s = \left(\frac{2\pi}{360°}\right)\theta r$. In other words, $s = r\theta$, if $\theta$ is measured in radians.

**WORTHY OF NOTE**

We will adopt the convention that unless degree measure is explicitly implied or noted with the ° symbol, radian measure is being used. In other words, $\theta = \frac{\pi}{2}$, $\theta = 2$, and $\theta = 32.76$ all indicate angles measured in radians.

Measuring a central angle $\theta$ in radians also simplifies the formula for the **area of a circular sector.** The area of a circle encompassing 360° is $A = \pi r^2$. To find the area of a circular segment (a pie slice), we use a proportional part of the 360°, which gives $\frac{\theta}{360°} = \frac{A}{\pi r^2}$ for $\theta$ in degrees. This can be written as $A = \left(\frac{\pi}{360°}\right)r^2\theta$ or $\left(\frac{2\pi}{360°}\right)\frac{1}{2}r^2\theta$ after regrouping. With $\theta$ in radians the formula becomes

**Figure 6.13**



$A = \frac{1}{2}r^2\theta$ by substituting as before. Finally, the ratio $\frac{2\pi}{360°} = 1$ *gives an explicit connection between degrees and radians:* $2\pi = 360°$. This means $\pi = 180°$ and with this relationship, we can easily state the radian measure of the standard angles using a simple division (see Figure 6.13). For $\pi = 180°$, we have:

$$\text{division by 2: } \frac{\pi}{2} = 90° \qquad \text{division by 3: } \frac{\pi}{3} = 60°$$

$$\text{division by 4: } \frac{\pi}{4} = 45° \qquad \text{division by 6: } \frac{\pi}{6} = 30°.$$

The radian measure of these standard angles play a major role in this chapter, and you are encouraged to become very familiar with them. Additional conversions can quickly be found using multiples of these four. For example, multiplying both sides of $\frac{\pi}{3} = 60°$ by two gives $\frac{2\pi}{3} = 120°$.

**ARC LENGTH AND AREA IN RADIANS**

Given a circle of radius $r$, with central angle $\theta$ subtended by arc $\widehat{BC}$.

| *Arc-Length Formula* | *The Area of a Circular Sector* |
|---|---|
| The length $s$ of the arc $\widehat{BC}$ is | The area of sector $BAC$ is |
| $s = r\theta$, | $A = \frac{1}{2}r^2\theta$, |
| *provided $\theta$ is in radians.* | *provided $\theta$ is in radians.* |

**EXAMPLE 5** For a circle with radius 72 cm and a central angle of 45°, find (a) the length of the subtended arc and (b) the area of the circular sector.

**Solution:**

**a.** Using the formula $s = r\theta$ with $45° = \frac{\pi}{4}$, we have $s = 72\left(\frac{\pi}{4}\right)$ or $18\pi$ cm in exact form. The arc is about 56.5 cm long.

**b.** The area formula gives

$$\begin{aligned} A &= \left(\frac{1}{2}\right)(72)^2\left(\frac{\pi}{4}\right) && \text{substitute 72 for } r \text{ and } \frac{\pi}{4} \text{ for } \theta \\ &= 648\pi \text{ cm}^2 && \text{result} \end{aligned}$$

The area of this sector is approximately 2036 $\text{cm}^2$.

**NOW TRY EXERCISES 37 THROUGH 60**

## D. Converting Between Degrees and Radians

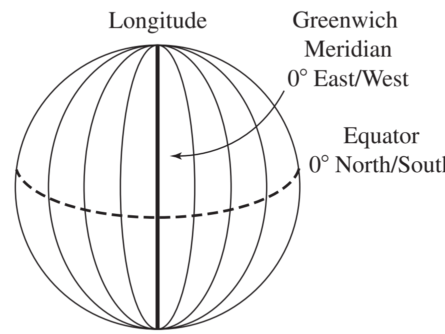
The relationship $\pi = 180°$ also gives the factors needed for conversions when $\theta$ is a non-standard angle. Dividing by $\pi$ we have $1 \text{ rad} = \frac{180°}{\pi}$, while division by 180° shows $1° = \frac{\pi}{180}$ radians. Multiplying by the appropriate factor gives an equivalent measure.

**DEGREES/RADIANS CONVERSION FACTORS**

To convert from radians to degrees: $1 \text{ rad} = \left(\dfrac{180}{\pi}\right)^\circ$ $\left(\text{multiply by } \dfrac{180}{\pi}\right)$.

To convert from degrees to radians: $1^\circ = \left(\dfrac{\pi}{180}\right)$ rad $\left(\text{multiply by } \dfrac{\pi}{180}\right)$.

**Figure 6.14**




One example where these conversions are useful is in applications involving longitude and latitude (see Figure 6.14). The **latitude** of a fixed point on the Earth's surface tells how many degrees north or south of the equator the point is, as measured from the center of the Earth. The **longitude** of a fixed point on the Earth's surface tells how many degrees east or west of the Greenwich Meridian (in England) the point is, as measured along the equator to the north/south line going through the point. For instance, Memphis, Tennessee, and New Orleans, Louisiana, are directly north/south of each other along 90° west longitude (different latitudes-see Figure 6.15). The cities of Zurich, Switzerland, and Budapest, Hungary, are east/west of each other at about 47.2° north latitude (different longitudes-see Figure 6.16).

**Figure 6.15**

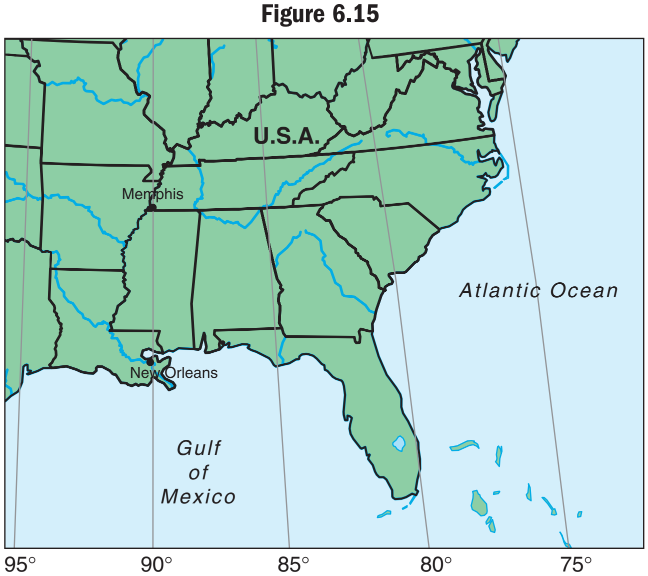



**Figure 6.16**


**EXAMPLE 6** The cities of Quito, Ecuador, and Macapá, Brazil, both lie very near the equator, at a latitude of 0°. However, Quito is at approximately 78° west longitude, while Macapá is at 51° west longitude. Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

**WORTHY OF NOTE**

Note that $r = 3960$ mi was used because Quito and Macapá *are both on the equator.* For other cities sharing the same longitude but not on the equator, the radius of the Earth *at that longitude* must be used. See Exercise 108.

**Solution:** First we note that $(78 - 51)° = 27°$ of longitude separate the two cities. Using the conversion factor $1° = \left(\frac{\pi}{180}\right)$ rad, we find the equivalent radian measure is $27°\left(\frac{\pi}{180}\right) = \frac{3\pi}{20}$. The arc length formula gives:

$$s = r\theta \quad \text{arc length formula; } \theta \text{ in radians}$$

$$= 3960\left(\frac{3\pi}{20}\right) \quad \text{substitute 3960 for } r \text{ and } \frac{3\pi}{20} \text{ for } \theta$$

$$= 594\pi \quad \text{result}$$

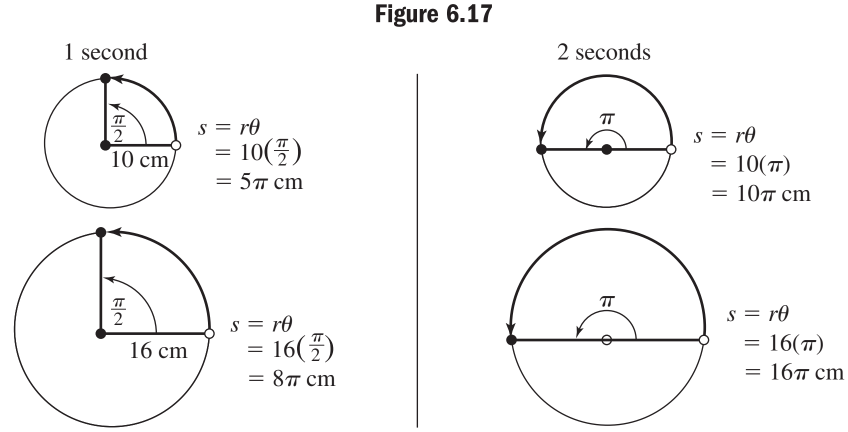
Quito and Macapá are approximately 1866 mi apart.

**NOW TRY EXERCISES 61 THROUGH 92**

## E. Angular and Linear Velocity

When the formula for uniform motion (Distance = Rate × Time: $D = RT$) is expressed in terms of $R$, we have $R = \frac{D}{T}$, where the rate is measured as a distance per unit time. We can also apply this concept to a *rate of rotation* per unit time, called **angular velocity.** In this context, we use the symbol $\omega$ to represent the rate of rotation and $\theta$ to represent the angle through which the terminal side has moved, measured in radians. The result is a formula for angular velocity: $\omega = \frac{\theta}{t}$ ($\theta$ in radians). For instance, a bicycle wheel turning at 150 revolutions per minute (rpm) or $\frac{150 \text{ revolutions}}{1 \text{ min}}$ has an angular velocity of $\omega = \frac{150(2\pi)}{1 \text{ min}} = \frac{300\pi \text{ rad}}{1 \text{ min}}$ since 1 revolution $= 2\pi$. We can also use $\omega = \frac{\theta}{t}$ to find the **linear velocity** of this bicycle. Figure 6.17 shows two disks rotating on a spindle at the same speed. The first disk has a radius 10 cm, while the second has radius of 16 cm. Assume both disks are rotating slowly at $\omega = \frac{\pi/2 \text{ rad}}{1 \text{ sec}}$. We'll track the distance traveled by a point on the circumference of each disk for 2 sec, using the arc length formula $s = r\theta$.

**Figure 6.17**



As you can see, the angular velocity is independent of the radius, while the linear distance traveled per unit time, the linear velocity, depends very much on the radius. For $r = 10$ the point traveled $10\pi$ cm in 2 sec, while for $r = 16$ the point traveled $16\pi$ cm in the same amount of time. In the context of angular motion, the formula $R = \frac{D}{T}$ becomes $V = \frac{s}{t}$, and since $s = r\theta$, $V = \frac{r\theta}{t}$. The formula can also be written directly in terms of $\omega$, since $\omega = \frac{\theta}{t}$: $V = r\omega$. In summary, $V = r\omega$ gives the linear velocity of a point on the circumference of a rotating circular object, as long as $\omega$ is in radians/unit time.

**ANGULAR AND LINEAR VELOCITY**

Given a central circle of radius $r$ with point $P$ on the circumference, and central angle $\theta$ with $P$ on the terminal side. If $P$ moves along the circumference at a uniform rate:

(1) The rate at which $\theta$ changes is called the *angular velocity* $\omega$,

$$\omega = \frac{\theta}{t}.$$

(2) The rate at which the position of $P$ changes is called the *linear velocity V,*

$$V = \frac{r\theta}{t} \quad \Rightarrow \quad V = r\omega.$$

**EXAMPLE 7** The wheels on a racing bicycle have a radius of 13 in. How fast is the cyclist traveling in miles per hour, if the wheels are turning at 150 rpm?

**Solution:** Earlier we noted 150 rpm $= \frac{300\pi}{1 \text{ min}}$. Using the formula $V = r\omega$ gives a linear velocity of $V = (13 \text{ in.})\frac{300\pi}{1 \text{ min}} \approx \frac{12{,}252.2 \text{ in.}}{1 \text{ min}}$. To convert this to miles per hour we convert minutes to hours (1 hr = 60 min) and inches to miles (1 mi = 5280 × 12 in.): $\left(\frac{12{,}252.2 \text{ in.}}{1 \text{ min}}\right)\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)\left(\frac{1 \text{ mi}}{63{,}360 \text{ in.}}\right) \approx 11.6$ mph. The bicycle is traveling about 11.6 mph.

NOW TRY EXERCISES 95 THROUGH 98

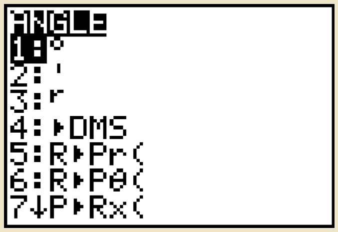
**TECHNOLOGY HIGHLIGHT**

## Decimal Degree and Radian Conversions

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

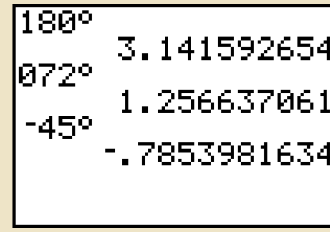
Most graphing calculators are programmed to compute conversions involving angle measure. On the TI-84 *Plus,* this is accomplished using the "°" feature for degree to radian conversions *while in radian* MODE, and the "r" feature for radian to degree conversions *while in degree* MODE. Both are found on the **ANGLE** submenu located at 2nd APPS, as is

Figure 6.18




the **4:▶DMS** feature used for conversion to the degrees, minutes, seconds format (see Figure 6.18). We'll illustrate by converting both standard and non-standard angles. To convert 180°, 72°, and −45° to radians, be sure you are in radian MODE then enter 180 2nd APPS ENTER (the 1:° feature is the default), then ENTER once again to execute the operation. The screen shows a value of 3.141592654, which we expected since $180° = \pi$. For 72° and −45°, we simply recall 180° ( 2nd ENTER ) and overwrite the desired value (see Figure 6.19). For radian-to-degree conversions, be sure you are in degree MODE and complete the conversions in a similar manner, using 2nd APPS **3:ʳ** instead of **1:°**.

Figure 6.19




Exercise 1: Use your graphing calculator to convert the radian measures $\frac{\pi}{2}$, $\pi$, $\frac{25\pi}{12}$, and 2.37 to degrees, then verify each using standard angles or a conversion factor.

Exercise 2: Experiment with the 2nd APPS **3:▶DMS** feature, and use it to convert 108.716° to the DMS format. Verify the result manually.

# 6.1 EXERCISES

## CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** ______________ angles sum to 90°. Supplementary angles sum to ____°. Acute angles are ___ than 90°. Obtuse angles are _______ than 90°.

**2.** The expression "theta equals two degrees" is written ______ using the ° notation. The expression, "theta equals two radians" is simply written _____.

**3.** The formula for arc length is $s =$ __. The area of a sector is $A =$ _____. For both formulas, $\theta$ must be in _______.

**4.** If $\theta$ is not a standard angle, multiply by _____ to convert radians to degrees. To convert degrees to radians, multiply by _____.

**5.** Discuss/explain the difference between angular velocity and linear velocity. In particular, why does one depend on the radius while the other does not?

**6.** Discuss/explain the difference between 1° and 1 radian. Exactly what is a radian? Without any conversions, explain why an angle of 4 rad terminates in QIII.

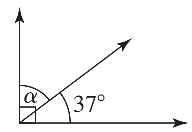
## DEVELOPING YOUR SKILLS

Determine the measure of the angle described or indicated.

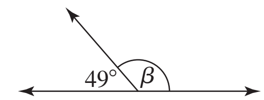
**7.** **a.** The complement of a 12.5° angle **b.** The supplement of a 149.2° angle

**8** **a.** The complement of a 62.4° angle **b.** The supplement of a 74.7° angle

**9.** The measure of angle $\alpha$




**10.** The measure of angle $\beta$




Convert from DMS (degree/minute/sec) notation to decimal degrees. Round to hundredths of a degree.

**11.** $42°30'$ **12.** $125°45'$ **13.** $67°22'42''$ **14.** $9°08'58''$

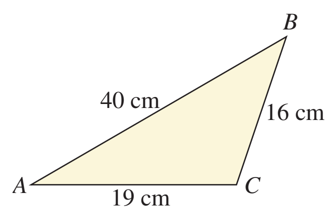
**15.** $285°00'48''$ **16.** $312°00'24''$ **17.** $45°45'45''$ **18.** $30°30'30''$

Convert the angles from decimal degrees to DMS (degree/minute/sec) notation.

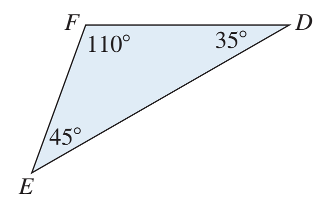
**19.** 20.25° **20.** 40.75° **21.** 67.30° **22.** 83.5°

**23.** 275.33° **24.** 330.45 **25.** 5.4506° **26.** 12.3228°

**27.** Is the triangle shown possible? Why/why not?

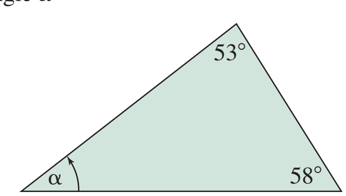



**28.** Is the triangle below possible? Why/why not?

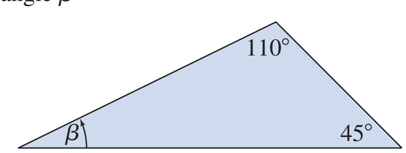



Determine the measure of the angle indicated.

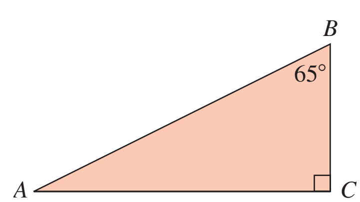
**29.** angle $\alpha$



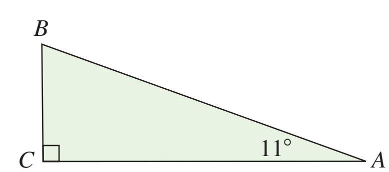

**30.** angle $\beta$




**31.** $\angle A$




**32.** $\angle B$

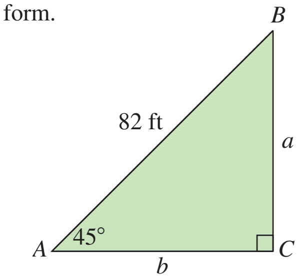



**33.** **Similar triangles:** A helicopter is hovering over a crowd of people watching a police standoff in a parking garage across the street. Stewart notices the shadow of the helicopter is lagging approximately 50 m behind a point directly below the helicopter. If he is 2 m tall and is casting a shadow of 1.6 m at this time, what is the altitude of the helicopter?

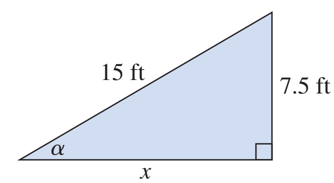
**34.** **Similar triangles:** Near Fort Macloud, Alberta (Canada), there is a famous cliff known as *Head Smashed in Buffalo Jump.* The area is now a Canadian National Park, but at one time the Native Americans hunted buffalo by steering a part of the herd over the cliff. While visiting the park late one afternoon, Denise notices that its shadow reaches 201 ft from the foot of the cliff, at the same time she is casting a shadow of $12'1''$. If Denise is $5'4''$ tall, what is the height of the cliff?

Solve using special triangles. Answer in both exact and approximate form.

**35. Special triangles:** A ladder-truck arrives at a high-rise apartment complex where a fire has broken out. If the maximum length the ladder extends is 82 ft and the angle of inclination is 45°, how high up the side of the building does the ladder reach?




**36. Special triangles:** A heavy-duty ramp is used to winch heavy appliances from street level up to a warehouse loading dock. If the ramp is 7.5 ft high and the incline is 15 ft long, (a) what angle α does the dock make with the street? (b) How long is the base of the ramp?




Use the formula for arc length to find the value of the remaining unknown: $s = r\theta$.

**37.** $\theta = 3.5$; $r = 280$ m

**38.** $\theta = 2.3$; $r = 129$ cm

**39.** $s = 2007$ mi; $r = 2676$ mi

**40.** $s = 4435.2$ km; $r = 12{,}320$ km

**41.** $\theta = \frac{3\pi}{4}$; $s = 4146.9$ yd

**42.** $\theta = \frac{11\pi}{6}$; $s = 28.8$ nautical miles

**43.** $\theta = \frac{4\pi}{3}$; $r = 2$ mi

**44.** $\theta = \frac{3\pi}{2}$; $r = 424$ in.

**45.** $s = 252.35$ ft; $r = 980$ ft

**46.** $s = 942.3$ mm; $r = 1800$ mm

**47.** $\theta = 320°$; $s = 52.5$ km

**48.** $\theta = 202.5°$; $s = 7627$ m

Use the formula for area of a circular sector to find the value of the remaining unknown: $A = \frac{1}{2}r^2\theta$.

**49.** $\theta = 5$; $r = 6.8$ km

**50.** $\theta = 3$; $r = 45$ mi

**51.** $A = 1080$ mi$^2$; $r = 60$ mi

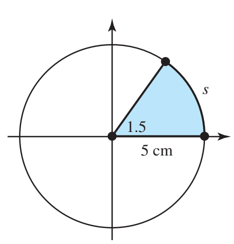
**52.** $A = 437.5$ cm$^2$; $r = 12.5$ cm

**53.** $\theta = \frac{7\pi}{6}$; $A = 16.5$ m$^2$

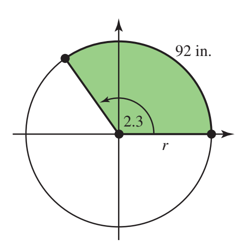
**54.** $\theta = \frac{19\pi}{12}$; $A = 753$ cm$^2$

Find the angle, radius, arc length, and/or area as needed, until all values are known.

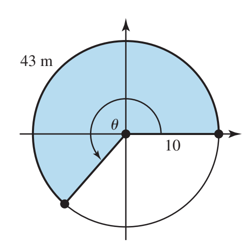
**55.**



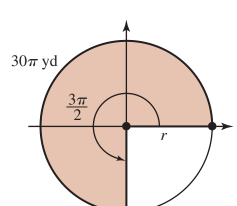

**56.**



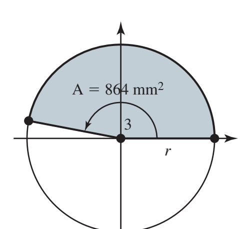

**57.**



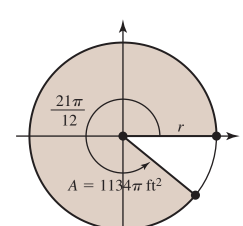

**58.**




**59.**




**60.**



Convert the following degree measures to radians in exact form, without the use of a calculator.

**61.** $\theta = 360°$ **62.** $\theta = 180°$ **63.** $\theta = 45°$ **64.** $\theta = 30°$

**65.** $\theta = 210°$ **66.** $\theta = 330°$ **67.** $\theta = -120°$ **68.** $\theta = -225°$

Convert each degree measure to radians. Round to the nearest ten-thousandth.

**69.** $\theta = 27°$ **70.** $\theta = 52°$ **71.** $\theta = 227.9°$ **72.** $\theta = 154.4°$

Convert each radian measure to degrees, without the use of a calculator.

**73.** $\theta = \frac{\pi}{3}$ **74.** $\theta = \frac{\pi}{4}$ **75.** $\theta = \frac{\pi}{6}$ **76.** $\theta = \frac{\pi}{2}$

**77.** $\theta = \frac{2\pi}{3}$ **78.** $\theta = \frac{5\pi}{6}$ **79.** $\theta = 4\pi$ **80.** $\theta = 6\pi$

Convert each radian measure to degrees. Round to the nearest tenth.

**81.** $\theta = \frac{11\pi}{12}$ **82.** $\theta = \frac{17\pi}{36}$ **83.** $\theta = 3.2541$ **84.** $\theta = 1.0257$

**85.** $\theta = 3$ **86.** $\theta = 5$ **87.** $\theta = -2.5$ **88.** $\theta = -3.7$

**89.** **Arc length:** The city of Pittsburgh, Pennsylvania, is directly north of West Palm Beach, Florida. Pittsburg is at 40.3° north latitude, while West Palm Beach is at 26.4° north latitude. Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

**90.** **Arc length:** Both Libreville, Gabon, and Jamame, Somalia, lie near the equator, but on opposite ends of the African continent. If Libreville is at 9.3° east longitude and Jamame is 42.5° east longitude, how wide is the continent of Africa at the equator?

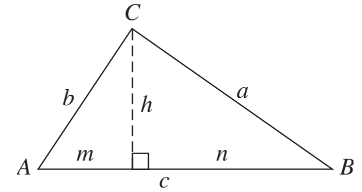
**91.** **Area of a sector:** A water sprinkler is set to shoot a stream of water 12 m long and rotate through an angle of 40°. (a) What is the area of the lawn it waters? (b) For $r = 12$ m, what angle is required to water twice as much area? (c) For $\theta = 40°$, what range for the water stream is required to water twice as much area?

**92.** **Area of a sector:** A motion detector can detect movement up to 25 m away through an angle of 75°. (a) What area can the motion detector monitor? (b) For $r = 25$ m, what angle is required to monitor 50% more area? (c) For $\theta = 75°$, what range is required for the detector to monitor 50% more area?

## WORKING WITH FORMULAS

**93. Additional relationships in a right triangle:** $h = \frac{ab}{c}$, $m = \frac{b^2}{c}$, **and** $n = \frac{a^2}{c}$

Given $\angle C$ is a right angle, and $h$ is the altitude of $\triangle ABC$, then $h$, $m$, and $n$ can all be expressed directly in terms of $a$, $b$, and $c$ by the relationships shown here. Compute the value of $h$, $m$, and $n$ for a right triangle with sides of 8, 15, and 17 cm.




**94. The height of an equilateral triangle:** $H = \frac{\sqrt{3}}{2}S$

Given an equilateral triangle with sides of length $S$, the height of the triangle is given by the formula shown. Once the height is known the area of the triangle can easily be found (also see Exercise 93). The Gateway Arch in St. Louis, Missouri, is actually composed of stainless steel sections that are equilateral triangles. At the base of the arch the length of the sides is 54 ft. The smallest cross section at the top of the arch has sides of 17 ft. Find the area of these cross sections.

## APPLICATIONS

**95. Riding a round-a-bout:** At the park two blocks from our home, the kid's round-a-bout has a radius of 56 in. About the time the kids stop screaming, "Faster, Daddy, faster!" I estimate the round-a-bout is turning at $\frac{3}{4}$ revolutions per second. (a) What is the related angular velocity? (b) What is the linear velocity (in miles per hour) of Eli and Reno, who are "hanging on for dear life" at the rim of the round-a-bout?

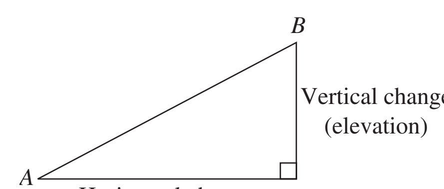
**96. Carnival rides:** At carnivals and fairs, the *Gravity Drum* is a popular ride. People stand along the wall of a circular drum with radius 12 ft, which begins spinning very fast, pinning them against the wall. The drum is then turned on its side by an armature, with the riders screaming and squealing with delight. As the drum is raised to a near-vertical position, it is spinning at a rate of 35 rpm. (a) What is the related angular velocity? (b) What is the linear velocity (in miles per hour) of a person on this ride?

**97. Speed of a winch:** A winch is being used to lift a turbine off the ground so that a tractor-trailer can back under it and load it up for transport. The winch drum has a radius of 3 in. and is turning at 20 rpm. Find (a) the angular velocity of the drum, (b) the linear velocity of the turbine in feet per second as it is being raised, and (c) how long it will take to get the load to the desired height of 6 ft (ignore the fact that the cable may wind over itself on the drum).

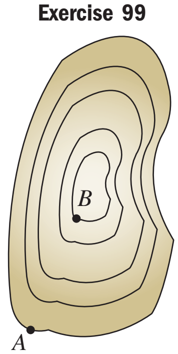
**98. Speed of a current:** An instrument called a *fluviometer* is used to measure the speed of flowing water, like that in a river or stream. A cruder method involves placing a paddle wheel in the current, and using the wheel's radius and angular velocity to calculate the speed of water flow. If the paddle wheel has a radius of 5.6 ft and is turning at 30 rpm, find (a) the angular velocity of the wheel and (b) the linear velocity of the water current in miles per hour.

On topographical maps, each concentric figure (a figure within a figure) represents a fixed change in elevation (the vertical change) according to a given *scale of elevation.* The *measured distance* on the map from point $A$ to point $B$ indicates the horizontal distance or the horizontal change between point $A$ and a location directly beneath point $B$, according to a given *scale of distances.*

**Exercises 99 and 100**



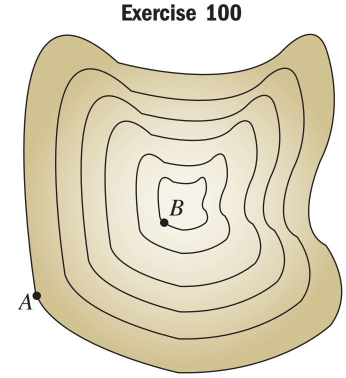

**Exercise 99**




**99. Special triangles:** In the figure shown, the *scale of elevation* is 1:250 (each concentric line indicates an increase of 250 m in elevation) and the scale of distances is 1 cm = 625 m. (a) Find the change of elevation from $A$ to $B$; (b) use a proportion to find the horizontal change between points $A$ and $B$ if the measured distance between them is 1.6 cm; and (c) Draw the corresponding right triangle and use a special triangle relationship to find the length of the trail up the mountain side which connects $A$ and $B$.

**100. Special triangles:** As part of park maintenance, the 2 by 4 handrail alongside a mountain trail leading to the summit of Mount Marilyn must be replaced. In the figure, the *scale of elevation* is 1:200 (each concentric line indicates an increase of 200 m in elevation) and the scale of distances is 1 cm = 400 m. (a) Find the change of elevation from $A$ to $B$; (b) use a proportion to find the horizontal change between $A$ and $B$ if the measured distance between them is 4.33 cm; and (c) draw the corresponding right triangle and use a special triangle relationship to find the length needed to replace the handrail (recall that $\sqrt{3} \approx 1.732$).

**Exercise 100**



**101. Special triangles:** Two light planes are flying in formation at 100 mph, doing some reconnaissance work. As they approach a mountain, one pilot breaks to the left at an angle of 45° to the other plane and increases speed to 141.4 mph. Assuming they keep the same altitude and the first plane continues to fly at 100 mph, use a special triangle to find the distance between them after 0.5 hr. Note $\sqrt{2} \approx 1.414$.

**102. Special triangles:** Two ships are cruising together on the open ocean at 10 nautical miles per hour. One of them turns to make a 90° angle with the first and increases speed to 17.32 nautical miles per hour, heading for port. Assuming the first ship continues traveling at 10 knots, use a special triangle to find the distance between the ships in 1 hr, just as the second ship reaches port. Note $\sqrt{3} \approx 1.732$.

**103. Angular and linear velocity:** The planet Jupiter's largest moon, Ganymede, rotates around the planet at a distance of about 656,000 miles, in an orbit that is perfectly circular. If the moon completes one rotation about Jupiter in 7.15 days, (a) find the angle $\theta$ that the moon moves through in 1 day, in both degrees and radians, (b) find the angular velocity of the moon in radians per hour, and (c) find the moon's linear velocity in miles per second as it orbits Jupiter.

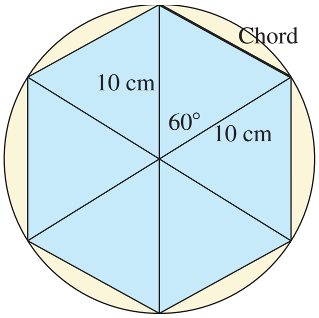
**104. Angular and linear velocity:** The planet Neptune has a very low eccentricity and an orbit that is nearly circular. It orbits the Sun at a distance of 4497 million kilometers and completes one revolution every 165 yr. (a) Find the angle $\theta$ that the planet moves through in one year in both degrees and radians and (b) find the linear velocity (km/hr) as it orbits the Sun.

## WRITING, RESEARCH, AND DECISION MAKING

**105.** As mentioned in the *Point of Interest,* many methods have been used for angle measure over the centuries, some more logical or meaningful than what is popular today. Do some research on the evolution of angle measure, and compare/contrast the benefits and limitations of each method. In particular, try to locate information on the history of degrees, radians, mils, and gradients, and identify those still in use.

**106.** Use the diagram given to develop the fixed ratios for the sides of a 30-60-90 triangle. Ancient geometers knew that a hexagon (six sides) could be inscribed in a circle by laying out six consecutive chords equal in length to the radius ($r = 10$ cm for illustration). After connecting the diagonals of the hexagon, six equilateral triangles are formed with sides of 10 cm.

**Exercise 106**



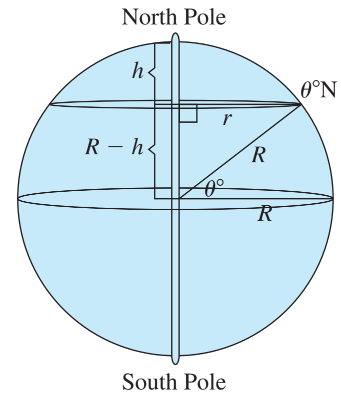

## EXTENDING THE CONCEPT

**107.** The Duvall family is out on a family bicycle ride around Creve Couer Lake. The adult bikes have a pedal sprocket with a 4-in. radius, wheel sprocket with 2-in. radius, and tires with a 13-in. radius. The kid's bikes have pedal sprockets with a 2.5-in. radius, wheel sprockets with 1.5-in. radius, and tires with a 9-in. radius. (a) If adults and kids both pedal at 50 rpm, how far ahead (in yards) are the adults after 2 min? (b) If adults pedal at 50 rpm, how fast do the kids have to pedal to keep up?

**108.** Suppose two cities *not on the equator* shared the same latitude but different longitude. To find the distance between them we can no longer use a circumference of $C \approx 24{,}881$ mi or a radius of $R = 3960$ mi, since this only applies for east/west measurements along the equator. For the circumference $c$ and the related "radius" $r$ at other latitudes, consider the diagram shown. (a) Use the diagram to find a formula for $r$ in terms of $R$ and $h$;

(b) given $h \approx 1054$ mi at 47.2° north latitude, use the formula to help find the distance between Zurich, Switzerland, and Budapest, Hungary (see Figure 6.16), if both are at 47.2° north latitude, with Zurich at 8.3° east longitude, and Budapest at 19° east longitude; (c) given $h \approx 1446$ mi at 39.4° north latitude, use the formula to find the distance between Denver, Colorado, and Indianapolis, Indiana, if both at 39.4° N latitude, with Denver at 105° west longitude and Indianapolis at 86° west longitude.

Note: Later in this chapter we'll note how basic trigonometry can be used to find $h$.



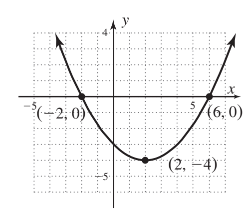

## MAINTAINING YOUR SKILLS

**109.** (4.3) Use the rational roots theorem to help find the domain of $y = \sqrt{x^3 - 7x - 6}$.

**110.** (3.3) Describe how the graph of $g(x) = -2\sqrt{x + 3} - 1$ can be obtained from transformations of $y = \sqrt{x}$.

**111.** (4.5) Sketch the graph of $h(x) = \dfrac{x^2 - 3}{x^2 - 1}$.

**112.** (3.4) Find the equation of the function whose graph is shown.




**113.** (5.4) Find the interest rate required for \$1000 to grow to \$1500 if the money is compounded monthly and remains on deposit for 5 yr.

**114.** (2.3) Given a line segment with endpoints $(-2, 3)$ and $(6, -1)$, find the equation of the line that bisects and is perpendicular to this segment.

# 6.2 Unit Circles and the Trigonometry of Real Numbers

### LEARNING OBJECTIVES

***In Section 6.2 you will learn how to:***

**A.** **Locate points on a unit circle and use symmetry to locate other points**

**B.** **Use standard triangles to find points on a unit circle and locate other points using symmetry**

**C.** **Define the six trig functions in terms of a point on the unit circle**

**D.** **Define the six trig functions in terms of a real number *t***

**E.** **Find the real number *t* corresponding to given values of sin *t*, cos *t*, and tan *t***

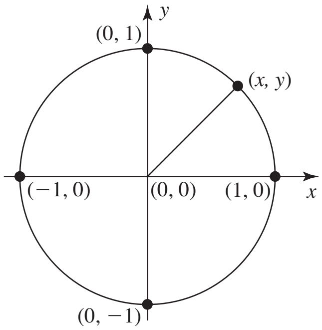
### INTRODUCTION

In this section we introduce the **trigonometry of real numbers,** a view of trig that can exist free of the traditional right triangle view. In fact, the ultimate value of the trig functions lies not in its classical study, but in the input/output nature of the trig functions and the cyclic values they generate. These functions have powerful implications in our continuing study, and important applications in this course and those that follow.

### POINT OF INTEREST

As functions of a real number, trigonometry has applications in some surprising areas: (a) blood pressure, (b) predator/prey models, (c) electric generators, (d) tidal motion, (e) meteorology, (f) planetary studies, (g) engine design, and (h) intensity of light/sound. Actually the list is endless, but perhaps these are sufficient to engage and intrigue, drawing us into our current study.

Figure 6.20




## A. The Unit Circle

A circle is defined as the set of all points in a plane that are a *fixed distance* called the **radius** from a *fixed point* called the **center.** Since the definition involves distance, we can construct the general equation of a circle using the distance formula. Assume the center has coordinates $(h, k)$ and let $(x, y)$ represent any point on the graph. Since the distance between these points is the radius $r$, the distance formula yields $\sqrt{(x-h)^2 + (y-k)^2} = r$. Squaring both sides gives $(x-h)^2 + (y-k)^2 = r^2$. For central circles both $h$ and $k$ are zero, and the result is the equation for a **central circle** of radius $r$: $x^2 + y^2 = r^2$. The **unit circle** is defined as a central circle with radius 1 unit. As such, the figure can easily be graphed by drawing a circle through the four **quadrantal points** $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$ as in Figure 6.20. To find other points on the circle, we simply select any value of $x$, where $|x| < 1$, then substitute and solve for $y$; or any value of $y$, where $|y| < 1$, then solve for $x$.

**EXAMPLE 1** Find a point on the unit circle given $y = \frac{1}{2}$ with $(x, y)$ in QII.

**Solution:** Using the equation of a unit circle, we have:

$$x^2 + y^2 = 1 \quad \text{unit circle equation}$$

$$x^2 + \left(\frac{1}{2}\right)^2 = 1 \quad \text{substitute } \frac{1}{2} \text{ for } y$$

$$x^2 + \frac{1}{4} = 1 \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$x^2 = \frac{3}{4} \quad \text{subtract } \frac{1}{4}$$

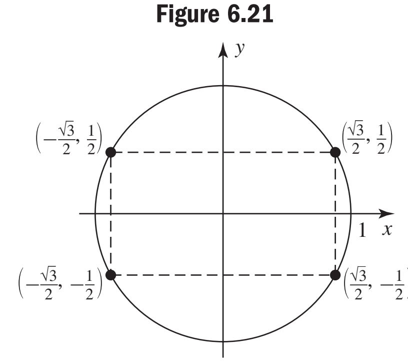
$$x = \pm\frac{\sqrt{3}}{2} \quad \text{result}$$

With $(x, y)$ in QII, we choose $x = -\frac{\sqrt{3}}{2}$. The point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

NOW TRY EXERCISES 7 THROUGH 18

Additional points on the unit circle can be found using symmetry. The simplest examples come from the quadrantal points, where $(1, 0)$ and $(-1, 0)$ are on opposite sides of the $y$-axis, and $(0, 1)$ and $(0, -1)$ are on opposite sides of the $x$-axis. In general, if $a$ and $b$ are positive real numbers and $(a, b)$ is on the unit circle, then $(-a, b)$, $(a, -b)$, and $(-a, -b)$ are also on the circle *because a circle is symmetric to both axes and the origin!* For the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ from Example 1, three other points can quickly be located, since the coordinates will differ only in sign. They are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ in QIII, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ in QIV and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ in QI. See Figure 6.21.

Figure 6.21



**CIRCLES AND SYMMETRY**

Given a unit circle and positive real numbers $a$ and $b$, if point $(a, b)$ is on the circle, then:

$(-a, b)$ must also be on the circle due to *y-axis symmetry*,

$(a, -b)$ must also be on the circle due to *x-axis symmetry*,

$(-a, -b)$ must also be on the circle due to *origin symmetry*.

**EXAMPLE 2** ▶ Name the quadrant containing $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ and verify it's on a unit circle. Then use symmetry to find three other points on the circle.

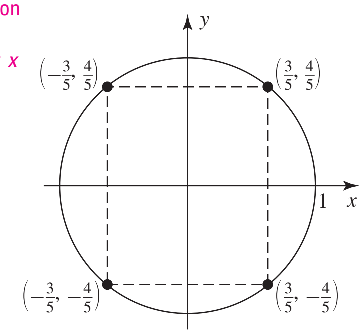
**Solution:** ▶ Since both coordinates are negative, $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ is in QIII. Substituting into the equation for a unit circle yields:

$$x^2 + y^2 = 1 \quad \text{unit circle equation}$$

$$\left(\frac{-3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 = 1 \quad \text{substitute } \tfrac{-3}{5} \text{ for } x \text{ and } \tfrac{-4}{5} \text{ for } y$$

$$\frac{9}{25} + \frac{16}{25} = 1 \quad \text{simplify}$$

$$\frac{25}{25} = 1 \quad \text{result checks}$$




Since $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ is on the unit circle, $\left(\frac{3}{5}, \frac{-4}{5}\right)$ $\left(\frac{-3}{5}, \frac{4}{5}\right)$, and $\left(\frac{3}{5}, \frac{4}{5}\right)$ are also on the circle due to symmetry (see figure).

NOW TRY EXERCISES 19 THROUGH 26 ▶

## B. Standard Triangles and the Unit Circle

The standard triangles can also be used to find points on a unit circle. As usually written, the triangles state a proportional relationship between its sides after assigning a value of 1 to the shortest side. However, precisely due to this proportional relationship, *we can divide all sides by the length of the hypotenuse,* giving *it* a length of 1 unit (see Figures 6.22 and 6.23). We then place the triangle within the unit circle, and reflect it from quadrant to quadrant to find additional points. We use *the sides* of the triangle to determine the absolute value of each coordinate, and *the quadrant* to

**Figure 6.22**

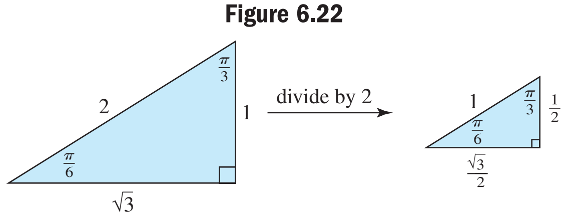
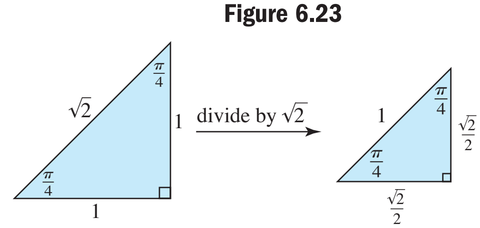



**Figure 6.23**



give each coordinate the appropriate sign. Note the standard triangles are now expressed in radians.

**EXAMPLE 3** Use the $\frac{\pi}{4} : \frac{\pi}{4} : \frac{\pi}{2}$ triangle from Figure 6.23 to find four points on the unit circle.

**Solution:** Begin by superimposing the triangle in QI, noting it gives the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ shown in Figure 6.24. By reflecting the triangle into QII, we find the additional point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on this circle. Realizing we can simply apply the circle's remaining symmetries, we obtain the two additional points $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ shown in Figure 6.25

**Figure 6.24**

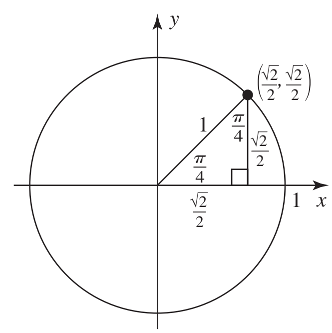
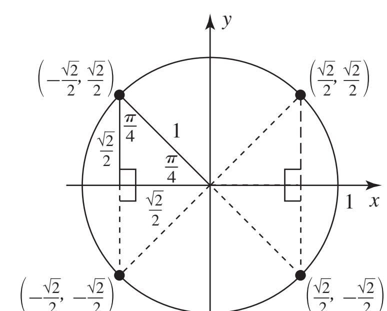


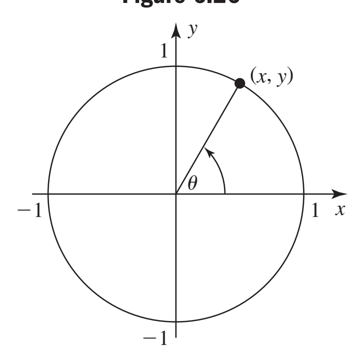

**Figure 6.25**




NOW TRY EXERCISES 27 AND 28

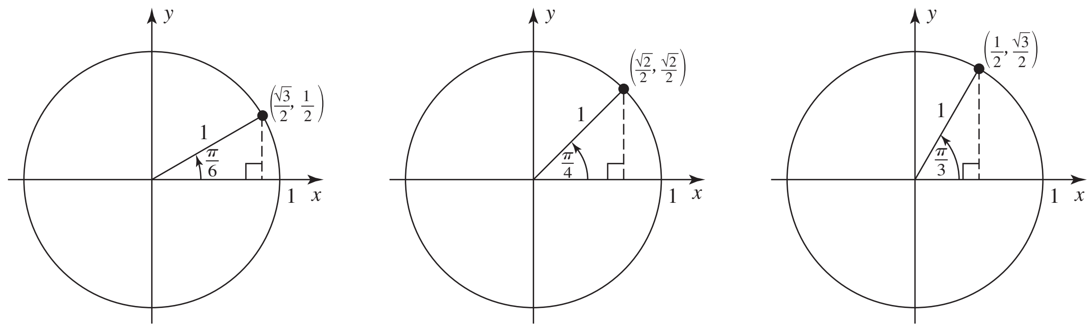
Applying the same idea to a $\frac{\pi}{6} : \frac{\pi}{3} : \frac{\pi}{2}$ triangle would give the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, *the same points we found in Example 1.*

**Figure 6.26**




When a central angle $\theta$ is viewed as a rotation, each rotation can be associated with a unique point $(x, y)$ on the terminal side, where it intersects the unit circle (see Figure 6.26). For the quadrantal angles $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, and $2\pi$, we associate the points $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(0, 0)$, respectively. When this rotation results in a standard angle $\theta$, the association can be found using a standard triangle in a manner similar to Example 3. Figure 6.27 shows we associate the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ with $\theta = \frac{\pi}{6}$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ with $\theta = \frac{\pi}{4}$, and by reorienting the $\frac{\pi}{6} : \frac{\pi}{3} : \frac{\pi}{2}$ triangle, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is associated with a rotation of $\theta = \frac{\pi}{3}$.

Figure 6.27




For standard rotations from $\theta = 0$ to $\theta = \frac{\pi}{2}$ we have the following:

| Rotation $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| Associated Point $(x, y)$ | (0, 0) | $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ | (0, 1) |

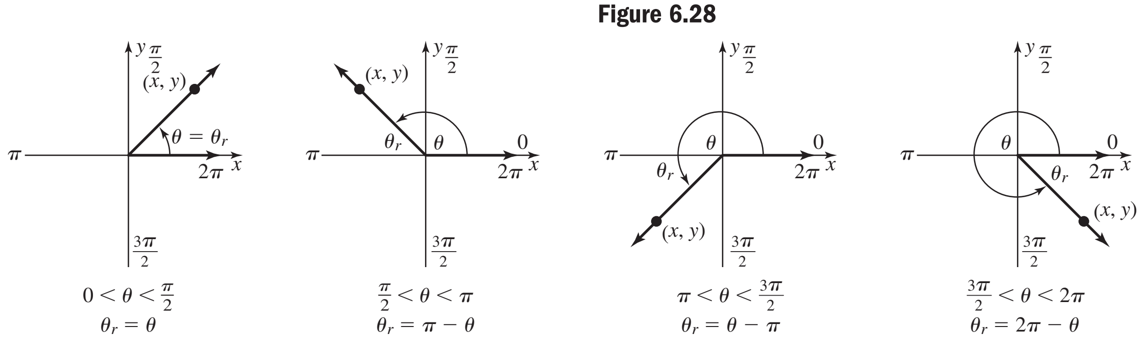
Each of these points give rise to three others using the symmetry of the circle. By defining a reference angle $\theta_r$, we can associate the additional points with the related rotation $\theta > \frac{\pi}{2}$.

**REFERENCE ANGLES**

For any angle $\theta$ in standard position, the acute angle $\theta_r$ formed by the terminal side and the nearest $x$-axis is called the *reference angle* for $\theta$.

Several examples of the reference angle concept are shown in Figure 6.28.

Figure 6.28



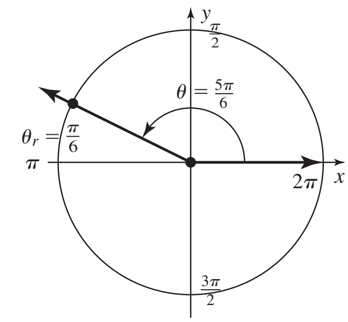

Due to the symmetries of the circle, reference angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ serve to fix the absolute value of the coordinates for $x$ and $y$, so all that remains is to *use the appropriate sign for each coordinate* ($r$ is always positive). This depends solely on the quadrant of the terminal side.

**EXAMPLE 4** Determine the reference angle for each rotation given, then find the associated point $(x, y)$ on the unit circle.

**a.** $\theta = \frac{5\pi}{6}$ **b.** $\theta = \frac{4\pi}{3}$ **c.** $\theta = \frac{7\pi}{4}$

**Solution:** **a.** A rotation of $\frac{5\pi}{6}$ terminates in QII: $\theta_r = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$. The associated point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ since $x < 0$ in QII. See Figure 6.29.

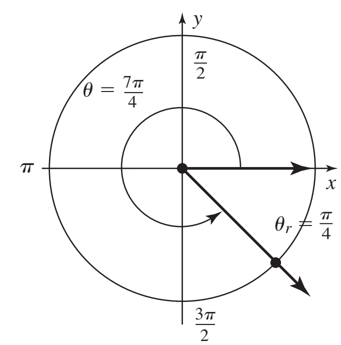
**Figure 6.29**




**b.** A rotation of $\frac{4\pi}{3}$ terminates in QIII: $\theta_r = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$. The associated point is $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ since $x < 0$ and $y < 0$ in QIII.

**c.** A rotation of $\frac{7\pi}{4}$ terminates in QIV: $\theta_r = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$. The associated point is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $y < 0$ in QIV. See Figure 6.30.

**Figure 6.30**




NOW TRY EXERCISES 29 THROUGH 36

## C. Trigonometric Functions and Points on the Unit Circle

While there is more than one approach to a study of trigonometry, we will initially define the six trig functions in terms of a point $(x, y)$ on the unit circle associated with a given rotation $\theta$. For this reason they are sometimes called the **circular functions.** These trig functions have word names rather than letter names like $f$, $g$, or $h$, and are often written in abbreviated form. They are the cosine function $\cos\theta$, sine function $\sin\theta$, tangent function $\tan\theta$, and the secant, cosecant, and cotangent functions, written $\csc\theta$, $\sec\theta$, and $\cot\theta$, respectively. We define them as follows:

**THE CIRCULAR FUNCTIONS**

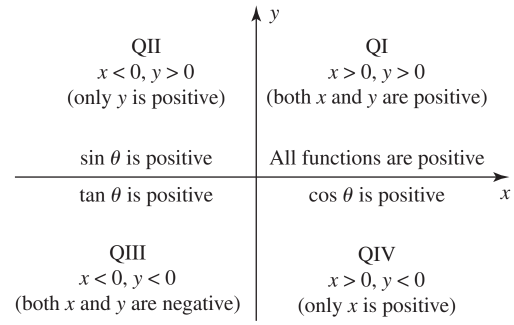
For any rotation $\theta$ and point $P(x, y)$ on the unit circle associated with $\theta$,

$\cos\theta = x$ $\sin\theta = y$ $\tan\theta = \frac{y}{x}; x \neq 0$

$\sec\theta = \frac{1}{x}; x \neq 0$ $\csc\theta = \frac{1}{y}; y \neq 0$ $\cot\theta = \frac{x}{y}; y \neq 0$

As these definitions are applied it will be helpful to note that sec $\theta$ is the reciprocal of cos $\theta$, csc $\theta$ is the reciprocal of sin $\theta$, and cot $\theta$ is the reciprocal of tan $\theta$. This tells us that once sin $\theta$, cos $\theta$ and tan $\theta$ are known, the values of csc $\theta$, sec $\theta$, and cot $\theta$ follow automatically since a number and its reciprocal always have the same sign. See Figure 6.31.

**Figure 6.31**




**EXAMPLE 5** Evaluate the six trig functions for $\theta = \frac{5\pi}{4}$.

**Solution:** A rotation of $\frac{5\pi}{4}$ terminates in QIII, so $\theta_r = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The associated point is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $x < 0$ and $y < 0$ in QIII. This yields

$$\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \tan\left(\frac{5\pi}{4}\right) = 1$$

Noting the reciprocal of $-\frac{\sqrt{2}}{2} = -\sqrt{2}$ after rationalizing, we have

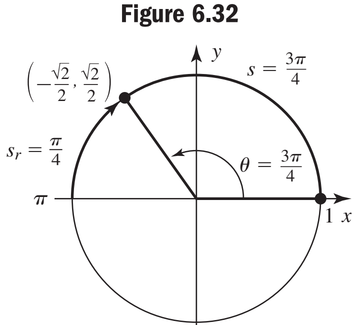
$$\sec\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \csc\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \cot\left(\frac{5\pi}{4}\right) = 1$$

**NOW TRY EXERCISES 37 THROUGH 40**

## D. The Trigonometry of Real Numbers

Defining the trig functions in terms of a point on the unit circle is precisely what we needed to work with them as functions of a real number. This is because when $r = 1$ and $\theta$ is in radians, *the length of the subtended arc is numerically the same as the measure of the angle:* $s = (1)\theta \rightarrow s = \theta$*!* This means we can view any function of $\theta$ as a like function of arc length $s$, where $s \in \mathbb{R}$. As a compromise the variable $t$ is commonly used, with $t$ representing *either* the amount of rotation *or* the length of the arc. As such we will assume $t$ is a dimensionless quantity, although there are other reasons for this assumption. In Figure 6.32, a rotation of $\theta = \frac{3\pi}{4}$ is subtended by an arc length of $s = \frac{3\pi}{4}$ (about 2.356 units). The reference angle for $\theta$ is $\frac{\pi}{4}$, which we will now refer to as a **reference arc.** As you work through the remaining examples and the exercises that follow, it will often help to draw a quick sketch similar to that in Figure 6.32 to determine the quadrant of the terminal side, the reference arc, and the signs of each function.

**Figure 6.32**



**WORTHY OF NOTE**

Once again, to evaluate sec *t*, csc *t*, and cot *t*, we simply reciprocate the values found for cos *t*, sin *t*, and tan *t*, respectively. For Example 6(a), these reciprocals resulted in a radical denominator, and the expression was rationalized to give the values shown.

**EXAMPLE 6** Evaluate the six trig functions for the given value of $t$.

**a.** $t = \frac{11\pi}{6}$ **b.** $t = \frac{3\pi}{2}$ **c.** $t = -\frac{10\pi}{3}$

**Solution:** **a.** For $t = \frac{11\pi}{6}$, the arc terminates in QIV, where $x > 0$ and $y < 0$. The reference arc is $\frac{\pi}{6}$ and from our previous work we know the point $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. This gives:

$$\cos\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{5\pi}{6}\right) = -\frac{1}{2} \qquad \tan\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

$$\sec\left(\frac{5\pi}{6}\right) = \frac{2\sqrt{3}}{3} \qquad \csc\left(\frac{5\pi}{6}\right) = 2 \qquad \cot\left(\frac{5\pi}{6}\right) = -\sqrt{3}$$

**b.** $t = \frac{3\pi}{2}$ is a quadrantal angle and the associated point is $(0, -1)$. This yields:

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1 \qquad \tan\left(\frac{3\pi}{2}\right) = \text{undefined}$$

$$\sec\left(\frac{3\pi}{2}\right) = \text{undefined} \qquad \csc\left(\frac{3\pi}{2}\right) = -1 \qquad \cot\left(\frac{3\pi}{2}\right) = 0$$

**c.** For $t = -\frac{10\pi}{3}$, the arc terminates in QII after completing one clockwise revolution of $-\frac{6\pi}{3}$ and continuing for another $-\frac{4\pi}{3}$. The reference arc is $\frac{\pi}{3}$ with associated point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. With $x < 0$ and $y > 0$ in QII, we have:

$$\cos\left(-\frac{10\pi}{3}\right) = -\frac{1}{2} \qquad \sin\left(-\frac{10\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \tan\left(-\frac{10\pi}{3}\right) = -\sqrt{3}$$

$$\sec\left(-\frac{10\pi}{3}\right) = -2 \qquad \csc\left(-\frac{10\pi}{3}\right) = \frac{2\sqrt{3}}{3} \qquad \cot\left(-\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{3}$$

NOW TRY EXERCISES 41 THROUGH 44

As Example 6(b) indicates, as functions of a real number the concept of domain comes into play. From their definition it is apparent there are no restrictions on the domain of cosine and sine, but the domains of the other functions must be restricted to exclude division by zero. For functions with $x$ in the denominator, we cast out the odd multiples of $\frac{\pi}{2}$, since the $x$-coordinate of the related quadrantal points are zero: $\frac{\pi}{2} \rightarrow (0, 1)$, $\frac{3\pi}{2} \rightarrow (0, -1)$, and so on. The excluded values can be stated as $t \neq \frac{\pi}{2} + \pi k$ for all integers $k$. For functions with $y$ in the denominator, we cast out all multiples of $\pi$ ($t \neq \pi k$ for all integers $k$) since the $y$-coordinate of these points are zero: $0 \rightarrow (1, 0)$, $\pi \rightarrow (-1, 0)$, $2\pi \rightarrow (1, 0)$, and so on.

**THE DOMAIN OF THE TRIG FUNCTIONS AS FUNCTIONS OF A REAL NUMBER**

For real numbers $t$ and integers $k$, the domains of the trig functions are:

| | | |
|---|---|---|
| $\cos t = x$ | $\sin t = y$ | $\tan t = \frac{y}{x};\ x \neq 0$ |
| $t \in \mathbb{R}$ | $t \in \mathbb{R}$ | $t \neq \frac{\pi}{2} + \pi k$ |
| $\sec t = \frac{1}{x};\ x \neq 0$ | $\csc t = \frac{1}{y};\ y \neq 0$ | $\cot t = \frac{x}{y};\ y \neq 0$ |
| $t \neq \frac{\pi}{2} + \pi k$ | $t \neq \pi k$ | $t \neq \pi k$ |

For any point $(x, y)$ on the unit circle, the definition of the trig functions can still be applied even if $t$ is unknown.

**EXAMPLE 7** Given $\left(\frac{-7}{25}, \frac{24}{25}\right)$ is a point on the unit circle corresponding to a real number $t$, find the value of all six trig functions of $t$.

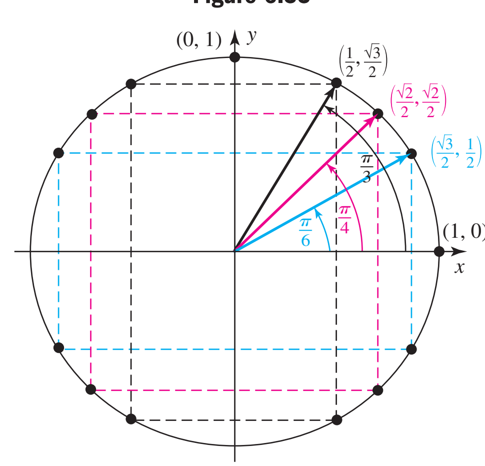
**Solution:** Using the definitions above we have $\cos t = \frac{-7}{25}$, $\sin t = \frac{24}{25}$, and $\tan t = \frac{24}{-7}$. The values of the reciprocal functions are then $\sec t = \frac{25}{-7}$, $\csc t = \frac{25}{24}$, and $\cot t = \frac{-7}{24}$.

NOW TRY EXERCISES 45 THROUGH 70

## E. Finding a Real Number *t* Whose Function Value is Known

In Example 7, we were able to apply the definition of the trig functions even though $t$ was unknown. In many cases, however, we need to find the related value of $t$. For instance, what is the value of $t$ given $\cos t = -\frac{\sqrt{3}}{2}$ with $t$ in QII? Exercises of this type fall into two broad categories: (1) you recognize the given number as one of the standard values: $\pm\left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, 1\right\}$; or (2) you don't. If you recognize a standard value you can often name the real number $t$ after a careful consideration of the related quadrant and required sign. The diagram in Figure 6.33 reviews these standard values for $0 \leq t \leq \frac{\pi}{2}$ but remember—all other standard values can be found using reference arcs and the symmetry of the circle.

**Figure 6.33**



**EXAMPLE 8** Find the value of $t$ that corresponds to the following functions:

**a.** $\cos t = -\frac{\sqrt{2}}{2}$; $t$ in QII **b.** $\tan t = \sqrt{3}$; $t$ in QIII

**c.** $\sin t = -\frac{1}{2}$; $t$ in QIV

**Solution:**

**a.** The cosine function is negative in QII and QIII. We recognize $-\frac{\sqrt{2}}{2}$ as a standard value for sine and cosine, related to certain multiples of $t = \frac{\pi}{4}$. In QII, we have $t = \frac{3\pi}{4}$.

**b.** The tangent function is positive in QI and QIII. We recognize $\sqrt{3}$ as a standard value for tangent and cotangent, related to certain multiples of $t = \frac{\pi}{6}$. For tangent in QIII, we have $t = \frac{8\pi}{6} = \frac{4\pi}{3}$.

**c.** The sine function is negative in QIII and QIV. We recognize $\frac{1}{2}$ as a standard value for sine and cosine, related to certain multiples of $t = \frac{\pi}{6}$. For sine in QIV, we have $t = \frac{11\pi}{6}$.

**NOW TRY EXERCISES 71 THROUGH 86**

If you don't recognize the given value, most calculators are programmed to give a reference value for $t$ that corresponds to the given function value, using the $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ keys. However, the quadrant, signs, and reference arcs must still be considered to correctly name $t$.

**EXAMPLE 9** Find the value of $t$ that corresponds to the following functions:

**a.** $\sin t = \frac{\sqrt{13}}{7}$; $t$ in QI **b.** $\tan t = \frac{4}{3}$; $t$ in QIII

**c.** $\cos t = -0.2621$; $t$ in QIII

**Solution:**

**a.** The sine function is positive in QI and QII. With your calculator in radian mode, the expression $\sin^{-1}\left(\frac{\sqrt{13}}{7}\right)$ [ 2nd SIN ($\sqrt{13}$)/7 ENTER ] gives $t \approx 0.5411$ to four decimal places, which is the desired QI value since $0 < 0.5411 < \frac{\pi}{2} \approx 1.57$.

**b.** The tangent function is positive in QI and QIII. Since $\tan t = \frac{4}{3}$ is likewise not a standard value, we use 2nd TAN 4/3 ENTER which gives $t \approx 0.9273$. We note this is the value from QI, since $0 < 0.9273 < \frac{\pi}{2} \approx 1.57$. The QIII value is then $t = \pi + 0.9273 \approx 4.0689$.

**c.** The cosine function is negative in QII and QIII. As $-0.2621$ is again a nonstandard value, we use 2nd COS $-0.2621$ ENTER, which gives $t \approx 1.8360$. This is the value from QII, since $\frac{\pi}{2} \approx 1.57 < 1.8360 < \pi \approx 3.14$, with a reference arc of $\pi - 1.8360 \approx 1.3056$. The value in QIII is $\pi + 1.3056 = 4.4472$.

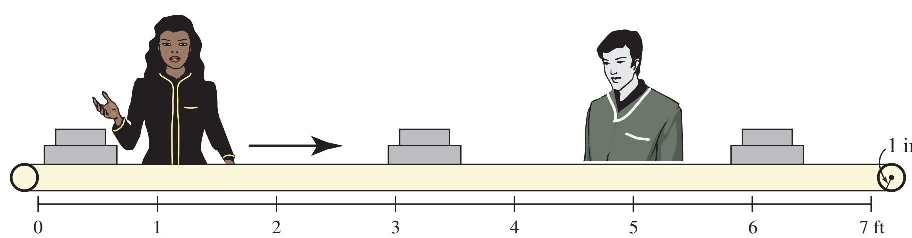
NOW TRY EXERCISES 87 THROUGH 102

Using radian measure and the unit circle is much more than a simple convenience to trigonometry. Whether the unit is 1 cm, 1 m, 1 km, or even 1 light-year, using 1 unit designations serves to simply a great many practical applications.

**EXAMPLE 10** The conveyor belt on an assembly line has rollers with 1-in. radius on each end.

**a.** Through what angle (in radians) must the rollers turn to move a product from Miriam to Melvin if they are sitting 4 ft apart?

**b.** If Melvin has fallen asleep on the job, and the rollers turn another 25 rad after passing his position, what happens to the product?




**Solution:** **a.** Since the rollers have a radius of 1 in. (a unit circle), the product will move exactly as many inches as the roller turns in radians. Hence the product moves from Miriam to Melvin after turning $4(12) = 48$ rad.

**b.** The conveyor belt ends 2 ft after passing Melvin, so if he is asleep and the rollers turn another 25 rad, the product will fall to the floor.

NOW TRY EXERCISES 105 THROUGH 112

TECHNOLOGY HIGHLIGHT

## Graphing the Unit Circle

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

When using a graphing calculator to study the unit circle, it's important to keep two things in mind. First, most graphing calculators are only capable of graphing *functions*, which means we must modify the equation of the circle (and relations like ellipses, hyperbolas, horizontal parabolas, and so on) before it can be graphed. Second, most standard viewing windows have the $x$- and $y$-values preset at $[-10, 10]$ even though the calculator screen is 94 pixels wide and 64 pixels high. This tends to compress the $y$-values and give a skewed image of the graph. Consider the equation $x^2 + y^2 = 1$. From our work in this section, we know this is the equation of a circle centered at (0, 0) with radius $r = 1$. For the calculator to graph this relation, we must define it in two pieces, each of which is a function, by solving for $y$:

$$x^2 + y^2 = 1 \quad \text{original equation}$$
$$y^2 = 1 - x^2 \quad \text{isolate } y^2$$
$$y = \pm\sqrt{1 - x^2} \quad \text{solve for } y$$

Note that we can separate this result into two parts, each of which is a function. The graph of $Y_1 = \sqrt{1 - x^2}$ gives the "upper half" of the circle, while $Y_2 = -\sqrt{1 - x^2}$ gives the "lower half". We can enter these on the Y= screen as shown, using the expression $-Y_1$ instead of reentering the entire expression. The function variables $Y_1$, $Y_2$, $Y_3$, and so on, can be accessed using VARS ▶ **(Y-VARS)** ENTER **(1:Function)**. Graphing $Y_1$ and $Y_2$ on the standard screen, the result appears very small and more elliptical than circular (Figure 6.34). One way to fix this (there are many others), is to use the ZOOM **4:ZDecimal** option, which places the tic marks equally spaced on both axes, instead of trying to force both to display points from −10 to 10. Using this option gives the screen shown in Figure 6.35. Although it is a much improved graph, there is still a great deal of "wasted space" on the screen. A better graph can be obtained using the ZOOM **2:Zoom In** option or by manually resetting the window size. Using the TRACE feature allows us to view points on the unit circle, but recall that this image is actually the union of two graphs and we may need to jump between the upper and lower halves using the up ▲ or down ▼ arrows.

**Figure 6.34**

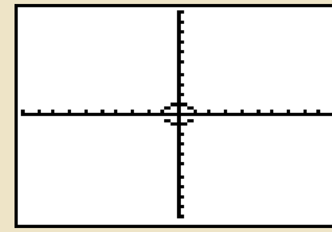
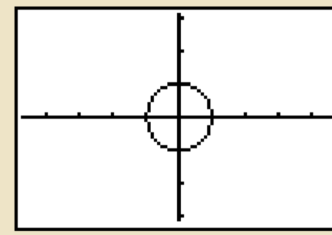


**Figure 6.35**



Exercise 1: Use the TRACE feature to verify the point (0.6, 0.8) is on the unit circle, as well as the other three related points given by symmetry (as shown in Example 2).

Exercise 2: Use the 2nd TRACE **(CALC)** feature to evaluate the function at $\frac{\sqrt{2}}{2}$. What do you notice about the output? For cos $t$ or sin $t$, what value of $t$ can we associate with this point?

Exercise 3: What other standard values can you identify as you TRACE around the circle?

# 6.2 EXERCISES

## CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** A central circle is symmetric to the ______ axis, the ______ axis and to the ______.

**2.** Since $\left(\frac{5}{13}, -\frac{12}{13}\right)$ is on the unit circle, the point __________ in QII is also on the circle.

**3.** On a unit circle, $\cos t =$ ______, $\sin t =$ ______ and $\tan t =$ ______; while $\frac{1}{x} =$ ______, $\frac{1}{y} =$ ______, and $\frac{x}{y} =$ ______.

**4.** On a unit circle with $\theta$ in radians, the length of a(n) ________ ______ is numerically the same as the measure of the ______, since for $s = r\theta$, $s = \theta$ when $r = 1$.

**5.** Discuss/explain how knowing only one point on the unit circle, actually gives the location of four points. Why is this helpful to a study of the circular functions?

**6.** A student is asked to find $t$ using a calculator, given $\sin t \approx 0.5592$ with $t$ in QII. The answer submitted is $t = \sin^{-1} 0.5592 \approx 34°$. Discuss/explain why this answer is not correct. What is the correct response?

## DEVELOPING YOUR SKILLS

Given the point is on a unit circle, complete the ordered pair $(x, y)$ for the quadrant indicated. For Exercises 7 to 14, answer in radical form as needed. For Exercises 15 to 18, round results to four decimal places.

**7.** $(x, -0.8)$; QIII **8.** $(-0.6, y)$; QII **9.** $\left(\frac{5}{13}, y\right)$; QIV **10.** $\left(x, -\frac{8}{17}\right)$; QIV

**11.** $\left(\frac{\sqrt{11}}{6}, y\right)$; QI **12.** $\left(x, -\frac{\sqrt{13}}{7}\right)$; QIII **13.** $\left(-\frac{\sqrt{11}}{4}, y\right)$; QII **14.** $\left(x, \frac{\sqrt{6}}{5}\right)$; QI

**15.** $(x, -0.2137)$; QIII **16.** $(0.9909, y)$; QIV **17.** $(x, 0.1198)$; QII **18.** $(0.5449, y)$; QI

Verify the point given is on a unit circle, then use symmetry to find three more points on the circle. Results for Exercises 19 to 22 are exact, results for Exercises 23 to 26 are approximate.

**19.** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **20.** $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$ **21.** $\left(\frac{\sqrt{11}}{6}, -\frac{5}{6}\right)$

**22.** $\left(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right)$ **23.** $(0.3325, 0.9431)$ **24.** $(0.7707, -0.6372)$

**25.** $(0.9937, -0.1121)$ **26.** $(-0.2029, 0.9792)$

**27.** Use a $\frac{\pi}{6} : \frac{\pi}{3} : \frac{\pi}{2}$ triangle with a hypotenuse of length 1 to verify that $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle.

**28.** Use the results from Exercise 27 to find three additional points on the circle and name the quadrant of each point.

Find the reference angle associated with each rotation, then find the associated point $(x, y)$.

**29.** $\theta = \frac{5\pi}{4}$ **30.** $\theta = \frac{5\pi}{3}$ **31.** $\theta = -\frac{5\pi}{6}$ **32.** $\theta = -\frac{7\pi}{4}$

**33.** $\theta = \frac{11\pi}{4}$ **34.** $\theta = \frac{11\pi}{3}$ **35.** $\theta = \frac{25\pi}{6}$ **36.** $\theta = \frac{39\pi}{4}$

Use the symmetry of the circle and reference angles as needed to state the exact value of the trig functions for the given angle, without the use of a calculator. A diagram may help.

**37. a.** $\sin\left(\frac{\pi}{4}\right)$ **b.** $\sin\left(\frac{3\pi}{4}\right)$ **c.** $\sin\left(\frac{5\pi}{4}\right)$ **d.** $\sin\left(\frac{7\pi}{4}\right)$

**e.** $\sin\left(\frac{9\pi}{4}\right)$ **f.** $\sin\left(-\frac{\pi}{4}\right)$ **g.** $\sin\left(-\frac{5\pi}{4}\right)$ **h.** $\sin\left(-\frac{11\pi}{4}\right)$

**38. a.** $\tan\left(\frac{\pi}{3}\right)$ **b.** $\tan\left(\frac{2\pi}{3}\right)$ **c.** $\tan\left(\frac{4\pi}{3}\right)$ **d.** $\tan\left(\frac{5\pi}{3}\right)$

**e.** $\tan\left(\frac{7\pi}{3}\right)$ **f.** $\tan\left(-\frac{\pi}{3}\right)$ **g.** $\tan\left(-\frac{4\pi}{3}\right)$ **h.** $\tan\left(-\frac{10\pi}{3}\right)$

**39. a.** $\cos \pi$ **b.** $\cos 0$ **c.** $\cos\left(\frac{\pi}{2}\right)$ **d.** $\cos\left(\frac{3\pi}{2}\right)$

**40. a.** $\sin \pi$ **b.** $\sin 0$ **c.** $\sin\left(\frac{\pi}{2}\right)$ **d.** $\sin\left(\frac{3\pi}{2}\right)$

Use the symmetry of the circle and reference arcs as needed to state the exact value of the trig functions for the given real number, without the use of a calculator. A diagram may help.

**41. a.** $\cos\left(\frac{\pi}{6}\right)$ **b.** $\cos\left(\frac{5\pi}{6}\right)$ **c.** $\cos\left(\frac{7\pi}{6}\right)$ **d.** $\cos\left(\frac{11\pi}{6}\right)$

**e.** $\cos\left(\frac{13\pi}{6}\right)$ **f.** $\cos\left(-\frac{\pi}{6}\right)$ **g.** $\cos\left(-\frac{5\pi}{6}\right)$ **h.** $\cos\left(-\frac{23\pi}{6}\right)$

**42.** **a.** $\csc\left(\frac{\pi}{6}\right)$ **b.** $\csc\left(\frac{5\pi}{6}\right)$ **c.** $\csc\left(\frac{7\pi}{6}\right)$ **d.** $\csc\left(\frac{11\pi}{6}\right)$

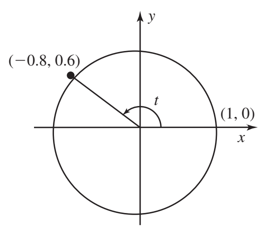
**e.** $\csc\left(\frac{13\pi}{6}\right)$ **f.** $\csc\left(-\frac{\pi}{6}\right)$ **g.** $\csc\left(-\frac{11\pi}{6}\right)$ **h.** $\csc\left(-\frac{17\pi}{6}\right)$

**43.** **a.** $\tan \pi$ **b.** $\tan 0$ **c.** $\tan\left(\frac{\pi}{2}\right)$ **d.** $\tan\left(\frac{3\pi}{2}\right)$

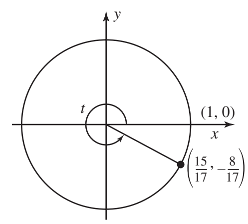
**44.** **a.** $\cot \pi$ **b.** $\cot 0$ **c.** $\cot\left(\frac{\pi}{2}\right)$ **d.** $\cot\left(\frac{3\pi}{2}\right)$

Given $(x, y)$ is a point on a unit circle corresponding to $t$, find the value of all six circular functions of $t$.

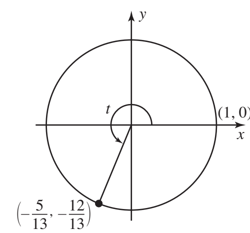
**45.**



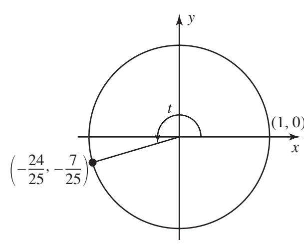

**46.**



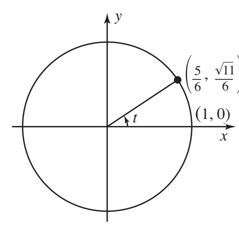

**47.**



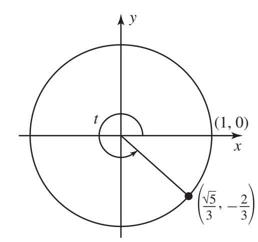

**48.**




**49.**




**50.**

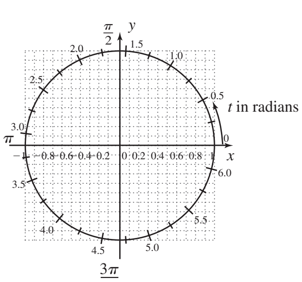



**51.** $\left(-\frac{2}{5}, \frac{\sqrt{21}}{5}\right)$ **52.** $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$ **53.** $\left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ **54.** $\left(-\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$

**55.** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **56.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **57.** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **58.** $\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{7}}{3}\right)$

On a unit circle, the real number $t$ can represent either the amount of rotation or the *length of the arc* when associating $t$ with a point $(x, y)$ on the circle. In the circle diagram that follows, the real number $t$ in radians is marked off along the circumference. For Exercises 59 through 70, name the quadrant in which $t$ terminates and use the figure to estimate function values to one decimal place (use a straightedge). Check results using a calculator.

**Exercises 59 to 70**




**59.** $\sin 0.75$ **60.** $\cos 2.75$ **61.** $\cos 5.5$ **62.** $\sin 4.0$

**63.** $\tan 0.8$ **64.** $\sec 3.75$ **65.** $\csc 2.0$ **66.** $\cot 0.5$

**67.** $\cos\left(\frac{5\pi}{8}\right)$ **68.** $\sin\left(\frac{5\pi}{8}\right)$ **69.** $\tan\left(\frac{8\pi}{5}\right)$ **70.** $\sec\left(\frac{8\pi}{5}\right)$

Without using a calculator, find the value of $t$ in $[0, 2\pi)$ that corresponds to the following functions.

**71.** $\sin t = \frac{\sqrt{3}}{2}$; $t$ in QII **72.** $\cos t = \frac{1}{2}$; $t$ in QIV **73.** $\cos t = -\frac{\sqrt{3}}{2}$; $t$ in QIII

**74.** $\sin t = -\frac{1}{2}$; $t$ in QIV **75.** $\tan t = -\sqrt{3}$; $t$ in QII **76.** $\sec t = -2$; $t$ in QIII

**77.** $\sin t = 1$; $t$ is quadrantal **78.** $\cos t = -1$; $t$ is quadrantal

Without using a calculator, find the two values of $t$ (where possible) in $[0, 2\pi)$ that make each equation true.

**79.** $\sec t = -\sqrt{2}$ **80.** $\csc t = -\frac{2}{\sqrt{3}}$ **81.** $\tan t$ undefined **82.** $\csc t$ undefined

**83.** $\cos t = -\frac{\sqrt{2}}{2}$ **84.** $\sin t = \frac{\sqrt{2}}{2}$ **85.** $\sin t = 0$ **86.** $\cos t = -1$

Use a calculator to find the value of $t$ in $[0, 2\pi)$ that satisfies the equation and condition given. Round to four decimal places.

**87.** $\cos t = \frac{\sqrt{5}}{4}$; $t$ in QIV **88.** $\sin t = \frac{\sqrt{6}}{5}$; $t$ in QII **89.** $\tan t = -\frac{7}{8}$; $t$ in QIV

**90.** $\sec t = -\frac{7}{4}$; $t$ in QIII **91.** $\sin t = -0.9872$; $t$ in QIII **92.** $\cos t = -0.5467$; $t$ in QII

**93.** $\cot t = 6.4521$; $t$ in QI **94.** $\csc t = 2.2551$; $t$ in QII

Find an additional value of $t$ (to four decimal places) in $[0, 2\pi)$ that makes the equation true.

**95.** $\sin 0.8 \approx 0.6967$ **96.** $\cos 2.12 \approx -0.5220$ **97.** $\cos 4.5 \approx -0.2108$

**98.** $\sin 5.23 \approx -0.8690$ **99.** $\tan 0.4 \approx 0.4228$ **100.** $\sec 5.7 \approx 1.1980$

**101.** Given $\left(\frac{3}{4}, -\frac{4}{5}\right)$ is a point on the unit circle that corresponds to $t$. Find the coordinates of the point corresponding to (a) $-t$ and (b) $t + \pi$.

**102.** Given $\left(-\frac{7}{25}, \frac{24}{25}\right)$ is a point on the unit circle that corresponds to $t$. Find the coordinates of the point corresponding to (a) $-t + \pi$ and (b) $t - \pi$.

## WORKING WITH FORMULAS

**103. From Pythagorean triples to points on the unit circle: $(x, y, r) \rightarrow \left(\frac{x}{r}, \frac{y}{r}, 1\right)$**

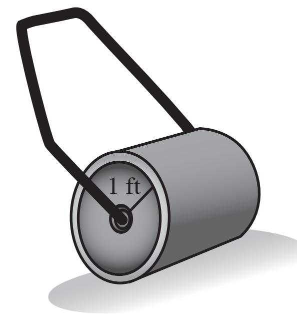
While not strictly a "formula," there is a simple algorithm for rewriting any Pythagorean triple as a triple with hypotenuse 1. This enables us to identify certain points on a unit circle, and to evaluate the six trig functions of the acute angles. Here are two common and two not-so-common Pythagorean triples. Rewrite each as a triple with hypotenuse 1, verify $\left(\frac{x}{r}, \frac{y}{r}\right)$ is a point on the unit circle, and evaluate the six trig functions using this point.

**a.** (5, 12, 13) **b.** (7, 24, 25) **c.** (12, 35, 37) **d.** (9, 40, 41)

**104. The sine and cosine of $(2k + 1)\frac{\pi}{4}$**

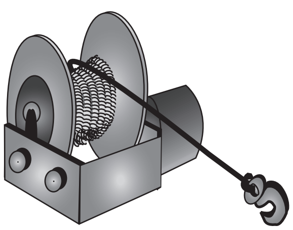
Prior to Example 8(a), we mentioned $\pm\frac{\sqrt{2}}{2}$ were standard values for sine and cosine, "related to certain multiples of $\frac{\pi}{4}$." Actually, we meant "odd multiples of $\frac{\pi}{4}$." The odd multiples of $\frac{\pi}{4}$ are given by the "formula" shown, where $k$ is any integer. (a) What multiples of $\frac{\pi}{4}$ are generated by $k = -3, -2, -1, 0, 1, 2, 3$? (b) Find similar formulas for Example 8(b), where $\sqrt{3}$ is a standard value for tangent and cotangent, "related to certain multiples of $\frac{\pi}{6}$."

## APPLICATIONS




**105. Laying new sod:** When new sod is laid, a heavy roller is used to press the sod down to insure good contact with the ground beneath. The radius of the roller is 1 ft. (a) Through what angle (in radians) has the roller turned after being pulled across 5 ft of yard? (b) What angle must the roller turn through to press a length of 30 ft?

Exercise 106



**106. Cable winch:** A large winch with a radius of 1 ft winds in 3 ft of cable. (a) Through what angle (in radians) has it turned? (b) What angle must it turn through in order to winch in 12.5 ft of cable?

**107. Wiring an apartment:** In the wiring of an apartment complex, electrical wire is being pulled from a spool with radius 1 decimeter (1 dm = 10 cm). (a) What length (in decimeters) is removed as the spool turns through 5 rad? (b) How many decimeters are removed in one complete turn ($t = 2\pi$) of the spool?

**108. Barrel races:** In the barrel races popular at some family reunions, contestants stand on a hard rubber barrel with a radius of 1 cubit (1 cubit = 18 in.), and try to "walk the barrel" from the start line to the finish line without falling. (a) What distance (in cubits) is traveled as the barrel is walked through an angle of 4.5 rad? (b) If the race is 25 cubits long, through what angle will the winning barrel walker, walk the barrel?

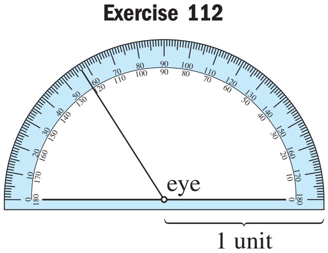
**Interplanetary measurement:** In around 1905, astronomers decided to begin using what are called astronomical units or AU to study the distances between the celestial bodies of our solar system. It represents the average distance between the Earth and the Sun, which is about 93 million miles. Pluto is roughly 39.24 AU from the Sun.

**109.** If the Earth travels through an angle of 2.5 rad about the Sun, (a) what distance in astronomical units (AU) has it traveled? (b) How many AU does it take for one complete orbit around the Sun?

**110.** Since Jupiter is actually the middle (fifth of nine) planet from the sun, suppose astronomers had decided to use *its* average distance from the Sun as 1 AU. In this case 1 AU would be 480 million miles. If Jupiter travels through an angle of 4 rad about the Sun, (a) what distance in the "new" astronomical units (AU) has it traveled? (b) how many of the new AU does it take to complete one-half an orbit about the Sun? (c) what distance in the new AU is Pluto from the Sun?

**111. Compact disc circumference:** A standard compact disk has a radius of 6 cm. Call this length "1 unit." Draw a long, straight line on a blank sheet of paper, then carefully roll the compact disc along this line without slippage, through one full revolution ($2\pi$ rad) and mark the spot. Take an accurate measurement of the resulting line segment. Is the result equal to $2\pi$ "units" ($2\pi \times 6$ cm)?

Exercise 112




**112. Verifying $s = r\theta$:** On a protractor, carefully measure the distance from the middle of the protractor's eye to the edge of the protractor along the 0° mark, to the nearest half-millimeter. Call this length "1 unit." Then use the ruler to draw a straight line on a blank sheet of paper, and with the protractor on edge, start the zero degree mark at one end of the line, carefully roll the protractor until it reaches 1 radian (57.3°), then mark this spot. Now measure the length of the line segment created. Is it very close to 1 "unit" long?

## WRITING, RESEARCH, AND DECISION MAKING

**113.** In this section, we discussed the *domain* of the circular functions, but said very little about their *range*. Review the concepts presented here and determine the range of $y = \cos t$ and $y = \sin t$. In other words, what are the largest and smallest output values we can expect? Since $\tan t = \dfrac{\sin t}{\cos t}$, what can you say about the range of the tangent function?

**114.** Another way to view the trig functions as functions of any real number is to compare the arc length from the circular functions, to the standard number line for the algebraic functions (actually there is little difference). In this regard, some texts introduce what is called the **wrapping function,** in which a number line is actually *wrapped* around the unit circle and a correspondence established between points on the number line, points on the unit circle, and the arc length $s$. Using the Internet or the resources of a local library, do some research on the Wrapping Function and how it is used to introduce the trig functions of a real number. Prepare a short summary on what you find.

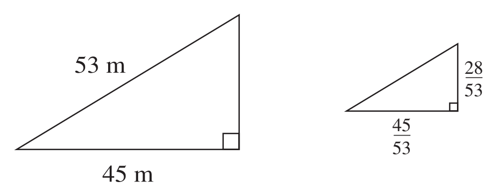
## EXTENDING THE CONCEPT

Use the radian grid with Exercises 59–70 to answer Exercises 115 and 116.

**115.** Given $\cos(2t) = -0.6$ with the terminal side of the arc in QII, (a) what is the value of $2t$? (b) What quadrant is $t$ in? (c) What is the value of $\cos t$? (d) Does $\cos(2t) = 2\cos t$?

**116.** Given $\sin(2t) = -0.8$ with the terminal side of the arc in QIII, (a) what is the value of $2t$? (b) What quadrant is $t$ in? (c) What is the value of $\sin t$? (d) Does $\sin(2t) = 2\sin t$?

**117.** Find the measure of all sides and all angles (in radians) for the triangles given. Round the angles to four decimal places and sides to the nearest tenth. (a) How are the two triangles related? (b) Which can be used to identify points on the unit circle? (c) What is the value of $t$ in $\sin t = \frac{28}{53}$?

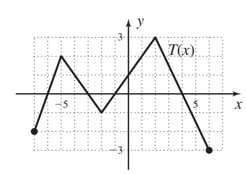



## MAINTAINING YOUR SKILLS

**118.** (R.7) State the formula for the (a) area of a circle, (b) perimeter of a rectangle, and (c) volume of a cylinder.

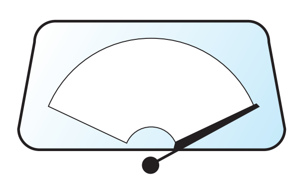
**119.** (1.3) Solve by factoring:
(a) $g^2 - 9g = 0$, (b) $g^2 - 9 = 0$,
(c) $g^2 - 9g - 10 = 0$,
(d) $g^2 + 9g - 10 = 0$, and
(e) $g^3 - 9g^2 - 10g + 90 = 0$.

**120.** (3.8) For the graph of $T(x)$ given, (a) name the local maximums and minimums, (b) the zeroes of $T$, (c) intervals where $T(x)\downarrow$ and $T(x)\uparrow$, and (d) intervals where $T(x) > 0$ and $T(x) < 0$.

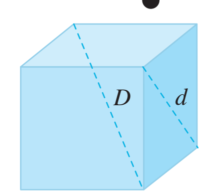



**121.** (3.2) Use a composition to show that for $f(x) = (x - 3)^2; x \geq 3$, the inverse function is $f^{-1}(x) = \sqrt{x} + 3$.

**122.** (6.1) The armature for the rear windshield wiper has a length of 24 in., with a rubber wiper blade that is 20 in. long. What area of my rear windshield is cleaned as the armature swings back-and-forth through an angle of 110°?



**123.** (6.1) The boxes used to ship some washing machines are perfect cubes with edges measuring 38 in. Use a special triangle to find the length of the diagonal $d$ of one side, and the length of the interior diagonal $D$ (through the middle of the box).



# 6.3 Graphs of the Sine and Cosine Functions

### LEARNING OBJECTIVES

***In Section 6.3 you will learn how to:***

**A.** **Graph $y = \sin t$ using standard values and symmetry**

**B.** **Graph $y = \cos t$ using standard values and symmetry**

**C.** **Graph sine and cosine functions with various amplitudes and periods**

**D.** **Investigate graphs of the reciprocal functions $y = \csc t$ and $y = \sec t$**

**E.** **Write the equation for a given graph**

### INTRODUCTION

As with the graphs of other functions, trigonometric graphs contribute a great deal toward the understanding of each trig function and its applications. For now, our primary interest is the general shape of each basic graph and some of the transformations that can be applied (much of what we learned in Chapter 3 can be used here). We will also learn to analyze each graph, and to capitalize on the features that enable us to apply the functions as real-world models.

### POINT OF INTEREST

The close of the third century B.C. marked the end of the glory years of Grecian mathematics. Ptolemy VII lacked the respect his predecessors had for math, science, and art, banishing to exile all scholars who would not swear loyalty to him. As things turned out, Alexandria's loss was the rest of Asia Minor's gain and the act contributed a great deal to the spread of mathematical and scientific knowledge. According to one Athenaeus of Naucratis, "The king sent many Alexandrians into exile, filling the islands and towns with . . . philologists, philosophers, mathematicians, musicians, painters, physicians and other professional men. The refugees, reduced by poverty to teaching what they knew, instructed many other men."

## A. Graphing $y = \sin t$

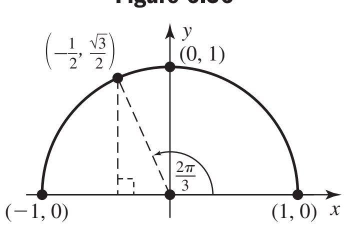
From our work in Section 6.2, we have the values for $y = \sin t$ in the interval $\left[0, \frac{\pi}{2}\right]$ shown in Table 6.1.

**Table 6.1**

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin t$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |

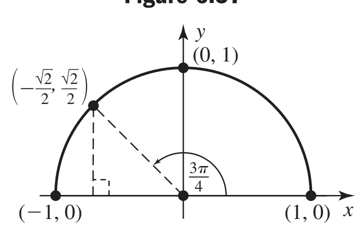
Observe that in this interval (representing Quadrant I), sine values are increasing from 0 to 1. From $\frac{\pi}{2}$ to $\pi$ (Quadrant II), standard values taken from the unit circle show sine values are decreasing from 1 to 0, *but through the same output values as in QI!* See Figures 6.36 through 6.38.

**Figure 6.36**



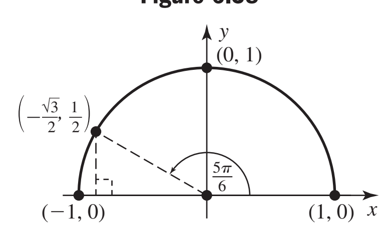

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

**Figure 6.37**




$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

**Figure 6.38**




$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

With this information we can extend our table of values through $\pi$, noting that $\sin \pi = 0$ (see Table 6.2). Note that both the table and unit circle show the range of the sine function is $y \in [-1, 1]$.

**Table 6.2**

| $t$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ |

Using the symmetry of the circle and the fact that $y$ is negative in QIII and QIV, we can complete the table for values between $\pi$ and $2\pi$.

**EXAMPLE 1** Use the symmetry of the unit circle and reference arcs of standard values to complete Table 6.3.

**Table 6.3**

| $t$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t$ | | | | | | | | | |

**Solution:** Symmetry shows that for any odd multiple of $t = \frac{\pi}{4}$, the function value will be $\pm\frac{\sqrt{2}}{2}$ depending on the quadrant of the terminal side. Similarly, for any value of $t$ having a reference arc of $\frac{\pi}{6}$, $\sin t = \pm\frac{1}{2}$. For a reference arc of $\frac{\pi}{3}$, $\sin t = \pm\frac{\sqrt{3}}{2}$ with the sign again depending on the quadrant of the terminal side. The completed table is shown in Table 6.4.

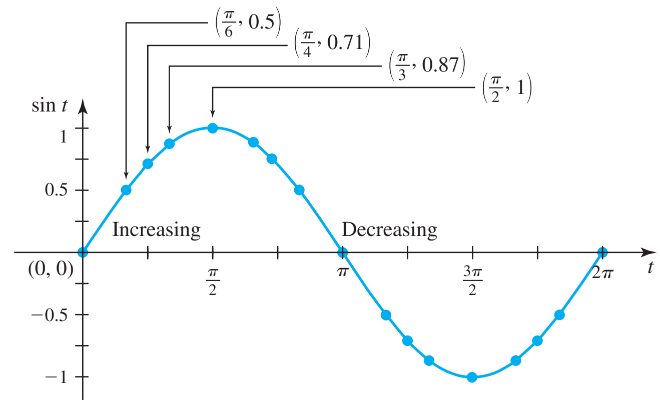
**Table 6.4**

| $t$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $0$ |

NOW TRY EXERCISES 7 AND 8

Noting that $\frac{1}{2} = 0.5$, $\frac{\sqrt{2}}{2} \approx 0.71$, and $\frac{\sqrt{3}}{2} \approx 0.87$, we plot the points above and connect them with a smooth curve to graph $y = \sin t$ for $t \in [0, 2\pi]$. The first five plotted points are labeled in Figure 6.39.

**Figure 6.39**




Expanding the table from $2\pi$ to $4\pi$ using reference arcs and the unit circle shows that function values begin to repeat. For example, $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)$ since $\theta_r = \frac{\pi}{6}$; $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$ since $\theta_r = \frac{\pi}{4}$, and so on. Functions that cycle through a set pattern of values are said to be **periodic functions,** which we more formally define here.

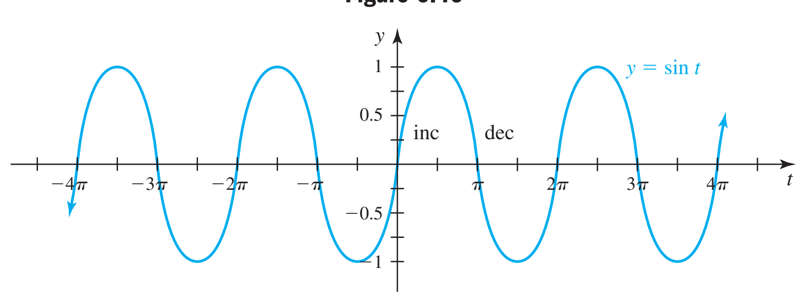
**PERIODIC FUNCTIONS**

A function $f$ is said to be periodic if there is a positive number $P$ such that $f(t + P) = f(t)$ for all $t$ in the domain. The smallest number $P$ for which this occurs is called the **period** of $f$.

For the sine function we have $\sin t = \sin(t + 2\pi)$, as seen in our illustrations above: $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right)$ and $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 2\pi\right)$, with the idea extending to all other real numbers $t$: $\sin t = \sin(t + 2\pi k)$ for all integers $k$. The sine function is periodic with period $P = 2\pi$.

Although we initially focused on positive values of $t$ in $[0, 2\pi]$, $t < 0$ and $k < 0$ are certainly possibilities and we note the graph of $y = \sin t$ extends infinitely in both directions (see Figure 6.40).

**Figure 6.40**



Finally, note that $y = \sin t$ is an odd function. For instance, we have $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$ and in general $\sin(-t) = -\sin t$ for all $t \in \mathbb{R}$. As a handy reference, the following box summarizes the main characteristics of $y = \sin t$ using features discussed here and in Section 6.2.

**CHARACTERISTICS OF $y = \sin t$**

| Unit Circle Definition | Domain | Symmetry | Maximum values | Increasing: $t \in (0, 2\pi)$ |
|---|---|---|---|---|
| $\sin t = y$ | $t \in \mathbb{R}$ | Odd: $\sin(-t) = -\sin t$ | $\sin t = 1$ at $t = \frac{\pi}{2} + 2\pi k$ | $t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ |
| **Period** | **Range** | **Zeroes:** | **Minimum values** | **Decreasing: $t \in (0, 2\pi)$** |
| $2\pi$ | $y \in [-1, 1]$ | $t = k\pi, k \in \mathbb{Z}$ | $\sin t = -1$ at $t = \frac{3\pi}{2} + 2\pi k$ | $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ |

Many of the transformations applied to algebraic graphs can also be applied to trigonometric graphs. As before, the transformations may stretch, reflect, or translate the graph, but it will maintain its same basic shape. In numerous applications it will help if you're able to draw a quick, accurate sketch of the transformations involving $y = \sin t$. To assist this effort, we'll begin with the standard interval $t \in [0, 2\pi]$, combine the characteristics just listed with some simple geometry, and offer a simple four-step process. Steps I through IV are illustrated in Figures 6.41 through 6.44.

Step I: Draw the $y$-axis, mark zero halfway up, with $-1$ and 1 an equal distance from this zero. Then draw an extended $t$-axis and tick mark $2\pi$ to the extreme right.

Step II: Mark halfway between the $y$-axis and $2\pi$ and label it "$\pi$," mark halfway between $\pi$ on either side and label the marks $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Halfway between these you can draw additional tic marks to represent the remaining multiples of $\frac{\pi}{4}$.

Step III: Next, lightly draw a rectangular frame, which we'll call the **reference rectangle,** $P = 2\pi$ units wide and 2 units tall, centered on the $t$-axis and with the $y$-axis along one side.

Step IV: Knowing $\sin t$ is positive and increasing in QI; that the zeroes are 0, $\pi$, and $2\pi$; and that maximum and minimum values *occur halfway between the zeroes* (since there is no vertical shift), we now can draw a reliable graph of $y = \sin t$. We will call this partitioning of the reference rectangle the **rule of fourths,** since its division into four equal parts helps us locate the zeroes and max/min values.

**Figure 6.41**

I.


**Figure 6.42**

II.

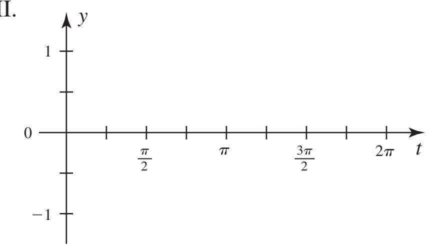



**Figure 6.43**

III.

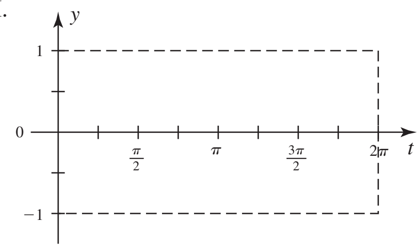
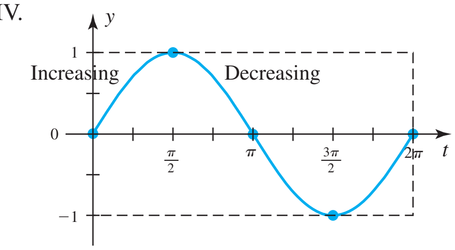


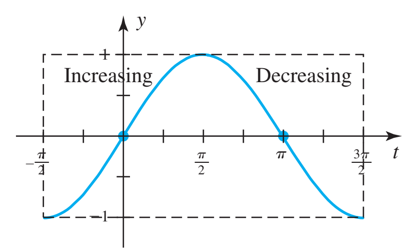

**Figure 6.44**

IV.




**EXAMPLE 2** Use steps I through IV to draw a sketch of $y = \sin t$ for $t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

**Solution:** Begin by completing steps I and II, then extend the $t$-axis to include $-\frac{\pi}{2}$. Beginning at $-\frac{\pi}{2}$, draw a reference rectangle 2 units high and $2\pi$ wide, ending at $\frac{3\pi}{2}$. The zeroes still occur at $t = 0$ and $t = \pi$, with the max/min values spaced equally between and on either side. See the figure.




**NOW TRY EXERCISES 9 AND 10**

## B. Graphing $y = \cos t$

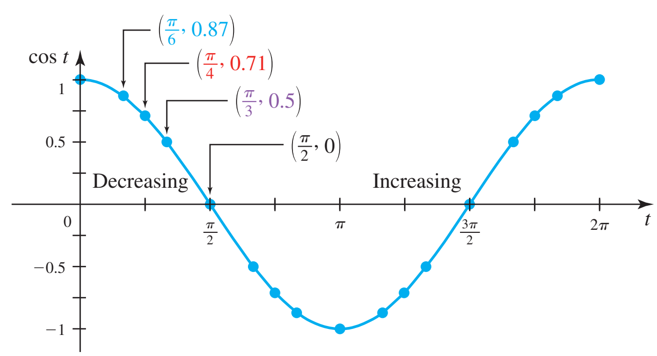
With the graph of $y = \sin t$ established, sketching the graph of $y = \cos t$ is a very natural next step. First, note that when $t = 0$, $\cos t = 1$ so the graph of $y = \cos t$ will begin at (0, 1) in the interval $[0, 2\pi]$. Second, we've seen $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ are all points on the unit circle since they satisfy $x^2 + y^2 = 1$. Since $\cos t = x$ and $\sin t = y$, the equation $\cos^2 t + \sin^2 t = 1$ can be obtained by direct substitution. This means if $\sin t = \pm\frac{1}{2}$, then $\cos t = \pm\frac{\sqrt{3}}{2}$ and vice versa, with the signs taken from the appropriate quadrant. The table of values for cosine then becomes a simple extension of the table for sine, as shown in Table 6.5 for $t \in [0, \pi]$.

**Table 6.5**

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t = y$ | 0 | $\frac{1}{2} = 0.5$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{\sqrt{3}}{2} \approx 0.87$ | 1 | $\frac{\sqrt{3}}{2} \approx 0.87$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{1}{2} = 0.5$ | 0 |
| $\cos t = x$ | 1 | $\frac{\sqrt{3}}{2} \approx 0.87$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{1}{2} = 0.5$ | 0 | $-\frac{1}{2} = -0.5$ | $\frac{\sqrt{2}}{2} \approx -0.71$ | $-\frac{\sqrt{3}}{2} \approx -0.87$ | $-1$ |

The same values can be taken from the unit circle, but this view requires much less effort and easily extends to values of $t$ in $[\pi, 2\pi]$. Using the points from Table 6.5 and its extension through $[\pi, 2\pi]$, we can draw the graph of $y = \cos t$ for $t \in [0, 2\pi]$ and identify where the function is increasing and decreasing in this interval. See Figure 6.45.

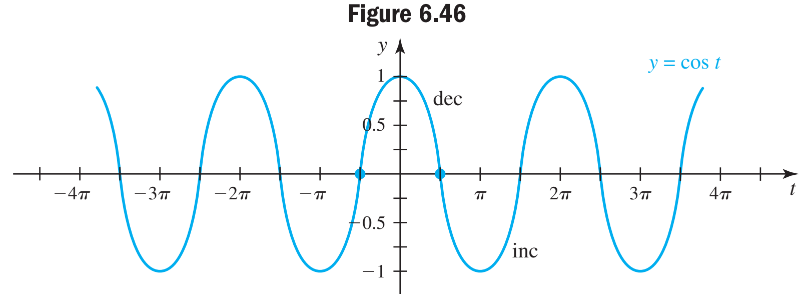
**Figure 6.45**




The function is decreasing from 0 to $\pi$, and increasing from $\pi$ to $2\pi$.

The end result appears to be the graph of $y = \sin t$, shifted to the left $\frac{\pi}{2}$ units. This is in fact the case, and is a relationship we will prove in Chapter 7. Like $y = \sin t$, the function $y = \cos t$ is periodic with period $P = 2\pi$, and extends infinitely in both directions. See Figure 6.46.

**Figure 6.46**



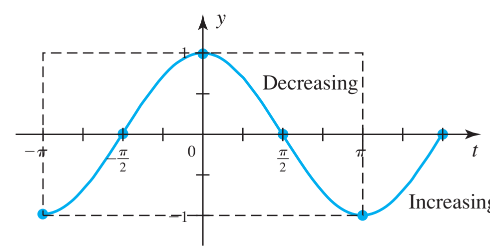

We further note that cosine is an **even function,** meaning $\cos(-t) = \cos t$ for all $t$ in the domain. For instance, from the graph we see that $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$. Here is a summary of important characteristics of the cosine function.

**CHARACTERISTICS OF $y = \cos t$**

| Unit Circle Definition | Domain | Symmetry | Maximum values | Increasing: $t \in (0, 2\pi)$ |
|---|---|---|---|---|
| $\cos t = x$ | $t \in \mathbb{R}$ | Even: $\cos(-t) = \cos t$ | $\cos t = 1$ at $t = 2\pi k$ | $t \in (\pi, 2\pi)$ |
| Period | Range | Zeroes | Minimum values | Decreasing: $t \in (0, 2\pi)$ |
| $2\pi$ | $y \in [-1, 1]$ | $t = \frac{\pi}{2} + \pi k;\ k \in \mathbb{Z}$ | $\cos t = -1$ at $t = \pi + 2\pi k$ | $t \in (0, \pi)$ |

**EXAMPLE 3** Draw a sketch of $y = \cos t$ for $t \in \left[-\pi, \frac{3\pi}{2}\right]$.

**Solution:** Using steps I through IV (scaling, frame, zeroes, and max/min values) produces this graph for $y = \cos t$ in $\left[-\pi, \frac{3\pi}{2}\right]$. Note the reference rectangle begins at $-\pi$ and is $2\pi$ units in length. The graph was then extended to the interval $\left[-\pi, \frac{3\pi}{2}\right]$.




NOW TRY EXERCISES 11 AND 12

## C. Graphing $y = A\sin(Bt)$ and $y = A\cos(Bt)$

In many applications, trig functions have maximum and minimum values other than 1 and $-1$, and periods other than $2\pi$. For instance, in tropical regions the maximum and minimum temperatures may vary by no more than 20°, while for desert regions this difference may be 40° or more. This variation is modeled by the *amplitude* of sine and cosine functions.

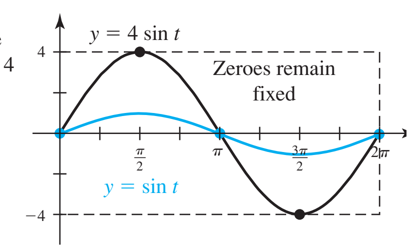
### Amplitude and the Coefficient *A*

For functions of the form $y = A\sin(Bt)$ and $y = A\cos(Bt)$, let $M$ represent the *M*aximum value and $m$ the *m*inimum value of the functions. Then the quantity $\frac{M + m}{2}$ gives the **average value** of the function, while $\frac{M - m}{2}$ gives the **amplitude** of the function. The amplitude is the maximum displacement from the average value in the positive or negative direction. Here, amplitude is represented by $|A|$, which plays a role similar to that seen for algebraic graphs, with $Af(x)$ vertically stretching or compressing the graph of $f$, and reflecting it across the $t$-axis if $A < 0$. While $A$ affects the amplitude and maximum/minimum values of sine and cosine, their graphs can quickly be sketched with any amplitude by noting that the *zeroes of the function remain fixed* (since $\sin t = 0$ implies

$A \sin t = 0$). In addition, the maximum and minimum values are simply multiplied by $A$, since $\sin t = 1$ or $-1$ implies $A \sin t = A$ or $-A$, respectively. Connecting the resulting points with a smooth curve will complete the graph.

**EXAMPLE 4** Draw a sketch of $y = 4 \sin t$ for $t \in [0, 2\pi]$.

**Solution:** This graph has the same zeroes as $y = \sin t$, but the maximum value is $4 \sin\left(\frac{\pi}{2}\right) = 4$, with a minimum value of $4 \sin\left(\frac{3\pi}{2}\right) = -4$ (the amplitude is $|A| = 4$). Connecting these points with a "sine curve" gives the graph shown ($y = \sin t$ is also shown for comparison).




NOW TRY EXERCISES 13 THROUGH 18

**Period and the Coefficient *B***

For basic sine and cosine functions the period is $2\pi$, but in many applications the period may be very long (tsunami's) or very short (electromagnetic waves). For the equations $y = A \sin(Bt)$ and $y = A \cos(Bt)$, the period depends on the value of $B$. To see why, consider the function $y = \cos(2t)$ and Table 6.6. Multiplying input values by 2 means each cycle will be completed twice as fast. The table shows that $y = \cos(2t)$ completes a full cycle in $[0, \pi]$, giving a period of $P = \pi$ (Figure 6.47, red graph).

**Table 6.6**

| $t$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|
| $2t$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| $\cos(2t)$ | 1 | 0 | $-1$ | 0 | 1 |

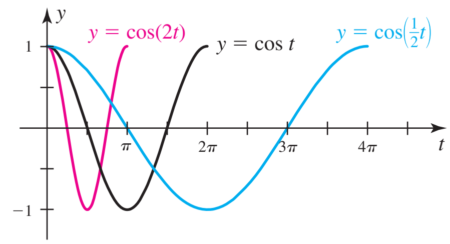
Dividing $B$ by 2 (or multiplying by $\frac{1}{2}$) will cause the function to complete a cycle only half as fast, doubling the time required to complete a full cycle. Table 6.7 shows $y = \cos\left(\frac{1}{2}t\right)$ completes only one-half cycle in $2\pi$ (Figure 6.47, blue graph).

**Table 6.7**

| $t$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}t$ | 0 | $\frac{\pi}{8}$ | $\frac{\pi}{4}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{2}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{4}$ | $\frac{7\pi}{8}$ | $\pi$ |
| $\cos\left(\frac{1}{2}t\right)$ | 1 | 0.92 | $\frac{\sqrt{2}}{2}$ | 0.38 | 0 | $-0.38$ | $-\frac{\sqrt{2}}{2}$ | $-0.92$ | $-1$ |

The graphs of $y = \cos t$, $y = \cos(2t)$, and $y = \cos\left(\frac{1}{2}t\right)$ shown in Figure 6.47 clearly illustrate this relationship and how the value of $B$ affects the period of a graph.

**Figure 6.47**




To find the period for arbitrary values of $B$, the formula $P = \frac{2\pi}{B}$ is used. Note for $y = \cos(2t)$, $B = 2$, and $P = \frac{2\pi}{2} = \pi$, as shown. For $y = \cos\left(\frac{1}{2}t\right)$, $B = \frac{1}{2}$, and $P = \frac{2\pi}{1/2} = 4\pi$. These changes in $B$ represent the horizontal stretching and shrinking studied in Section 3.3 (also see Exercise 74).

**PERIOD FORMULA FOR SINE AND COSINE**

For $B$ a real number and functions $y = A\sin(Bt)$ and $y = A\cos(Bt)$, the period is given by $P = \frac{2\pi}{B}$.

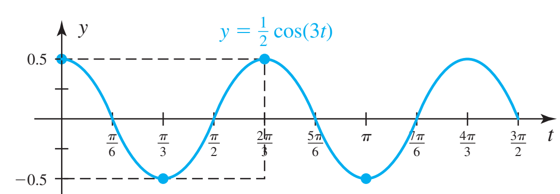
To sketch these functions for various periods, we use a reference rectangle of height $2A$ and length $P$, then break the enclosed $t$-axis in four equal parts to help find and graph the zeroes and maximum/minimum values. In general, if the period is "very large" one full cycle is appropriate for the graph. If the period is very small, graph at least two cycles.

**WORTHY OF NOTE**

When scaling the $t$-axis for graphing various periods, it helps to "count by" $\frac{P}{4}$ ($P$ divided by 4) as you place the tic marks, waiting until afterward to reduce the fractions. The scaling of Example 5 would then be $\frac{1\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \frac{6\pi}{6}$, and so on, which can be reduced to $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$, and $\pi$ afterward.

**EXAMPLE 5** Draw a sketch of $y = \frac{1}{2}\cos(3t)$ for $t \in \left[0, \frac{3\pi}{2}\right]$.

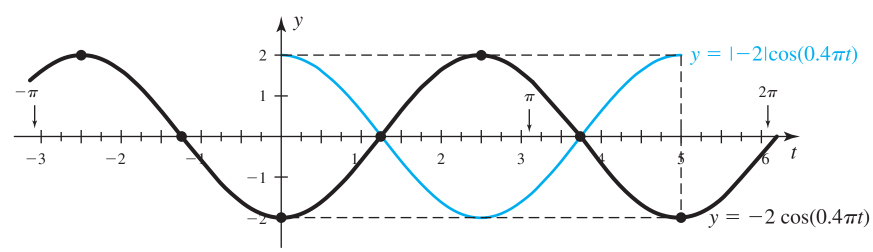
**Solution:** The amplitude of the graph is $|A| = \frac{1}{2}$, so the reference rectangle will be $2A = 1$ unit high. The period is $P = \frac{2\pi}{3}$ so the frame will be $\frac{2\pi}{3}$ units in length. Breaking the $t$-axis into four parts within the frame means each tick-mark will be $\left(\frac{1}{4}\right)\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$ units apart, indicating that we should scale the $t$-axis in multiples of $\frac{\pi}{6}$ rather than $\frac{\pi}{4}$ as before. The completed graph is shown here.




**NOW TRY EXERCISES 19 THROUGH 24**

**EXAMPLE 6** Draw a sketch of $y = -2\cos(0.4\pi t)$ for $t \in [-\pi, 2\pi]$.

**Solution:** The amplitude is $|A| = 2$, so the reference rectangle will be $2(2) = 4$ units high. Since $A < 0$ the *graph will be vertically reflected across the t-axis*. The period is $P = \frac{2\pi}{0.4\pi} = 5$ (note the factors of $\pi$ reduce to 1), so the frame will be 5 units in length. Breaking the $t$-axis into four parts within the frame gives $\left(\frac{1}{4}\right)5 = \frac{5}{4}$ units, indicating that we should scale the $t$-axis in multiples of $\frac{1}{4}$. In cases where the $\pi$ factor reduces, we scale the $t$-axis as a "standard" number line, and *estimate the location of multiples of* $\pi$. For practical reasons, we first draw the unreflected graph for guidance in drawing the reflected graph.




**NOW TRY EXERCISES 25 THROUGH 30**

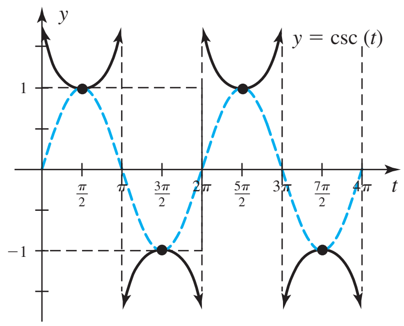
## D. Graphs of $y = \csc(Bt)$ and $y = \sec(Bt)$

The graphs of the reciprocal functions follow quite naturally from the graphs of $y = A\sin(Bt)$ and $y = A\cos(Bt)$, by using these observations: (1) you cannot divide by zero, (2) the reciprocal of a very small number is a very large number (and vice versa), and (3) the reciprocal of 1 is 1. Just as with rational functions, division by zero creates a vertical asymptote, so the graph of $y = \csc t = \frac{1}{\sin t}$ will have a vertical asymptote at every point where $\sin t = 0$. This occurs at $t = \pi k$, where $k$ is an integer $(\ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots)$. Further, the graph of $y = \csc(Bt)$ will share the maximums and minimums of $y = \sin(Bt)$, since the reciprocal of 1 and $-1$ are still 1 and $-1$. Finally, due to observation (2), the graph of the cosecant function will be increasing when the sine function is decreasing, and decreasing when the sine function is increasing. In most cases, we graph $y = \csc(Bt)$ by drawing a sketch of $y = \sin(Bt)$, then using the preceding observations. In doing so, we discover that the period of the cosecant function is also $2\pi$.

**EXAMPLE 7** Graph the function $y = \csc t$ for $t \in [0, 4\pi]$.

**Solution:** The related sine function is $y = \sin t$, which means we'll draw a rectangular frame $2A = 2$ units high. The period is $P = \frac{2\pi}{1} = 2\pi$, so the reference frame will be $2\pi$ units in length. Breaking the $t$-axis into four parts within the frame means each tic mark will be $\left(\frac{1}{4}\right)\left(\frac{2\pi}{1}\right) = \frac{\pi}{2}$ units apart, with the asymptotes occurring at 0, $\pi$, and $2\pi$. A partial table and the resulting graph is shown.

- Vertical asymptotes where sine is zero
- Shares maximum and minimum values with sine
- Output values are reciprocated




| $t$ | $\sin t$ | $\csc t$ |
|---|---|---|
| 0 | 0 | $\frac{1}{0} \rightarrow$ undefined |
| $\frac{\pi}{6}$ | $\frac{1}{2} = 0.5$ | $\frac{2}{1} = 2$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{2}{\sqrt{2}} \approx 1.41$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2} \approx 0.87$ | $\frac{2}{\sqrt{3}} \approx 1.15$ |
| $\frac{\pi}{2}$ | 1 | 1 |

NOW TRY EXERCISES 31 AND 32

Similar observations can be made regarding $y = \sec(Bt)$ and its relationship to $y = \cos(Bt)$ (see Exercises 8, 33 and 34). The most important characteristics of the cosecant and secant functions are summarized in the table below. For these functions, there is no discussion of amplitude, and no mention is made of their zeroes since neither graph intersects the $t$-axis.

**CHARACTERISTICS OF $y = \csc t$ and $y = \sec t$**

**$y = \csc t$**

| Unit circle Definition | Asymptotes | → | Domain |
|---|---|---|---|
| $\csc t = \frac{1}{y}$ | $t = k\pi$ | | $t \neq k\pi$ |
| **Period** | **Symmetry** | | **Range** |
| $2\pi$ | Odd<br>$\csc(-t) = -\csc t$ | | $y \in (-\infty, -1] \cup [1, \infty)$ |

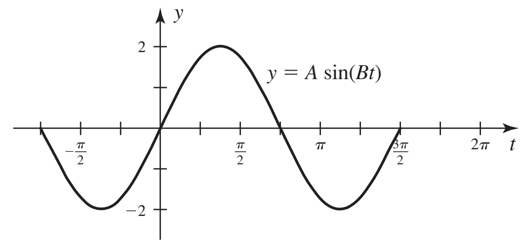
**$y = \sec t$**

| Unit circle Definition | Asymptotes | → | Domain |
|---|---|---|---|
| $\sec t = \frac{1}{x}$ | $t = \frac{\pi}{2} + \pi k$ | | $t \neq \frac{\pi}{2} + \pi k$ |
| **Period** | **Symmetry** | | **Range** |
| $2\pi$ | Even<br>$\sec(-t) = \sec t$ | | $y \in (-\infty, -1] \cup [1, \infty)$ |

## E. Writing Equations from Graphs and Solving Equations Graphically

Mathematical concepts are best reinforced by working with them in both "forward and reverse." Where graphs are concerned, this means we should attempt to find the equation of a given graph, rather than only using an equation to sketch the graph. Exercises of this type require that you become very familiar with the graph's basic characteristics and how each is expressed as part of the equation.

**EXAMPLE 8** ▶ The graph shown below is of the form $y = A\sin(Bt)$. Find the value of $A$ and $B$.




**Solution:** ▶ By inspection, the graph has an amplitude of $A = 2$ and a period of $P = \frac{3\pi}{2}$. To find $B$ we used the period formula $P = \frac{2\pi}{B}$, substituting $\frac{3\pi}{2}$ for $P$ and solving.

$$P = \frac{2\pi}{B} \quad \text{period formula}$$

$$\frac{3\pi}{2} = \frac{2\pi}{B} \quad \text{substitute } \frac{3\pi}{2} \text{ for } P$$

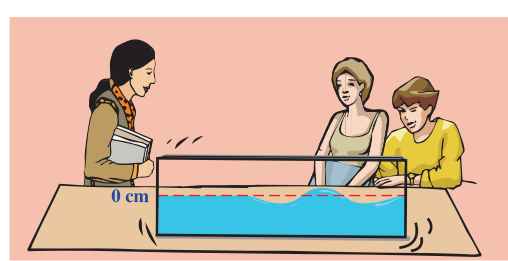
$$3\pi B = 4\pi \quad \text{cross multiply}$$

$$B = \frac{4}{3} \quad \text{solve for } B$$

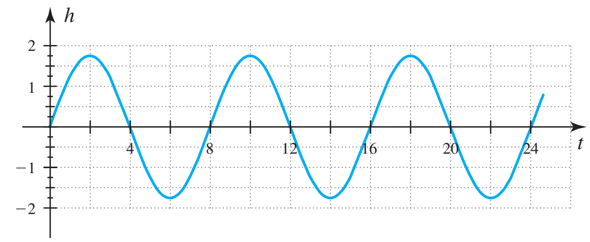
The result is $B = \frac{4}{3}$, which gives in the equation $y = 2\sin\left(\frac{4}{3}t\right)$.

NOW TRY EXERCISES 35 THROUGH 56 ▶

We end this section with an application involving sine. In the classroom, wave tanks can be used to study wave theory and its applications. Consider an open rectangular tank, with a straight "zero line" along the edge marking the level of still water. By bumping one end, a wave is created that will run the full length of the tank, with its height and depth (crest and trough) measured in relation to the zero line.




**EXAMPLE 9** ▶ Identical waves are generated in a wave tank. A photo taken through the translucent sides results in the graph shown, where $h$ is the height in centimeters and $t$ is the distance in centimeters from the end of the tank. Use the graph to (a) determine the amplitude and period of the wave, (b) find the equation modeling this wave, and (c) use the equation to find the height of the wave at $t = 19$ cm.




**Solution:** **a.** The graph has an amplitude $A = 1.75$ and period $P = 8$.

**b.** Using $h(t) = A\sin(Bt)$ as a model, we find $B$ using $P = \frac{2\pi}{B}$, which shows $B = \frac{2\pi}{P}$, so $B = \frac{2\pi}{8} = \frac{\pi}{4}$. The equation modeling this wave is represented by $h(t) = 1.75\sin\left(\frac{\pi}{4}t\right)$.

**c.** For $t = 19$, we have $h(19) = 1.75\sin\left[\frac{5\pi}{2}(19)\right]$ and a calculator indicates the wave's height will be about 1.24 cm.

NOW TRY EXERCISES 59 THROUGH 70

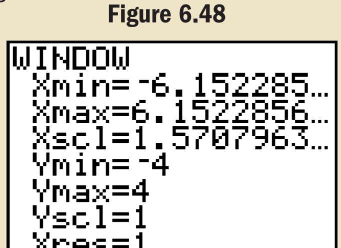
TECHNOLOGY HIGHLIGHT

## Exploring Amplitudes and Periods

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

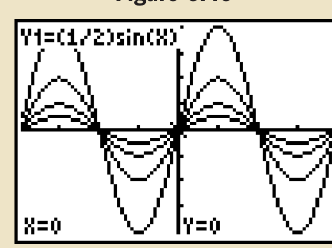
In practice, trig applications offer an immense range of coefficients, creating amplitudes that are sometimes very large and sometimes extremely small, as well as periods ranging from nanoseconds, to many years. This *Technology Highlight* is designed to help you use the calculator more effectively in the study of these functions. To begin, we note the TI-84 Plus offers a preset ZOOM option that automatically sets a window size convenient to many trig graphs. The resulting WINDOW after pressing ZOOM **7:ZTrig** is shown in Figure 6.48 for a calculator set in **Radian** MODE.

**Figure 6.48**




One important concept of Section 6.3 is that a change in amplitude will not change the location of the zeroes or max/min values. On the Y= screen, enter $Y_1 = \frac{1}{2}\sin x$, $Y_2 = \sin x$, $Y_3 = 2\sin x$, and $Y_4 = 4\sin x$, then use ZOOM **7:ZTrig** to graph the functions. As you see in Figure 6.49, each graph rises to the expected amplitude, while "holding on" to the zeroes (graph the functions in **Simul**taneous MODE ).

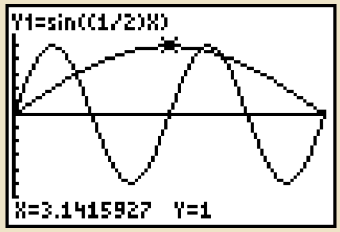
**Figure 6.49**




To explore concepts related to the coefficient $B$ and its effect on the period of a trig function, enter

$Y_1 = \sin\left(\frac{1}{2}x\right)$ and $Y_2 = \sin(2x)$ on the Y= screen and graph using ZOOM **7:ZTrig.** While the result is "acceptable," the graphs are difficult to read and compare, so we manually change the window to obtain a better view (Figure 6.50). After pressing GRAPH we can use TRACE and the up or down arrows to help identify each function.

**Figure 6.50**




A true test of effective calculator use comes when the amplitude or period is a very large or very small number. For instance, the tone you hear while pressing "5" on your telephone is actually a combination of the tones modeled by $Y_1 = \sin[2\pi(770)t]$ and $Y_2 = \sin[2\pi(1336)t]$. To graph these functions requires a careful analysis of the period, otherwise the graph can appear either garbled, misleading or difficult to read—try graphing $Y_1$ on the ZOOM **7:ZTrig** or ZOOM **6:ZStandard** screens (see Figure 6.51). First note the amplitude is $A = 1$, and the period is $P = \frac{2\pi}{2\pi 770}$ or $\frac{1}{770}$. With a period this short, even graphing the function from Xmin $= -1$ to Xmax $= 1$ gives a distorted graph. Setting Xmin to $-1/770$, Xmax to $1/770$, and Xscl to $(1/770)/10$ gives the graph in Figure 6.52, which can be used to investigate characteristics of the function.

**Figure 6.51**

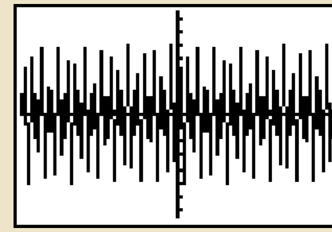
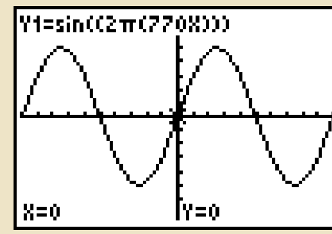


**Figure 6.52**




Exercise 1: Graph the second tone $Y_2 = \sin[2\pi(1336)t]$ mentioned here and find its value at $t = 0.00025$ sec.

Exercise 2: Graph the function $Y_1 = 950\sin(0.005t)$ on a "friendly" window and find the value at $x = 550$.

# 6.3 EXERCISES

## CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** For the sine function, output values are ________ in the interval $\left[0, \frac{\pi}{2}\right]$.

**2.** For the cosine function, output values are ________ in the interval $\left[0, \frac{\pi}{2}\right]$.

**3.** For the sine and cosine functions, the domain is ________ and the range is ________.

**4.** The amplitude of sine and cosine is defined to be the maximum ________ from the ________ value in the positive and negative directions.

**5.** Discuss/describe the four-step process outlined in this section for the graphing of basic trig functions. Include a worked-out example and a detailed explanation.

**6.** Discuss/explain how you would determine the domain and range of $y = \sec x$. Where is this function undefined? Why? Graph $y = 2\sec(2t)$ using $y = 2\cos(2t)$. What do you notice?

## DEVELOPING YOUR SKILLS

**7.** Use the symmetry of the unit circle and reference arcs of standard values to complete a table of values for $y = \cos t$ in the interval $t \in [\pi, 2\pi]$.

**8.** Use the standard values for $y = \cos t$ for $t \in [\pi, 2\pi]$ to create a table of values for $y = \sec t$ on the same interval.

Use steps I through IV given in this section to draw a sketch of:

**9.** $y = \sin t$ for $t \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$

**10.** $y = \sin t$ for $t \in [-\pi, \pi]$

**11.** $y = \cos t$ for $t \in \left[-\frac{\pi}{2}, 2\pi\right]$

**12.** $y = \cos t$ for $t \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of the following functions through at least one full period. Clearly state the amplitude and period as you begin.

**13.** $y = 3\sin t$

**14.** $y = 4\sin t$

**15.** $y = -2\cos t$

**16.** $y = -3\cos t$

**17.** $y = \frac{1}{2}\sin t$

**18.** $y = \frac{3}{4}\sin t$

**19.** $y = -\sin(2t)$

**20.** $y = -\cos(2t)$

**21.** $y = 0.8\cos(2t)$

**22.** $y = 1.7\sin(4t)$

**23.** $f(t) = 4\cos\left(\frac{1}{2}t\right)$

**24.** $y = -3\cos\left(\frac{3}{4}t\right)$

**25.** $f(t) = 3\sin(4\pi t)$

**26.** $g(t) = 5\cos(8\pi t)$

**27.** $y = 4\sin\left(\frac{5\pi}{3}t\right)$

**28.** $y = 2.5\cos\left(\frac{2\pi}{5}t\right)$

**29.** $f(t) = 2\sin(256\pi t)$

**30.** $g(t) = 3\cos(184\pi t)$

Draw the graph of each function by first sketching the related sine and cosine graphs, and applying the observations made in this section.

**31.** $y = 3\csc t$

**32.** $g(t) = 2\csc(4t)$

**33.** $y = 2\sec t$

**34.** $f(t) = 3\sec(2t)$

Clearly state the amplitude and period of each function, then match it with the correct graph a to l shown here.

**35.** $y = -2\cos(4t)$

**36.** $y = 2\sin(4t)$

**37.** $y = 3\sin(2t)$

**38.** $y = -3\cos(2t)$

**39.** $y = 2\csc\left(\frac{1}{2}t\right)$

**40.** $y = 2\sec\left(\frac{1}{4}t\right)$

**41.** $f(t) = \frac{3}{4}\cos(0.4t)$

**42.** $g(t) = \frac{7}{4}\cos(0.8t)$

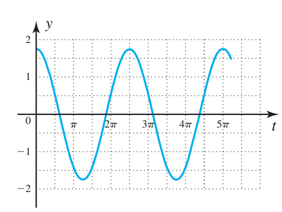
**43.** $y = \sec(8\pi t)$

**44.** $y = \csc(12\pi t)$

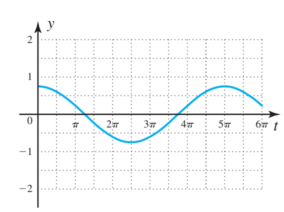
**45.** $y = 4\sin(144\pi t)$

**46.** $y = 4\cos(72\pi t)$

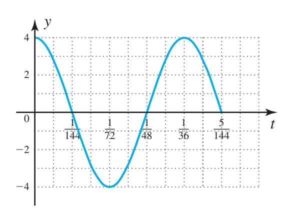
**a.**



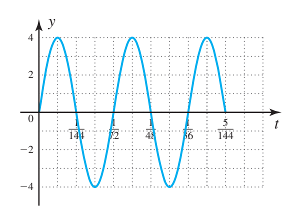

**b.**



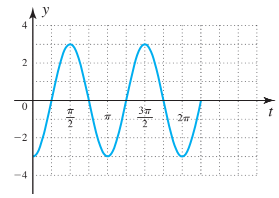

**c.**



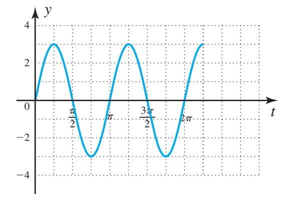

**d.**



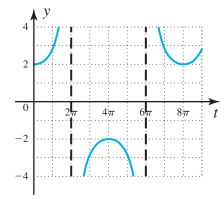
**e.**



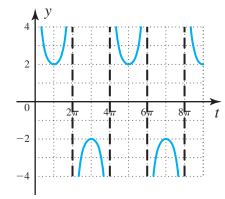
**f.**



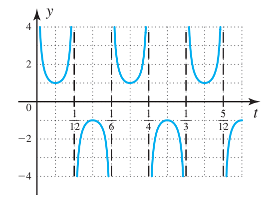
**g.**



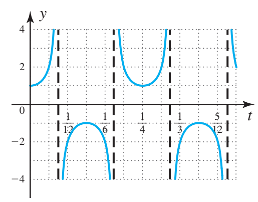
**h.**



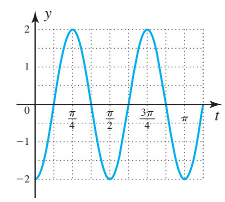
**i.**



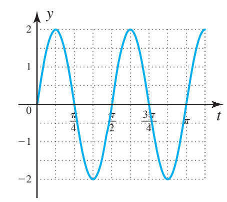
**j.**



**k.**

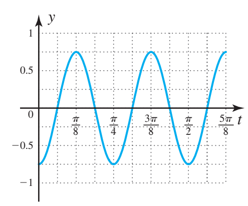


**l.**

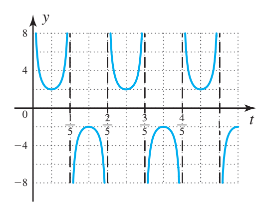


The graphs shown are of the form $y = A\cos(Bt)$ or $y = A\csc(Bt)$. Use the characteristics illustrated for each graph to determine its equation.

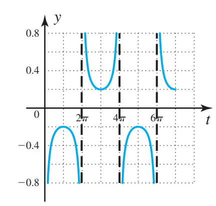
**47.**



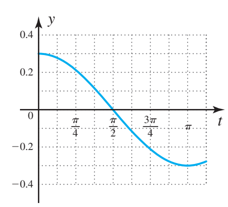
**48.**



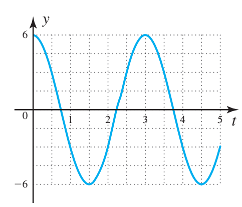
**49.**



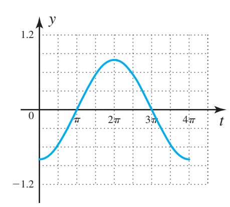
**50.**



**51.**

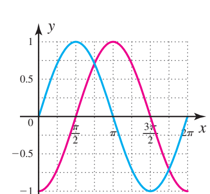


**52.**

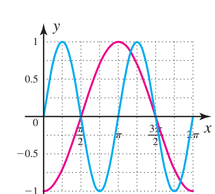


Match each graph to its equation, then graphically estimate the points of intersection. Confirm or contradict your estimate(s) by substituting the values into the given equations using a calculator.

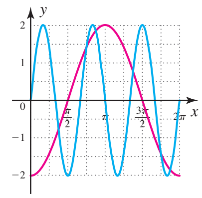
**53.** $y = -\cos x$; $y = \sin x$



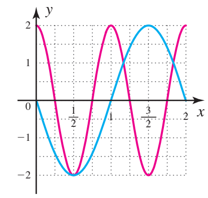
**54.** $y = -\cos x$; $y = \sin(2x)$



**55.** $y = -2\cos x$; $y = 2\sin(3x)$




**56.** $y = 2\cos(2\pi x)$; $y = -2\sin(\pi x)$

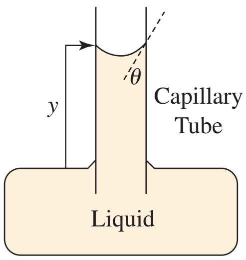



## WORKING WITH FORMULAS

**57. The Pythagorean theorem in trigonometric form: $\sin^2\theta + \cos^2\theta = 1$**

The formula shown is commonly known as a Pythagorean identity and is introduced more formally in Chapter 7. It is derived by noting that on a unit circle, $\cos t = x$ and $\sin t = y$, while $x^2 + y^2 = 1$. Given that $\sin t = \frac{15}{113}$, use the formula to find the value of $\cos t$ in Quadrant I. What is the Pythagorean triple associated with these values of $x$ and $y$?

**Exercise 58**



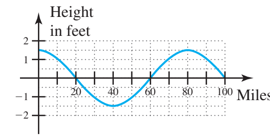

**58. Hydrostatics, surface tension, and contact angles: $y = \dfrac{2\gamma\cos\theta}{kr}$**

The height that a liquid will rise in a capillary tube is given by the formula shown, where $r$ is the radius of the tube, $\theta$ is the contact angle of the liquid with the side of the tube (the meniscus), $\gamma$ is the surface tension of the liquid-vapor film, and $k$ is a constant that depends on the weight-density of the liquid. How high will the liquid rise given that the surface tension $\gamma$ has a value of 0.2706, the tube has radius $r = 0.2$ cm, the contact angle $\theta$ is 22.5°, and the constant $k = 1.25$?

## APPLICATIONS

**Tidal waves:** Tsunamis, also known as tidal waves, are ocean waves produced by earthquakes or other upheavals in the Earth's crust and can move through the water undetected for hundreds of miles at great speed. While traveling in the open ocean, these waves can be represented by a sine graph with a very long wavelength (period) and a very small amplitude. Tsunami waves only attain a monstrous size as they approach the shore, and represent a very different phenomenon than the ocean swells created by heavy winds over an extended period of time.

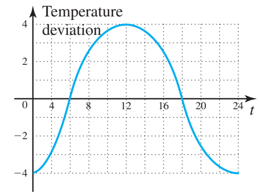
**59.** A graph modeling a tsunami wave is given in the figure. (a) What is the height of the tsunami wave (from crest to trough)? Note that $h = 0$ is considered the level of a calm ocean. (b) What is the tsunami's wavelength? (c) Find the equation for this wave.




**60.** A heavy wind is kicking up ocean swells approximately 10 ft high (from crest to trough), with wavelengths of 250 ft. (a) Find the equation that models these swells. (b) Graph the equation. (c) Determine the height of a wave measured 200 ft from the trough of the previous wave.

**Temperature studies:** The sine and cosine functions are of great importance to meteorological studies, as when modeling the temperature based on the time of day, the illumination of the Moon as it goes through its phases, or even the prediction of tidal motion.

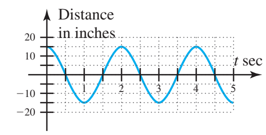
**61.** The graph given shows the deviation from the average daily temperature for the hours of a given day, with $t = 0$ corresponding to 6 A.M. (a) Use the graph to determine the related equation. (b) Use the equation to find the deviation at $t = 11$ (5 P.M.) and confirm that this point is on the graph. (c) If the average temperature for this day was 72°, what was the temperature at midnight?



**62.** The equation $y = 7\sin\left(\frac{\pi}{6}t\right)$ models the height of the tide along a certain coastal area, as compared to the average sea level. Assuming $t = 0$ is midnight, (a) graph this function over a 12-hr period. (b) What will the height of the tide be at 5 A.M.? (c) Is the tide rising or falling at this time?

**Sinusoidal movements:** Many animals exhibit a wavelike motion in their movements, as in the tail of a shark as it swims in a straight line or the wingtips of a large bird in flight. Such movements can be modeled by a sine or cosine function and will vary depending on the animal's size, speed, and other factors.

**63.** The graph shown models the position of a shark's tail at time *t*, as measured to the left (negative) and right (positive) of a straight line along its length. (a) Use the graph to determine the related equation. (b) Is the tail to the right, left, or at center when $t = 6.5$ sec? How far? (c) Would you say the shark is "swimming leisurely," or "chasing its prey"? Justify your answer.

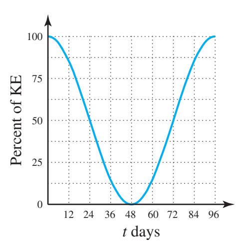



**64.** The State Fish of Hawaii is the *humuhumunukunukuapua'a*, a small colorful fish found abundantly in coastal waters. Suppose the tail motion of an adult fish is modeled by the equation $d(t) = \sin(15\pi t)$ with $d(t)$ representing the position of the fish's tail at time *t*, as measured in inches to the left (negative) or right (positive) of a straight line along its length. (a) Graph the equation over two periods. (b) Is the tail to the left or right of center at $t = 2.7$ sec? How far? (c) Would you say this fish is "swimming leisurely," or "running for cover"? Justify your answer.

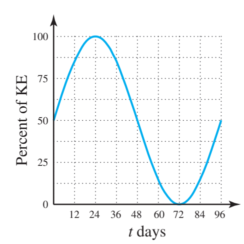
**Kinetic energy:** The kinetic energy a planet possesses as it orbits the Sun can be modeled by a cosine function. When the planet is at is apogee (greatest distance from the Sun), its kinetic energy is at its lowest point as it slows down and "turns around" to head back toward the Sun. The kinetic energy is at its highest when the planet "whips around the Sun" to begin a new orbit.

**65.** Two graphs are given below. (a) Which of the graphs could represent the kinetic energy of a planet orbiting the Sun if the planet is at its perigee (closest distance to the Sun) when $t = 0$? (b) For what value(s) of *t* does this planet possess 62.5% of its maximum kinetic energy with the kinetic energy increasing? (c) What is the orbital period of this planet?

**a.**

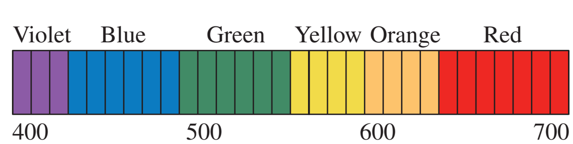



**b.**




**66.** The *potential energy* of the planet is the antipode of its kinetic energy, meaning when kinetic energy is at 100%, the potential energy is 0%, and when kinetic energy is at 0% the potential energy is at 100%. (a) How is the graph of the kinetic energy related to the graph of the potential energy? In other words, what transformation could be applied to the kinetic energy graph to obtain the potential energy graph? (b) If the kinetic energy is at 62.5% and increasing [as in Graph 65(b)], what can be said about the potential energy in the planet's orbit at this time?

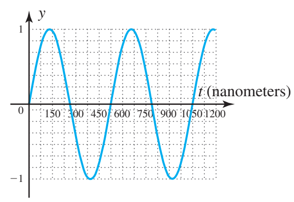
**Visible light:** One of the narrowest bands in the electromagnetic spectrum is the region involving visible light. The wavelengths (periods) of visible light vary from 400 nanometers (purple/violet colors) to 700 nanometers (bright red). The approximate wavelengths of the other colors are shown in the diagram below.



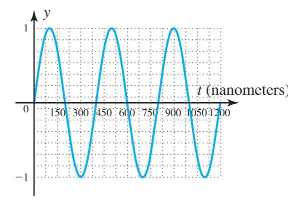
**67.** The equations for the colors in this spectrum have the form $y = \sin(\gamma t)$, where $\frac{2\pi}{\gamma}$ gives the length of the sine wave. (a) What color is represented by the equation $y = \sin\left(\frac{\pi}{240}t\right)$? (b) What color is represented by the equation $y = \sin\left(\frac{\pi}{310}t\right)$?

**68.** Name the color represented by each of the graphs a and b here and write the related equation.

**a.**



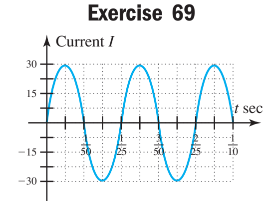

**b.**




**Alternating current:** Surprisingly, even characteristics of the electric current supplied to your home can be modeled by sine or cosine functions. For alternating current (AC), the amount of current $I$ (in amps) at time $t$ can be modeled by $I = A\sin(\omega t)$, where $A$ represents the maximum current that is produced, and $\omega$ is related to the frequency at which the generators turn to produce the current.

**69.** Find the equation of the household current modeled by the graph, then use the equation to determine $I$ when $t = 0.045$ sec. Verify that the resulting ordered pair is on the graph.

**Exercise 69**




**70.** If the *voltage* produced by an AC circuit is modeled by the equation $E = 155\sin(120\pi t)$, (a) what is the period and amplitude of the related graph? (b) What voltage is produced when $t = 0.2$?

## WRITING, RESEARCH, AND DECISION MAKING

**71.** For $y = A\sin(Bx)$ and $y = A\cos(Bx)$, the expression $\frac{M + m}{2}$ gives the average value of the function, where $M$ and $m$ represent the maximum and minimum values respectively. What was the average value of every function graphed in this section? Compute a table of values for the function $y = 2\sin t + 3$, and note its maximum and minimum values. What is the average value of this function? What transformation has been applied to change the average value of the function? Can you name the average value of $y = -2\cos t + 1$ by inspection? How is the amplitude related to this average value? (*Hint:* Graph the horizontal line $y = \frac{M + m}{2}$ on the same grid.)

**72.** Use the Internet or the resources of a local library to do some research on tsunamis (see Exercises 59 and 60). Attempt to find information on (a) exactly how they are generated; (b) some of the more notable tsunamis in history; (c) the average amplitude of a tsunami; (d) their average wavelength or period; and (e) the speeds they travel in the open ocean. Prepare a short summary of what you find.

**73.** To understand where the period formula $P = \frac{2\pi}{B}$ came from, consider that if $B = 1$, the graph of $y = \sin(Bt) = \sin(1t)$ completes one cycle from $1t = 0$ to $1t = 2\pi$. If $B \neq 1$, $y = \sin(Bt)$ completes one cycle from $Bt = 0$ to $Bt = 2\pi$. Discuss how this observation validates the period formula.

**74.** Horizontal stretches and compressions are remarkably similar for all functions. Use a graphing calculator to graph the functions $Y_1 = \sin x$ and $Y_2 = \sin(2x)$ on a ZOOM **7:ZTrig** screen (graph $Y_1$ in **bold**). Note that $Y_2$ completes two periods in the time in takes $Y_1$ to complete 1 period (the graph of $Y_2$ is horizontally compressed). Now enter $Y_1 = (x^2 - 1)(x^2 - 4)$ and $Y_2 = ([2x]^2 - 1)([2x]^2 - 4)$, substituting $2x$ for $x$ as before. After graphing these on a ZOOM **4:ZDecimal** screen, what do you notice? How do these algebraic graphs compare to the trigonometric graphs?

## EXTENDING THE CONCEPT

**75.** The tone you hear when pressing the digit "9" on your telephone is actually a combination of two separate tones, which can be modeled by the functions $f(t) = \sin[2\pi(852)t]$ and $g(t) = \sin[2\pi(1477)t]$. Which of the two functions has the shortest period? By carefully scaling the axes, graph the function having the shorter period using the steps I through IV discussed in this section.

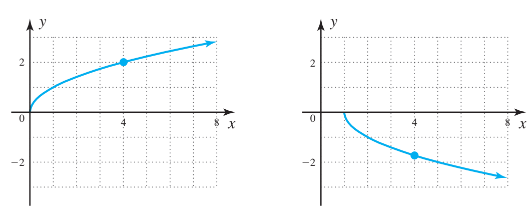
**76.** Consider the functions $f(x) = \sin(|t|)$ and $g(x) = |\sin t|$. For one of these functions, the portion of the graph below the $x$-axis is reflected above the $x$-axis, creating a series of "humps." For the other function, the graph obtained for $t > 0$ is reflected across the $y$-axis to form the graph for $t < 0$. After a thoughtful contemplation (without actually graphing), try to decide which description fits $f(x)$ and which fits $g(x)$, justifying your thinking. Then confirm or contradict your guess using a graphing calculator.

## MAINTAINING YOUR SKILLS

**77.** (6.2) Given $\tan(1.17) \approx 2.36$ with $t$ in QII, find an additional value of $t$ in $[0, 2\pi]$ that makes the equation $\tan t \approx 2.36$ true.

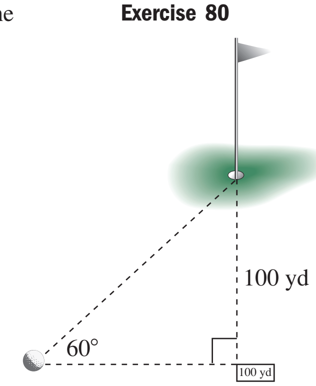
**78.** (6.1) Invercargill, New Zealand, is at 46°14′24″ south latitude. If the Earth has a radius of 3960 mi, how far is Invercargill from the equator?

**79.** (3.3) The graph on the left is $y = \sqrt{x}$. What is the equation of the graph on the right?




**80.** (6.1) Use a standard triangle to calculate the distance from the ball to the pin on the seventh hole, given the ball is in a straight line with the 100-yd plate, as shown in the figure.

**Exercise 80**




**81.** (1.4) Given $z_1 = 1 + i$ and $z_2 = 2 - 5i$, compute the following:

**a.** $z_1 + z_2$ **b.** $z_1 - z_2$ **c.** $z_1 z_2$ **d.** $\frac{z_2}{z_1}$

**82.** (3.6) According to health experts, a healthy body weight varies directly with a person's height. If the optimum body weight for a male 70 in. tall is 170 lb, what is the corresponding optimum weight for a male who is 75 in. tall?

# 6.4 Graphs of the Tangent and Cotangent Functions

**LEARNING OBJECTIVES**

***In Section 6.4 you will learn how to:***

**A.** **Graph $y = \tan t$ using asymptotes, zeroes, and the ratio $\frac{\sin t}{\cos t}$**

**B.** **Graph $y = \cot t$ using asymptotes, zeroes, and the ratio $\frac{\cos t}{\sin t}$**

**C.** **Identify and discuss important characteristics of $y = \tan t$ and $y = \cot t$**

**D.** **Graph $y = A\tan(Bt)$ and $y = A\cot(Bt)$ with various values of $A$ and $B$**

**E.** **Solve applications of $y = \tan t$ and $y = \cot t$**

**INTRODUCTION**

Unlike the other four trig functions, tangent and cotangent have no maximum or minimum value on any open interval of their domain. However, it is precisely this unique feature that adds to their value as mathematical models. Collectively, the six functions give scientists from many fields the tools they need to study, explore, and investigate a wide range of phenomena, extending our understanding of the world around us.

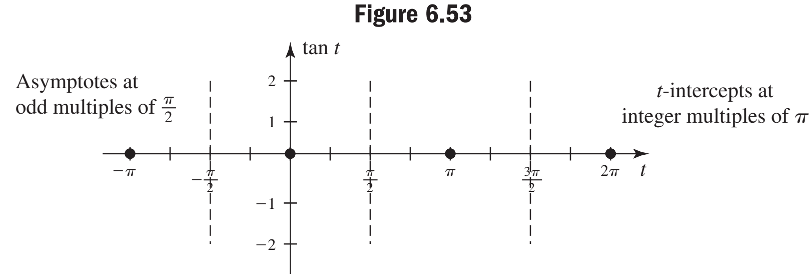
**POINT OF INTEREST**

The sine and cosine functions evolved from studies of the chord lengths within a circle and their applications to astronomy. Although we know today they are related to the tangent and cotangent functions, it appears the latter two developed quite independently of this context. In a day where time was measured by checking the length of the shadow cast by a vertical stick, observers noticed the shadow was extremely long at sunrise, cast no shadow at noon, and returned to extreme length at sunset. Records tracking shadow length in this way date back as far as 1500 B.C. in ancient Egypt, and are the precursor of our modern tangent and cotangent functions.

## A. The Graph of $y = \tan t$

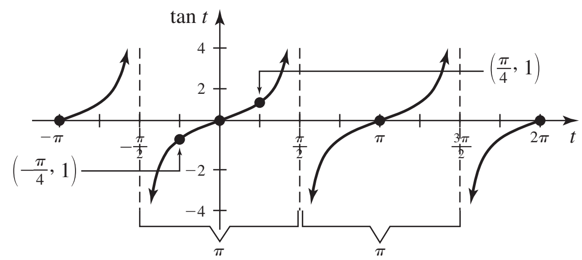
Like the secant and cosecant functions, tangent is defined in terms of a ratio that creates asymptotic behavior at the zeroes of the denominator. In terms of the unit circle, $\tan t = \frac{y}{x}$, which means in $[-\pi, 2\pi]$, vertical asymptotes occur at $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$, and $\frac{3\pi}{2}$, since the $x$-coordinate on the unit circle is zero. We further note $\tan t = 0$ when the $y$-coordinate is zero, so the function will have $t$-intercepts at $t = -\pi, 0, \pi$, and $2\pi$ in the same interval. This produces the framework for graphing the tangent function shown in Figure 6.53, where we note $y = \tan t$ has a period of $P = \pi$.

**Figure 6.53**




Knowing the graph must go through these zeroes and approach the asymptotes, we are left with determining the *direction of the approach.* This can be discovered by noting that in QI, the $y$-coordinates of points on the unit circle start at 0 and increase, while the $x$-values start at 1 and decrease. This means the ratio $\frac{y}{x}$ defining $\tan t$ is increasing, and in fact becomes infinitely large as $t$ gets very close to $\frac{\pi}{2}$. A similar observation can be made for a negative rotation of $t$ in QIV. Using the additional points provided by $\tan\left(-\frac{\pi}{4}\right) = -1$ and $\tan\left(\frac{\pi}{4}\right) = 1$, we find the graph of $\tan t$ is increasing throughout the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that the function has a period of $\pi$. We also note $y = \tan t$ is an odd function (symmetric about the origin), since $\tan(-t) = -\tan t$ as evidenced by the two points just computed. The completed graph is shown in Figure 6.54.

**Figure 6.54**




The graph can also be developed by noting the ratio relationship that exists between $\sin t$, $\cos t$, and $\tan t$. In particular, since $\sin t = y$, $\cos t = x$, and $\tan t = \frac{y}{x}$, we have $\tan t = \frac{\sin t}{\cos t}$ by direct substitution. These and other relationships between the trig functions will be fully explored in Chapter 7.

**EXAMPLE 1** ▶ Complete Table 6.8 shown for $\tan t = \frac{y}{x}$ using the values given for $\sin t$, and $\cos t$, then graph the function by plotting points. Verify that your graph and the graph shown in Figure 6.54 are identical.

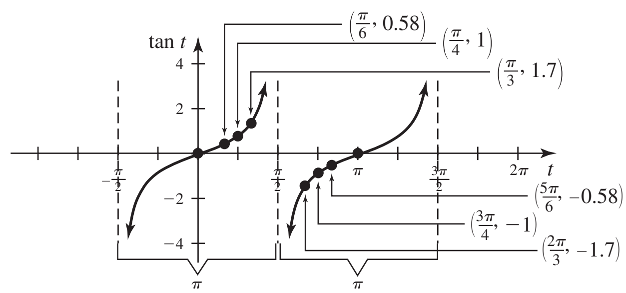
**Table 6.8**

| $t$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t = y$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ |
| $\cos t = x$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| $\tan t = \frac{y}{x}$ | | | | | | | | | |

**Solution:** ▶ For the noninteger values of $x$ and $y$, the "two's will cancel" each time we compute $\frac{y}{x}$. This means we can simply list the ratio of numerators. The resulting table (Table 6.9), plotted points, and graph are shown. The graph was completed using symmetry and the previous observations.

**Table 6.9**

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t = y$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $\cos t = x$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| $\tan t = \frac{y}{x}$ | 0 | $\frac{1}{\sqrt{3}} \approx 0.58$ | 1 | $\sqrt{3} \approx 1.7$ | undefined | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | 0 |



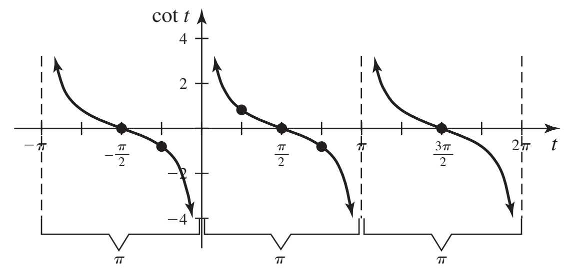

NOW TRY EXERCISES 7 AND 8

Additional values can be found using a calculator as needed. For future use and reference, it will help to recognize the approximate decimal equivalent of all standard values and radian angles. In particular, note that $\sqrt{3} \approx 1.73$ and $\frac{1}{\sqrt{3}} \approx 0.58$. See Exercises 9 through 14.

## B. The Graph of $y = \cot t$

Since the cotangent function is also defined in terms of a ratio, it too displays asymptotic behavior at the zeroes of the denominator. For $\cot t = \frac{x}{y}$ in $[-\pi, 2\pi]$, vertical asymptotes occur at $t = 0, \pi$, and $2\pi$, where $y = 0$ on the unit circle. The function will have $t$-intercepts at $t = -\frac{\pi}{2}, \frac{\pi}{2}$, and $\frac{3\pi}{2}$ in the same interval, where $x = 0$ on the unit circle. In QI, the $y$-values start at 0 and increase, while the $x$-values start at 1 and decrease. This means the ratio $\frac{x}{y}$ defining $\cot t$ is decreasing. A similar observation can be made as the rotation continues into QII (function values continue to decrease). Using the additional points provided by $\cot\left(\frac{\pi}{4}\right) = 1$ and $\cot\left(\frac{3\pi}{4}\right) = -1$, we find the graph is decreasing throughout the interval $(0, \pi)$, and has a period of $\pi$. Knowing the graph must go through the zeroes and plotted points, approach the asymptotes, and has period $\pi$, produces the graph in Figure 6.55, where we note the function is odd: $\cot(-t) = -\cot t$. Like the tangent function, $\cot t = \frac{x}{y}$ can be written in terms of $\cos t = x$ and $\sin t = y$: $\cot t = \frac{\cos t}{\sin t}$, and the graph could also be obtained by plotting points as in Example 2.

**Figure 6.55**




**EXAMPLE 2** Complete a table of values for $\cot t = \frac{x}{y}$ for $t \in [0, \pi]$ using its ratio relationship with $\cos t$ and $\sin t$. Then use the results to verify the location of the asymptotes and plotted points of the graph in Figure 6.55.

**Solution:** The completed table is shown here. In this interval, the cotangent function has asymptotes at 0 and $\pi$, since $y = 0$ at these points, and has a $t$-intercept at $\frac{\pi}{2}$ since $x = 0$. The points and asymptotes indicated in the table support the graph of $\cot t$ in Figure 6.55.

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t = y$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $\cos t = x$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| $\cot t = \frac{x}{y}$ | — | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $-1$ | $-\sqrt{3}$ | — |

NOW TRY EXERCISES 15 AND 16

## C. Characteristics of $y = \tan t$ and $y = \cot t$

The most important characteristics of the tangent and cotangent functions are summarized in the box. There is no discussion of amplitude, maximum, or minimum values, since maximum or minimum values do not exist. For future use and reference, perhaps the most significant characteristic distinguishing $\tan t$ from $\cot t$ is that *tan t increases,* while *cot t decreases* over their respective domains. Also note that due to symmetry, the zeroes of each function are always located halfway between the asymptotes.

**CHARACTERISTICS OF $y = \tan t$ and $y = \cot t$**

**$y = \tan t$**

| Unit Circle Definition | Domain | Range |
|---|---|---|
| $\tan t = \frac{y}{x}$ | $t \neq \frac{(2k+1)\pi}{2}$; $k \in \mathbb{Z}$ | $y \in (-\infty, \infty)$ |
| **Period** | **Behavior** | **Symmetry** |
| $\pi$ | increasing | Odd<br>$\tan(-t) = -\tan t$ |

**$y = \cot t$**

| Unit Circle Definition | Domain | Range |
|---|---|---|
| $\cot t = \frac{x}{y}$ | $t \neq k\pi$; $k \in \mathbb{Z}$ | $y \in (-\infty, \infty)$ |
| **Period** | **Behavior** | **Symmetry** |
| $\pi$ | decreasing | Odd<br>$\cot(-t) = -\cot t$ |

**EXAMPLE 3** For $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, what can you say about $\tan\left(\frac{7\pi}{6}\right)$, $\tan\left(\frac{13\pi}{6}\right)$, and $\tan\left(-\frac{5\pi}{6}\right)$?

**Solution:** Each of the arguments differs by a multiple of $\pi$: $\tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{\pi}{6} + \pi\right)$, $\tan\left(\frac{13\pi}{6}\right) = \tan\left(\frac{\pi}{6} + 2\pi\right)$ and $\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6} - \pi\right)$. Since the period of the tangent function is $p = \pi$, all of these expressions have a value of $\frac{1}{\sqrt{3}}$.

NOW TRY EXERCISES 17 THROUGH 20

Since the tangent function is more common than the cotangent and few calculators have a cotangent key, many needed calculations will first be done using the tangent function and its properties, then reciprocated. For instance, to evaluate $\cot\left(-\frac{\pi}{6}\right)$ we reason that $\cot t$ is an odd function, so $\cot\left(-\frac{\pi}{6}\right) = -\cot\left(\frac{\pi}{6}\right)$. Since cotangent is the reciprocal of tangent and $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $-\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$. Note how some of these ideas are used in Example 4.

**EXAMPLE 4** Given $t = \frac{\pi}{8}$ is a solution to $\cot t = 1 + \sqrt{2}$, use the period of the cotangent function to name all real roots. Check two of these roots using a calculator.

**Solution:** Since the period is $P = \pi$, solutions are given by $t = \frac{\pi}{8} + \pi k$ for all integers $k$. Using $k = -1$ and $k = 1$ yields $\cot\left(\frac{-7\pi}{8}\right)$ and $\cot\left(\frac{9\pi}{8}\right)$, respectively. A calculator shows $\tan\left(\frac{-7\pi}{8}\right) = \tan\left(\frac{9\pi}{8}\right) = \sqrt{2} - 1$ (approximately 0.4142), with the reciprocal value being $1 + \sqrt{2}$ (approximately 2.4142). The solutions check.

NOW TRY EXERCISES 21 THROUGH 24

## D. Graphing $y = A\tan(Bt)$ and $y = A\cot(Bt)$

### The Coefficient $A$: Vertical Stretches and Compressions

For the tangent and cotangent functions, the role of coefficient $A$ is best seen through an analogy from basic algebra (the concept of amplitude is foreign to the tangent and cotangent functions). Consider the graph of $y = x^3$ (Figure 6.56), which you may recall had the appearance of a vertical propeller (Section 3.3). Comparing the parent function

$y = x^3$ with functions $y = Ax^3$, the graph is stretched vertically if $|A| > 1$ (see Figure 6.57) and compressed if $0 < |A| < 1$. In the latter case the graph becomes very "flat" near the zeroes, as shown in Figure 6.58.

**Figure 6.56**

$y = x^3$

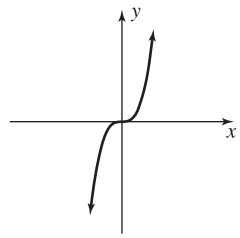



**Figure 6.57**

$y = 4x^3;\ A = 4$

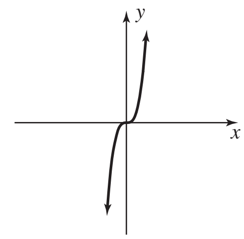
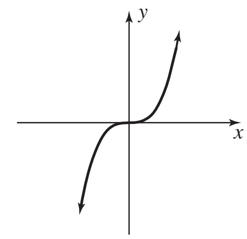



**Figure 6.58**

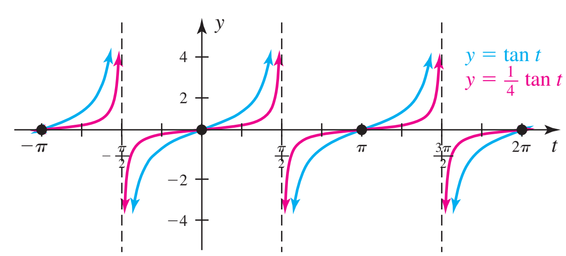
$y = \frac{1}{4}x^3;\ A = \frac{1}{4}$




While ***cubic functions are not asymptotic,*** they are a good illustration of $A$'s effect on the tangent and cotangent functions. Fractional values of $A$ compress the graph, flattening it out near its zeroes. Numerically, this is because a fractional part of a small quantity is an even smaller quantity. For instance, compare $\tan\left(\frac{\pi}{6}\right)$ with $\frac{1}{4}\tan\left(\frac{\pi}{6}\right)$. To two decimal places, $\tan\left(\frac{\pi}{6}\right) = 0.57$, while $\frac{1}{4}\tan\left(\frac{\pi}{6}\right) = 0.14$ so the graph must be "nearer the $t$-axis" at this value.

**EXAMPLE 5** Draw a "comparative sketch" of $y = \tan t$ and $y = \frac{1}{4}\tan t$ on the same axis and discuss similarities and differences. Use the interval $t \in [-\pi, 2\pi]$.

**Solution:** Both graphs will maintain their essential features (zeroes, asymptotes, increasing, and so on). However the graph of $y = \frac{1}{4}\tan t$ is vertically compressed, causing it to flatten out near its zeroes and changing how the graph approaches its asymptotes in each interval.




**NOW TRY EXERCISES 25 THROUGH 28**

**WORTHY OF NOTE**

It may be easier to interpret the phrase "twice as fast" as $2P = \pi$ and "one-half as fast" as $\frac{1}{2}P = \pi$. In each case, solving for $P$ gives the correct interval for the period of the new function.

### The Coefficient $B$: The Period of Tangent and Cotangent

Like the other trig functions, the value of $B$ has a material impact on the period of the function, and with the same effect. The graph of $y = \cot(2t)$ completes a cycle twice as fast as $y = \cot t$ $\left(P = \frac{\pi}{2} \text{ versus } P = \pi\right)$, while $y = \cot\left(\frac{1}{2}t\right)$ completes a cycle one-half as fast ($P = 2\pi$ versus $P = \pi$).

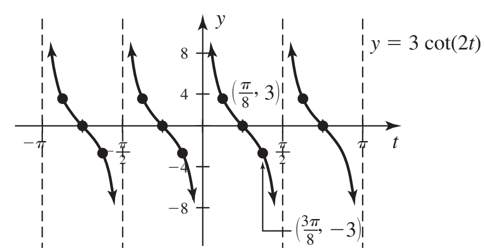
**WORTHY OF NOTE**

Similar to the four-step process used to graph sine and cosine functions, we can graph tangent and cotangent functions using a rectangle $P = \frac{\pi}{B}$ units in length and 2A units high, centered on the primary interval. After dividing the length of the rectangle into fourths, the *t*-intercept will always be the halfway point, with *y*-values of $|A|$ occuring at the $\frac{1}{4}$ and $\frac{3}{4}$ marks. See the last paragraph on page 642.

This type of reasoning leads us to a **period formula** for tangent and cotangent, namely, $P = \frac{\pi}{B}$, where $B$ is the coefficient of the input variable.

**EXAMPLE 6** Sketch the graph of $y = 3\cot(2t)$ over the interval $[-\pi, \pi]$.

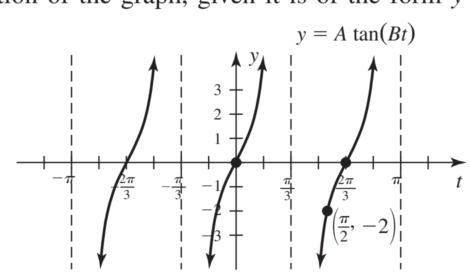
**Solution:** For $y = 3\cot(2t)$, $A = 3$ which results in a vertical stretch, and $B = 2$ which gives a period of $\frac{\pi}{2}$. The function is still undefined at $t = 0$ and is asymptotic there, then at all integer multiples of $P = \frac{\pi}{2}$. Selecting the inputs $t = \frac{\pi}{8}$ and $t = \frac{3\pi}{8}$ yields the points $\left(\frac{\pi}{8}, 3\right)$ and $\left(\frac{3\pi}{8}, -3\right)$, which we'll use along with the period and symmetry of the function to complete the graph:




**NOW TRY EXERCISES 29 THROUGH 40**

As with the trig functions from Section 6.3, it is possible to determine the equation of a tangent or cotangent function from a given graph. Where previously we noted the amplitude, period, and max/min values to obtain our equation, here we first determine the period of the function by calculating the "distance" between asymptotes, then choose any convenient point on the graph (other than an *x*-intercept) and substitute in the equation to solve for $A$.

**EXAMPLE 7** Find the equation of the graph, given it is of the form $y = A\tan(Bt)$.




**Solution:** Using the interval centered at the origin and the asymptotes at $t = -\frac{\pi}{3}$ and $t = \frac{\pi}{3}$, we find the period of the function is $P = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$. To find the value of $B$ we substitute in

$P = \frac{\pi}{B}$ and find $B = \frac{3}{2}$. This gives the equation $y = A\tan\left(\frac{3}{2}t\right)$. To find $A$, we take the point $\left(\frac{\pi}{2}, -2\right)$ given, and use $t = \frac{\pi}{2}$ with $y = -2$ to solve for $A$:

$$y = A\tan\left(\frac{3}{2}t\right) \quad \text{substitute } \frac{3}{2} \text{ for } B$$

$$-2 = A\tan\left[\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right)\right] \quad \text{substitute } -2 \text{ for } y \text{ and } \frac{\pi}{2} \text{ for } t$$

$$-2 = A\tan\left(\frac{3\pi}{4}\right) \quad \text{multiply}$$

$$A = \frac{-2}{\tan\left(\frac{3\pi}{4}\right)} \quad \text{solve for } A$$

$$= 2 \quad \text{result}$$

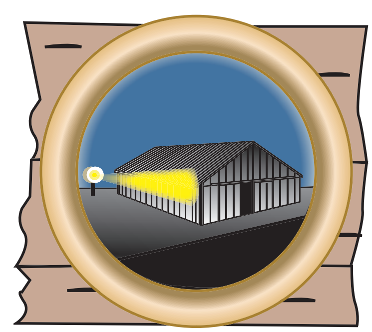
The equation of the graph is $y = 2\tan\left(\frac{3}{2}t\right)$.

NOW TRY EXERCISES 41 THROUGH 46

## E. Applications of Tangent and Cotangent Functions

We end this section with one example of how tangent and cotangent functions can be applied. Numerous others can be found in the exercise set.

**EXAMPLE 8** One evening, in port during a *Semester at Sea,* Marlon is debating a project choice for his Precalculus class. Looking out his porthole, he notices a revolving light turning at a constant speed near the corner of a long warehouse. The light throws its beam along the length of the warehouse, then disappears into the air, and then returns time and time again. Suddenly—Marlon has his project. He notes the time it takes the beam to traverse the warehouse wall is exactly 4 sec, and in the morning he measures the wall's length at 127.26 m. His project? Modeling the distance of the beam from the corner of the warehouse with a tangent function. Can you help?



**Solution:** The equation model will have the form $D(t) = A\tan(Bt)$, where $D(t)$ is the distance (in meters) of the beam from the corner after $t$ sec. The distance along the wall is measured in positive values so we're using only $\frac{1}{2}$ the period of the function, giving $\frac{1}{2}P = 4$ (the beam "disappears" at $t = 4$) and $P = 8$. Substitution in the period formula gives $B = \frac{\pi}{8}$ and the equation $D = A\tan\left(\frac{\pi}{8}t\right)$. Knowing the beam travels 127.26 m in $t = 4$ sec (when it disappears into infinity), we'll use $t = 3.9$ and 127.26 for $D$ in order to solve for $A$ and complete our equation model (see note following).

$$A \tan\left(\frac{\pi}{8}t\right) = D \qquad \text{equation model}$$

$$A \tan\left[\frac{\pi}{8}(3.9)\right] = 127.26 \qquad \text{substitute 127.26 for } D \text{ and 3.9 for } t$$

$$A = \frac{127.26}{\tan\left[\frac{\pi}{8}(3.9)\right]} \qquad \text{solve for } A$$

$$\approx 5 \qquad \text{result}$$

One equation modeling the distance of the beam from the corner of the warehouse is $D(t) = 5\tan\left(\frac{\pi}{8}t\right)$.

**NOW TRY EXERCISES 49 THROUGH 52**

For Example 8, we should note the choice of 3.9 for $t$ was very arbitrary, and while we obtained an "acceptable" model, different values of $A$ would be generated for other choices. For instance, $t = 3.95$ gives $A \approx 2.5$, while $t = 3.99$ gives $A \approx 0.5$. The true value of $A$ depends on the distance of the light from the corner of the warehouse wall. In any case, it's interesting to note that at $t = 2$ sec (one-half the time it takes the beam to disappear), the beam has traveled only 5 m from the corner of the building: $D(2) = 5\tan\left(\frac{\pi}{4}\right) = 5$ m. Although the light is rotating at a constant angular speed, the speed of the beam along the wall increases *dramatically* as it moves along and $t$ gets very close to 4 sec.

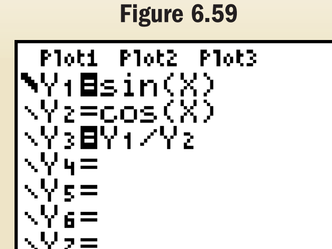
## TECHNOLOGY HIGHLIGHT

### Zeroes, Asymptotes, and the Tangent/Cotangent Functions

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

In this *Technology Highlight* we'll explore the tangent and cotangent functions from the perspective of their ratio definition. While we could easily use $Y_1 = \tan x$ to generate and explore the graph, we would miss an opportunity to note the many important connections that emerge from a ratio definition perspective. To begin, enter $Y_1 = \sin x$, $Y_2 = \cos x$, and $Y_3 = \frac{Y_1}{Y_2}$, as shown in Figure 6.59 [recall that function variables are accessed using VARS ▶ **(Y-VARS)** ENTER **(1:Function)]**. Note that $Y_2$ has been disabled by overlaying the cursor on the equal sign and pressing ENTER. In addition, note the slash next to $Y_1$ is more **bold** than the other slashes. The TI-84 Plus offers options that help distinguish between graphs when more that one is being displayed on the GRAPH

**Figure 6.59**



screen, and we selected a **bold** line for $Y_1$ by moving the cursor to the far left position and repeatedly pressing ENTER until the desired option appeared. Pressing ZOOM **7:ZTrig** at this point produces the screen shown in Figure 6.60, where we immediately note that tan $x$ is zero everywhere that sin $x$ is zero. This is hardly surprising since $\tan x = \frac{\sin x}{\cos x}$, but is a point that is often overlooked. Going back to the Y= and disabling $Y_1$ while enabling $Y_2$ will produce the graph shown in Figure 6.61 where similar observations can be made.

**Figure 6.60**

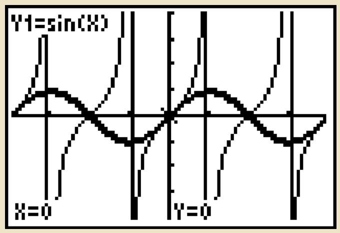
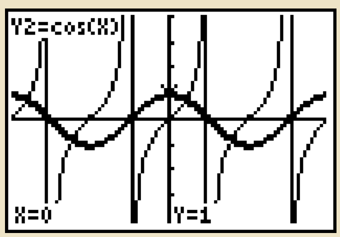



**Figure 6.61**




Exercise 1: What do you notice about the zeroes of cos $x$ as they relate to the graph of $Y_3 = \tan x$?

Exercise 2: Going back to the graph of $Y_1$ and $Y_3$, from $Y_3$ we note the tangent function is increasing everywhere it is defined. What do you notice about the increasing/decreasing intervals for sin $x$ as they relate to tan $x$? What do you notice about the intervals where each function is positive or negative?

Exercise 3: Go to the Y= screen and change $Y_3$ from $\frac{Y_1}{Y_2}$ (tangent) to $\frac{Y_2}{Y_1}$ (cotangent), then graph $Y_2$ and $Y_3$ on the same screen. From the graph of $Y_3$ we note the cotangent function in decreasing everywhere it is defined. What do you notice about the increasing/decreasing intervals for cos $x$ as they relate to cot $x$? What do you notice about the intervals where each function is positive or negative?

# 6.4 EXERCISES

## CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** The period of $y = \tan t$ and $y = \cot t$ is __________. To find the period of $y = \tan(Bt)$ and $y = \cot(Bt)$, the formula __________ is used.

**2.** The function $y = \tan t$ is __________ everywhere it is defined. The function $y = \cot t$ is __________ everywhere it is defined.

**3.** Tan $t$ and cot $t$ are _______ functions, so $f(-t) =$ __________. If $\tan\left(-\frac{11\pi}{12}\right)$, $\approx 0.268$, then $\tan\left(\frac{11\pi}{12}\right) \approx$ _______.

**4.** The asymptotes of $y =$ _______ are located at odd multiples of $\frac{\pi}{2}$. The asymptotes of $y =$ _______ are located at integer multiples of $\pi$.

**5.** Discuss/explain how you can obtain a table of values for $y = \cot t$ (a) given the values for $y = \sin t$ and $y = \cos t$ and (b) given the values for $y = \tan t$.

**6.** Explain/discuss how the zeroes of $y = \sin t$ and $y = \cos t$ are related to the graphs of $y = \tan t$ and $y = \cot t$. How can these relationships help graph functions of the form $y = A\tan(Bt)$ and $y = A\cot(Bt)$?

## DEVELOPING YOUR SKILLS

Use the values given for sin $t$ and cos $t$ to complete the table.

**7.**

| $t$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin t = y$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| $\cos t = x$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |
| $\tan t = \frac{y}{x}$ | | | | | |

**8.**

| | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|
| $\sin t = y$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |
| $\cos t = x$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\tan t = \frac{y}{x}$ | | | | | |

**9.** Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$\frac{\pi}{2}$ $\frac{\pi}{4}$ $\frac{\pi}{6}$ $\sqrt{2}$ $\frac{\sqrt{2}}{2}$ $\frac{2}{\sqrt{3}}$

**10.** Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$\frac{\pi}{3}$ $\pi$ $\frac{3\pi}{2}$ $\sqrt{3}$ $\frac{\sqrt{3}}{2}$ $\frac{1}{\sqrt{3}}$

**11.** State the value of each expression without the use of a calculator.

**a.** $\tan\left(-\frac{\pi}{4}\right)$ **b.** $\cot\left(\frac{\pi}{6}\right)$

**c.** $\cot\left(\frac{3\pi}{4}\right)$ **d.** $\tan\left(\frac{\pi}{3}\right)$

**12.** State the value of each expression without the use of a calculator.

**a.** $\cot\left(\frac{\pi}{2}\right)$ **b.** $\tan \pi$

**c.** $\tan\left(-\frac{5\pi}{4}\right)$ **d.** $\cot\left(-\frac{5\pi}{6}\right)$

**13.** State the value of each expression without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

**a.** $\tan^{-1}(-1)$; QIV

**b.** $\cot^{-1}\sqrt{3}$; QIII

**c.** $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; QIV

**d.** $\tan^{-1}(-1)$; QII

**14.** State the value of each expression without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

**a.** $\cot^{-1}1$; QI

**b.** $\tan^{-1}(-\sqrt{3})$; QII

**c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$; QI

**d.** $\cot^{-1}1$; QIII

Use the values given for sin $t$ and cos $t$ to complete the table.

**15.**

| $t$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin t = y$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| $\cos t = x$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |
| $\cot t = \frac{x}{y}$ | | | | | |

**16.**

| | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|
| $\sin t = y$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |
| $\cos t = x$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cot t = \frac{x}{y}$ | | | | | |

**17.** Given $t = \frac{11\pi}{24}$ is a solution to $\tan t \approx 7.6$, use the period of the function to name three additional solutions. Check your answer using a calculator.

**18.** Given $t = \frac{7\pi}{24}$ is a solution to $\cot t \approx 0.77$, use the period of the function to name three additional solutions. Check your answer using a calculator.

**19.** Given $t \approx 1.5$ is a solution to $\cot t = 0.07$, use the period of the function to name three additional solutions. Check your answer using a calculator.

**20.** Given $t \approx 1.25$ is a solution to $\tan t = 3$, use the period of the function to name three additional solutions. Check your answer using a calculator.

Verify the value shown for $t$ is a solution to the equation given, then use the period of the function to name all real roots. Check two of these roots on a calculator.

**21.** $t = \frac{\pi}{12}$; $\cot t = 2 + \sqrt{3}$

**22.** $t = \frac{5\pi}{12}$; $\cot t = 2 - \sqrt{3}$.

**23.** $t = \frac{\pi}{10}$; $\tan t \approx 0.3249$

**24.** $t = -\frac{\pi}{16}$; $\tan t \approx -0.1989$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and value of $A$. Include a comparative sketch of $y = \tan t$ or $y = \cot t$ as indicated.

**25.** $f(t) = 2\tan t$; $[-2\pi, 2\pi]$

**26.** $g(t) = \frac{1}{2}\tan t$; $[-2\pi, 2\pi]$

**27.** $h(t) = 3\cot t$; $[-2\pi, 2\pi]$

**28.** $r(t) = \frac{1}{4}\cot t$; $[-2\pi, 2\pi]$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and value of $A$.

**29.** $y = \tan(2t)$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

**30.** $y = \tan\left(\frac{1}{4}t\right)$; $[-4\pi, 4\pi]$

**31.** $y = \cot(4t)$; $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

**32.** $y = \cot\left(\frac{1}{2}t\right)$; $[-2\pi, 2\pi]$

**33.** $y = 2\tan(4t)$; $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

**34.** $y = 4\tan\left(\frac{1}{2}t\right)$; $[-2\pi, 2\pi]$

**35.** $y = 5\cot\left(\frac{1}{3}t\right)$; $[-3\pi, 3\pi]$

**36.** $y = \frac{1}{2}\cot(2t)$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

**37.** $y = 3\tan(2\pi t)$; $\left[-\frac{1}{2}, \frac{1}{2}\right]$

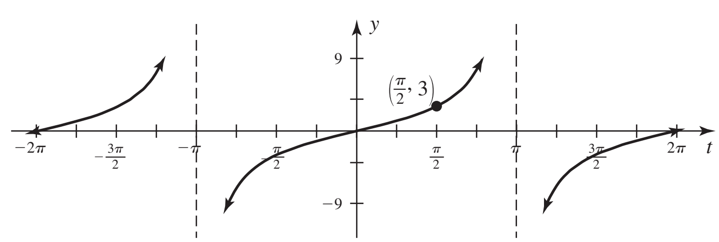
**38.** $y = 4\tan\left(\frac{\pi}{2}t\right)$; $[-2, 2]$

**39.** $f(t) = 2\cot(\pi t)$; $[-1, 1]$

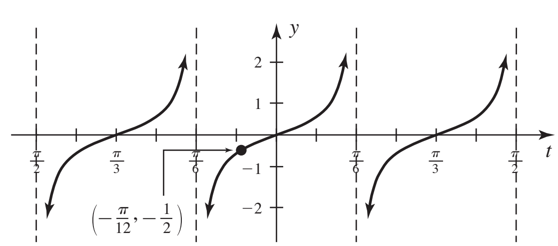
**40.** $p(t) = \frac{1}{2}\cot\left(\frac{\pi}{4}t\right)$; $[-4, 4]$

Find the equation of each graph, given it is of the form $y = A\tan(Bt)$.

**41.**

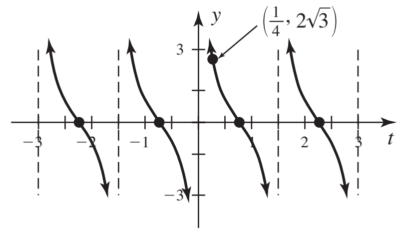



**42.**

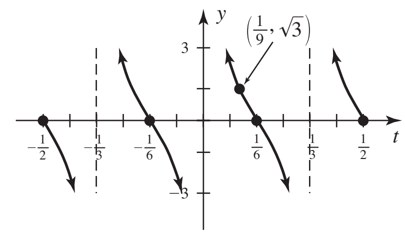


Find the equation of each graph, given it is of the form $y = A\cot(Bt)$.

**43.**




**44.**



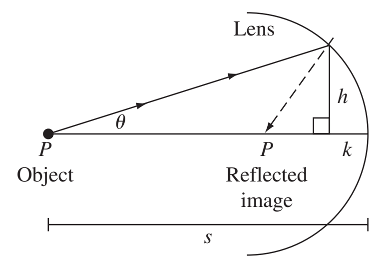

**45.** Given that $t = -\frac{\pi}{8}$ and $t = -\frac{3\pi}{8}$ are solutions to $\cot(3t) = \tan t$, use a graphing calculator to find two additional solutions in $[0, 2\pi]$.

**46.** Given $t = \frac{1}{6}$ is a solution to $\tan(2\pi t) = \cot(\pi t)$, use a graphing calculator to find two additional solutions in $[-1, 1]$.

## WORKING WITH FORMULAS

**47. Position of an image reflected from a spherical lens:** $\tan\theta = \dfrac{h}{s - k}$

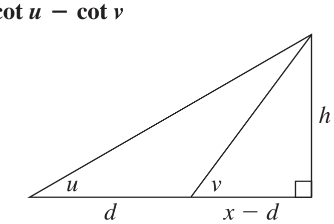
The equation shown is used to help locate the position of an image reflected by a spherical mirror, where $s$ is the distance of the object from the lens along a horizontal axis, $\theta$ is the angle of elevation from this axis, $h$ is the altitude of the right triangle indicated, and $k$ is distance from the lens to the foot of altitude $h$.




Find the distance $k$ given $h = 3$ mm, $\theta = \frac{\pi}{24}$, and that the object is 24 mm from the lens.

**48. The height of an object calculated from a distance:** $h = \dfrac{d}{\cot u - \cot v}$

The height $h$ of a tall structure can be computed using two angles of elevation measured some distance apart along a straight line with the object. This height is given by the formula shown, where $d$ is the distance between the two points from which angles $u$ and $v$ were measured. Find the height $h$ of a building if $u = 40°$, $v = 65°$, and $d = 100$ ft.




## APPLICATIONS

**Tangent function data models:** Model the data in Exercises 49 and 50 using the function $y = A\tan(Bx)$. State the period of the function, the location of the asymptotes, the value of $A$, and name the point $(x, y)$ used to calculate $A$ (answers may vary). Use your equation model to evaluate the function at $x = -2$ and $x = 2$. What observations can you make? Also see Exercise 58.

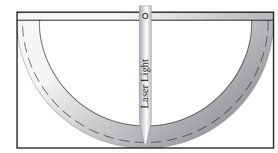
**49.**

| Input | Output | Input | Output |
|---|---|---|---|
| −6 | $-\infty$ | 1 | 1.4 |
| −5 | −20 | 2 | 3 |
| −4 | −9.7 | 3 | 5.2 |
| −3 | −5.2 | 4 | 9.7 |
| −2 | −3 | 5 | 20 |
| 1 | −1.4 | 6 | $\infty$ |
| 0 | 0 | | |

**50.**

| Input | Output | Input | Output |
|---|---|---|---|
| −3 | $-\infty$ | 0.5 | 6.4 |
| −2.5 | −91.3 | 1 | 13.7 |
| −2 | −44.3 | 1.5 | 23.7 |
| −1.5 | −23.7 | 2 | 44.3 |
| −1 | −13.7 | 2.5 | 91.3 |
| −0.5 | −6.4 | 3 | $\infty$ |
| 0 | 0 | | |

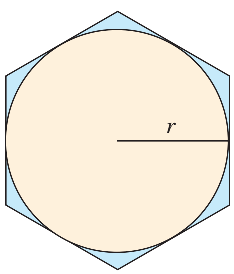
**51.** As part of a lab setup, a laser pen is made to swivel on a large protractor as illustrated in the figure. For their lab project, students are asked to take the instrument to one end of a long hallway and measure the distance of the projected beam relative to the angle the pen is being held, and collect the data in a table. Use the data to find a function of the form $y = A\tan(B\theta)$. State the period of the function, the location of the asymptotes, the value of $A$ and name the point $(\theta, y)$ you used to calculate $A$ (answers may vary). Based on the result, can you approximate the length of the laser pen? Note that in degrees, the period formula for tangent is $P = \frac{180°}{B}$.




| $\theta$ (degrees) | Distance (cm) |
|---|---|
| 0 | 0 |
| 10 | 2.1 |
| 20 | 4.4 |
| 30 | 6.9 |
| 40 | 10.1 |
| 50 | 14.3 |
| 60 | 20.8 |
| 70 | 33.0 |
| 80 | 68.1 |
| 89 | 687.5 |

**52.** Use the equation model obtained in Exercise 51 to compare the values given by the equation to the actual data. As a percentage, what was the largest deviation between the two?

Exercise 53




**53. Circumscribed polygons:** The *perimeter* of a regular polygon circumscribed about a circle of radius $r$ is given by $P = 2nr\tan\left(\frac{\pi}{n}\right)$, where $n$ is the number of sides $(n \geq 3)$ and $r$ is the radius of the circle. Given $r = 10$ cm, (a) What is the perimeter of the circle? (b) What is the perimeter of the polygon when $n = 4$? Why? (c) Calculate the perimeter of the polygon for $n = 10, 20, 30$, and 100. What do you notice?

**54. Circumscribed polygons:** The area of a regular polygon circumscribed about a circle of radius $r$ is given by $A = nr^2\tan\left(\frac{\pi}{n}\right)$, where $n$ is the number of sides $(n \geq 3)$ and $r$ is the radius of the circle. Given $r = 10$ cm,

**a.** What is the area of the circle?

**b.** What is the area of the polygon when $n = 4$? Why?

**c.** Calculate the area of the polygon for $n = 10, 20, 30$, and 100. What do you notice?

**Coefficients of friction:** Pulling someone on a sled is much easier during the winter than in the summer, due to a phenomenon known as the *coefficient of friction*. The friction between the sled's skids and the snow is much lower than the friction between the skids and the dry ground or pavement. Basically, the coefficient of friction is defined by the relationship $\mu = \tan\theta$, where $\theta$ is the angle at which a block composed of one material will slide down an inclined plane made of another material, with a constant velocity. Coefficients of friction have been established experimentally for many materials and a short list is shown here.

| Material | Coefficient |
|---|---|
| steel on steel | 0.74 |
| copper on glass | 0.53 |
| glass on glass | 0.94 |
| copper on steel | 0.68 |
| wood on wood | 0.5 |

**55.** Graph the function $\mu = \tan\theta$ with $\theta$ in degrees over the interval $[0°, 60°]$ and use the graph to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

**a.** A block of copper is placed on a sheet of steel, which is slowly inclined. Is the block of copper moving when the angle of inclination is 30°? At what angle of inclination will the copper block be moving with a constant velocity down the incline?

**b.** A block of copper is placed on a sheet of cast-iron. As the cast-iron sheet is slowly inclined, the copper block begins sliding at a constant velocity when the angle of inclination is approximately 46.5°. What is the coefficient of friction for copper on cast-iron?

**c.** Why do you suppose coefficients of friction greater than $\mu = 2.5$ are extremely rare? Give an example of two materials that likely have a high $\mu$-value.

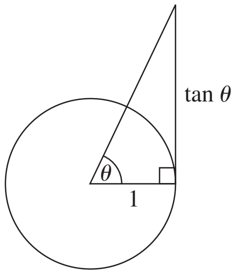
**56.** Graph the function $\mu = \tan\theta$ with $\theta$ in radians over the interval $\left[0, \frac{5\pi}{12}\right]$ and use the graph to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

**a.** A block of glass is placed on a sheet of glass, which is slowly inclined. Is the block of glass moving when the angle of inclination is $\frac{\pi}{4}$? What is the smallest angle of inclination for which the glass block will be moving with a constant velocity down the incline (rounded to four decimal places)?

**b.** A block of Teflon is placed on a sheet of steel. As the steel sheet is slowly inclined, the Teflon block begins sliding at a constant velocity when the angle of inclination is approximately 0.04. What is the coefficient of friction for Teflon on steel?

**c.** Why do you suppose coefficients of friction less than $\mu = 0.04$ are extremely rare for two solid materials? Give an example of two materials that likely have a very low $\mu$ value.

**57. Tangent lines:** The actual definition of the word *tangent* comes from the Latin *tangere*, meaning "to touch." In mathematics, a tangent line touches the graph of a circle at only one point and function values for $\tan\theta$ are obtained from the length of the line segment tangent to a unit circle.




**a.** What is the length of the line segment when $\theta = 80°$?

**b.** If the line segment is 16.35 units long, what is the value of $\theta$?

**c.** Can the line segment ever be greater than 100 units long? Why or why not?

**d.** How does your answer to (c) relate to the asymptotic behavior of the graph?

## WRITING, RESEARCH, AND DECISION MAKING

**58.** Rework Exercises 49 and 50, obtaining a new equation for the data using a different ordered pair to compute the value of $A$. What do you notice? Try yet another ordered pair and calculate $A$ once again for another equation $Y_2$. Complete a table of values using the given inputs, with the outputs of the three equations generated (original, $Y_1$, and $Y_2$). Does any one equation seem to model the data better than the others? Are all of the equation models "acceptable"? Please comment.

**59.** The golden ratio $\frac{-1 + \sqrt{5}}{2}$ has long been thought to be the most pleasing ratio in art and architecture. It is commonly believed that many forms of ancient architecture were constructed using this ratio as a guide. The ratio actually turns up in some surprising places, far removed from its original inception as a line segment cut in "mean and extreme" ratio. Do some research on the golden ratio and some of the equations that have been used to produce it. Given $x = 0.6662394325$, try to find a connection between $y = \cos x$, $y = \tan x$ and $y = \sin x$.

## EXTENDING THE CONCEPT

**60.** Regarding Example 8, we can use the standard distance/rate/time formula $D = RT$ to compute the average velocity of the beam of light along the wall in any interval of time: $R = \frac{D}{T}$. For example, using $D(t) = 5\tan\left(\frac{\pi}{8}t\right)$, the average velocity in the interval $[0, 2]$ is $\frac{D(2) - D(0)}{2 - 0} =$ 2.5 m/sec. Calculate the average velocity of the beam in the time intervals $[2, 3]$, $[3, 3.5]$, and $[3.5, 3.8]$ sec. What do you notice? How would the average velocity of the beam in the interval $[3.9, 3.99]$ sec compare?

**61.** Determine the slope of the line drawn *through* the parabola (called a **secant** line) in Figure I. Use the same method (any two points on the line) to calculate the slope of the line drawn **tangent** to the parabola in Figure II. Compare your calculations to the tangent of the angles $\alpha$ and $\beta$ that each line makes with the $x$-axis. What can you conclude? Write a formula for the point/slope equation of a line using $\tan\theta$ instead of $m$.

**Figure I**

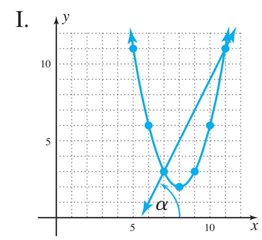
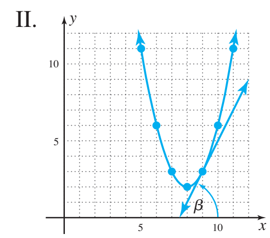


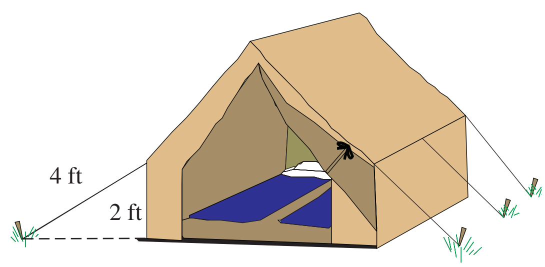

**Figure II**



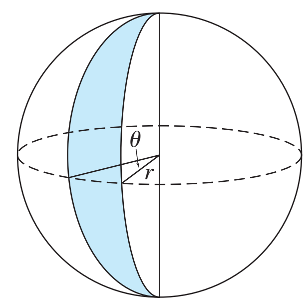

## MAINTAINING YOUR SKILLS

**62.** (2.1) A tent rope is 4 ft long and attached to the tent wall 2 ft above the ground. How far from the tent is the stake holding the rope?

**Exercise 62**




**Exercise 63**




**63.** (6.1) A lune is a section of surface area on a sphere, which is subtended by an angle $\theta$ at the circumference. For $\theta$ *in radians,* the surface area of a lune is $A = 2r^2\theta$, where $r$ is the radius of the sphere. Find the area of a lune on the surface of the earth which is subtended by an angle of 15°. Assume the radius of the Earth is 6373 km.

**64.** (4.4/4.5) Find the $y$-intercept, $x$-intercept(s), and all asymptotes of each function, but do not graph.

**a.** $h(x) = \dfrac{3x^2 - 9x}{2x^2 - 8}$ **b.** $t(x) = \dfrac{x + 1}{x^2 - 4x}$ **c.** $p(x) = \dfrac{x^2 - 1}{x + 2}$

**65.** (6.2) State the points on the unit circle that correspond to $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{4}, \frac{3\pi}{2}$, and $2\pi$. What is the value of $\tan\left(\frac{\pi}{2}\right)$? Why?

**66.** (6.3) Given $\sin 212° \approx -0.53$, find another angle $\theta$ in $[0°, 360°]$ that satisfies $\sin\theta \approx -0.53$ without using a calculator.

**67.** (5.1) The radioactive element potassium-42 is sometimes used as a tracer in certain biological experiments, and its decay can be modeled by the formula $Q(t) = Q_0e^{-0.055t}$, where $Q(t)$ is the amount that remains after $t$ hours. If 15 grams (g) of potassium-42 are initially present, how many hours until only 10 g remain?

# MID-CHAPTER CHECK

**1.** The city of Las Vegas, Nevada, is located at 36°06′36″ north latitude, 115°04′48″ west longitude. (a) Convert both measures to decimal degrees. (b) If the radius of the Earth is 3960 mi, how far north of the equator is Las Vegas?

**2.** Find the angle subtended by the arc shown in the figure, then determine the area of the sector.

**3.** Evaluate without using a calculator: (a) cot 60° and (b) $\sin\left(\frac{7\pi}{4}\right)$.

**4.** Evaluate using a calculator: (a) $\sec\left(\frac{\pi}{12}\right)$ and (b) tan 4.3.

**5.** Complete the ordered pair indicated on the unit circle in the figure and find the value of all six trigonometric functions at this point.

**6.** For the point on the unit circle in Exercise 5, find the related angle $t$ in both degrees (to tenths) and radians (to ten-thousandths).

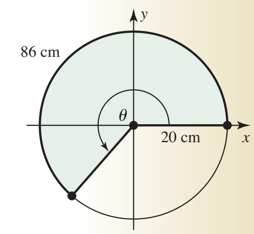
**7.** Name the location of the asymptotes and graph $y = 3\tan\left(\frac{\pi}{2}t\right)$ for $t \in [-2\pi, 2\pi]$.

**8.** Clearly state the amplitude and period, then sketch the graph: $y = -3\cos\left(\frac{\pi}{2}t\right)$.

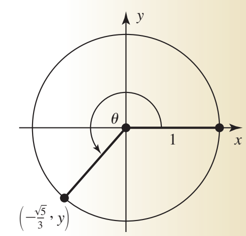
**9.** On a unit circle, if arc $t$ has length 5.94, (a) in what quadrant does it terminate? (b) What is its reference arc? (c) Of sin $t$, cos $t$, and tan $t$, which are negative for this value of $t$?

**10.** For the graph given here, (a) clearly state the amplitude and period; (b) find the equation of the graph; (c) graphically find $f(\pi)$ and then confirm/contradict your estimation using a calculator.

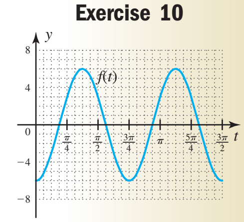
**Exercise 2**




**Exercise 5**




**Exercise 10**

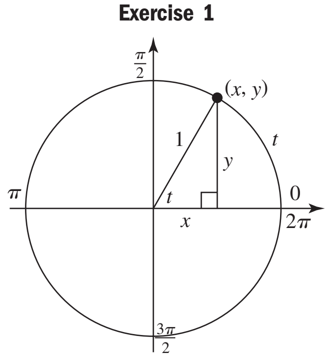



# REINFORCING BASIC CONCEPTS

## Trigonometric Potpourri

This *Reinforcing Basic Concepts* is simply a collection of patterns, observations, hints, and reminders connected with an introduction to trigonometry. Individually, the points may seem trivial, but taken together they tend to reinforce the core fundamentals of trig, and enable a student to sequence and store the ideas in their own way. Having these basic elements available for instant retrieval builds a stronger bridge to future concepts, assists in the discovery of additional connections, and enables a closer tie between these concepts and the real-world situations to which they will be applied. Just a little work now, pays big dividends later. As Louis Pasteur once said, "Fortune favors a prepared mind."

**Exercise 1**




**1.** The collection begins with an all-encompassing view of the trig functions, as seen by the imposition of a right triangle in a unit circle, on the coordinate grid. This allows all three approaches to trigonometry to be seen at one time:

$$\text{right triangle: } \cos t = \frac{\text{adj}}{\text{hyp}} \qquad \text{any angle: } \cos t = \frac{x}{r}$$

$$\text{real number: } \cos t = x$$

Should you ever forget the association of *x with cosine and y with sine,* just remember that *c* comes before *s* in the alphabet in the same way the *x* comes before y: $\cos t \to x$ and $\sin t \to y$.

**2.** *Know the standard angles and the standard values.* As mentioned earlier, they are used repeatedly throughout higher mathematics to introduce new concepts and skills without the clutter and distraction of large decimal values. It is interesting to note the standard values (from QI) can always be recreated using the *pattern of fourths*. Simply write the fractions with integer numerators from $\frac{0}{4}$ through $\frac{4}{4}$, and take their square root:

$$\sqrt{\frac{0}{4}} = 0 \qquad \sqrt{\frac{1}{4}} = \frac{1}{2} \qquad \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} \qquad \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \qquad \sqrt{\frac{4}{4}} = 1$$

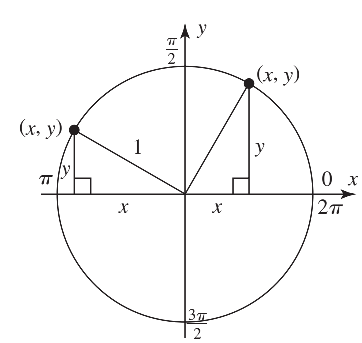
**3.** There are many *decimal equivalents* in elementary mathematics that contribute to concept building. In the same way you recognize the decimal values of $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and others, it helps *tremendously* to recognize or "know" decimal equivalents for values that are commonly used in a study of trig. They help to identify the quadrant of an arc/angle's terminal side when $t$ is expressed in radians, they assist in graphing and estimation skills, and are used extensively throughout this text and in other areas of mathematics. In terms of $\pi$ these are $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, and $2\pi$. In terms of radicals they are $\sqrt{2}$ and $\frac{\sqrt{2}}{2}$ and $\sqrt{3}$ and $\frac{\sqrt{3}}{2}$. Knowing $\pi \approx 3.14$ leads directly to $\frac{\pi}{2} \approx 1.57$ and $2\pi \approx 6.28$, while adding the decimal values for $\pi$ and $\frac{\pi}{2}$ gives $\frac{3\pi}{2} \approx 4.71$. Further, since $\sqrt{2} \approx 1.41$, $\frac{\sqrt{2}}{2} \approx 0.7$, and since $\sqrt{3} \approx 1.73$, $\frac{\sqrt{3}}{2} \approx 0.87$.

**4.** To specifically remember what standard value is associated with a standard angle or arc, recall that

**a.** If $t$ is a quadrantal arc/angle, it's easiest to use the coordinates $(x, y)$ of a point on a unit circle.

**b.** If $t$ is any odd multiple of $\frac{\pi}{4}$, $\sin t$ and $\cos t$ must be $-\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{2}}{2}$, with the choice depending on the quadrant of the terminal side. The value of the other functions can be found using these.

**c.** If $t$ is a multiple of $\frac{\pi}{6}$ (excluding the quadrantal angles), the value for sine and cosine must be either $\pm\frac{1}{2}$ or $\pm\frac{\sqrt{3}}{2}$, depending on the quadrant of the terminal side. If there's any hesitation about which value applies to sine and which to cosine, mental imagery can once again help. Since $\frac{\sqrt{3}}{2} \approx 0.87 > \frac{1}{2} = 0.5$, we simply apply



the larger value to the larger arc/angle. Note that for the triangle drawn in QI $\left(\frac{\pi}{3} = 60° \text{ angle at vertex}\right)$, $y$ is obviously longer than $x$, meaning the association must be $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$. For the triangle in QII $\left(\frac{5\pi}{6} = 150° \text{ whose reference angle is } 30°\right)$, $x$ is longer than $y$, meaning the association must be $\cos t = -\frac{\sqrt{3}}{2}$ and $\sin t = \frac{1}{2}$.

**5.** Although they are often neglected or treated lightly in a study of trig, the secant, cosecant, and cotangent functions play an integral role in more advanced mathematics classes. Be sure you're familiar with their reciprocal relationship to the more common cosine, sine, and tangent functions:

$$\sec t = \frac{1}{\cos t}, \quad \cos t = \frac{1}{\sec t}, \quad \sec t \cos t = 1; \quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \rightarrow \sec\left(\frac{2\pi}{3}\right) = -2$$

$$\csc t = \frac{1}{\sin t}, \quad \sin t = \frac{1}{\csc t}, \quad \csc t \sin t = 1; \quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \rightarrow \csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}}$$

$$\cot t = \frac{1}{\tan t}, \quad \tan t = \frac{1}{\cot t}, \quad \cot t \tan t = 1; \quad \tan 60° = \sqrt{3} \rightarrow \cot 60° = \frac{1}{\sqrt{3}}$$

**6.** Finally, the need to be very familiar with the basic graphs of the trig functions would be hard to overstate. As with transformations of the toolbox functions (from algebra), transformations of the basic trig graphs are a huge help to the understanding and solution of trig equations and inequalities, as well as to their application in the context of real-world phenomena.

# 6.5 Transformations and Applications of Trigonometric Graphs

### LEARNING OBJECTIVES

***In Section 6.5 you will learn how to:***

**A. Apply vertical translations in context**

**B. Apply horizontal translations in context**

**C. Solve applications involving harmonic motion**

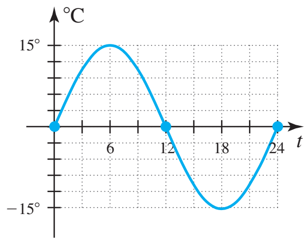
### INTRODUCTION

From your algebra experience, you may remember beginning with a study of linear graphs, then moving on to quadratic graphs and their characteristics. By combining and extending the knowledge you gained, you were able to investigate and understand a variety of polynomial graphs—along with some powerful applications. A study of trigonometry follows a similar pattern, and by "combining and extending" our understanding of the basic trig graphs, we'll look at some powerful applications in *this* section.

### POINT OF INTEREST

In some coastal areas, tidal motion is simple, predictable, and under perfect conditions can be approximated using a single sine function. However, at some of the northern latitudes and in other areas, a combination of factors make the motion much more complex, although still predictable. The mathematical model for these tides requires more than a single trig function, and serves to introduce a concept known as the *addition of ordinates.* Similar concepts also apply in many other areas, such as sound waves and musical tones.

**Figure 6.62**




## A. Vertical Translations: $y = A \sin(Bt) + D$

On any given day, outdoor temperatures tend to follow a **sinusoidal pattern,** or a pattern that can be modeled by a sine function. As the sun rises, the morning temperature begins to warm and rise until reaching its high in the late afternoon, then begins to cool over early evening and nighttime hours until falling to its nighttime low just prior to sunrise. Next morning, the cycle begins again. In the northern latitudes where the winters are very cold, it's not unreasonable to assume an average daily temperature of 0°C (32°F), and a temperature graph in degrees Celsius that looks like the one in Figure 6.62. For the moment, we'll assume that $t = 0$ corresponds to 12:00 noon.

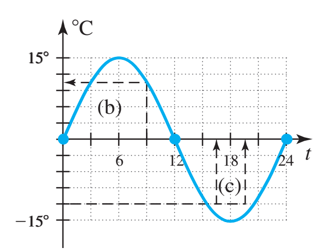
**EXAMPLE 1** Use the graph in Figure 6.62 to: (a) state the amplitude and period of the function, (b) estimate the temperature at 9:00 P.M., and (c) estimate the number of hours the temperature is below −12°C.

**Solution:**

**a.** By inspection we see the amplitude of the graph is $|A| = 15$, since this is the maximum displacement from the average value. As the context and graph indicate, the period is 24 hr.

**b.** Since 9:00 P.M. corresponds to $t = 9$, we read along the $t$-axis to 9, then move up to the graph to approximate the related temperature value on the °C axis (see figure). It appears the temperature will be close to 11°C.

**c.** In part (b) we're given the time and asked for a temperature. Here we're given a temperature and asked for a time. Begin on the °C axis at −12°, move horizontally until you intersect the graph, then move upward to the $t$-axis. The temperature is below 12°C for about 5 hr (from $t \approx 15.5$ to $t \approx 20.5$.)




NOW TRY EXERCISES 7 THROUGH 10

If you live in a more temperate area, the daily temperatures still follow a sinusoidal pattern, but the average temperature could be much higher. This is an example of a **vertical shift,** and is the role $D$ plays in the equation $y = A \sin(Bt) + D$. All other aspects of a graph remain the same, it is simply shifted $D$ units up if $D > 0$ and $D$ units down if $D < 0$. As in Section 6.3, for maximum value $M$ and minimum value $m$, $\frac{M - m}{2}$ gives the amplitude $A$ of the sine curve, while $\frac{M + m}{2}$ gives the **average value** $D$. From Example 1 we have $\frac{15 + (-15)}{2} = 0$ and $\frac{15 - (-15)}{2} = 15$✓.

**EXAMPLE 2** On a fine day in Galveston, Texas, the high temperature might be about 85°F with an overnight low of 61°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate what time(s) of day the temperature is 65°F. Assume $t = 0$ corresponds to 12:00 noon.

**Solution:**

**a.** We first note the period is still $P = 24$, giving $B = \frac{\pi}{12}$, and the equation model will have the form $y = A\sin\left(\frac{\pi}{12}t\right) + D$. Using $\frac{M+m}{2} = \frac{85+61}{2}$, we find the *average value* $D = 73$, with amplitude $A = \frac{85-61}{2} = 12$. The resulting equation is $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$.

**b.** To sketch the graph, use a reference rectangle $2A = 24$ units tall and $P = 24$ units wide, along with the rule of fourths to locate zeroes and max/min values (see Figure 6.63). Then lightly sketch a sine curve through these points and within the rectangle as in Figure 6.64. This is the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 0$.

**Figure 6.63**

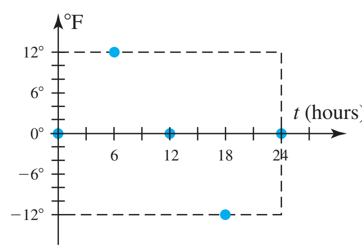
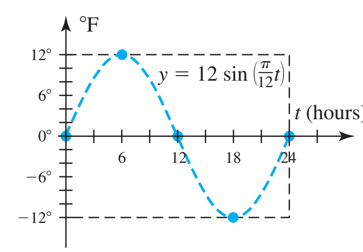
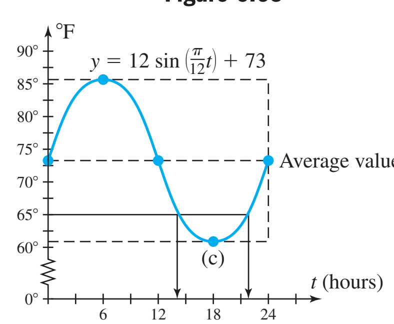



**Figure 6.64**




Using an appropriate scale, shift the rectangle and plotted points vertically upward 73 units and carefully draw the finished graph through the points and within the rectangle (see Figure 6.65).

**Figure 6.65**




This gives the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$. Note the broken-line notation "≷" in Figure 6.65 indicates that certain values along an axis are unused (in this case, we skipped 0° to 60°), and we began scaling the axis with the values needed.

**c.** As indicated in Figure 6.65, the temperature hits 65° twice, at about 15 and 21 hr after 12:00 noon, or at 3:00 A.M. and 9:00 A.M.

NOW TRY EXERCISES 11 THROUGH 18

Sinusoidal graphs actually include both sine and cosine graphs, the difference being that sine graphs begin at the average value, while cosine graphs begin at the maximum value. Sometimes it's more advantageous to use one over the other, but equivalent forms can easily be found. In Example 3, a cosine function is used to model an animal population that fluctuates sinusoidally due to changes in food supplies.

**EXAMPLE 3** The population of a certain animal species can be modeled by the function $P(t) = 1200\cos\left(\frac{\pi}{5}t\right) + 9000$, where $P(t)$ represents the population in year $t$. Use the model to: (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum values; and (d) estimate the number of years the population is less than 8000.

**Solution:**

**a.** Since $B = \frac{\pi}{5}$, the period is $P = \frac{2\pi}{\pi/5} = 10$, meaning the population of this animal species fluctuates over a 10-yr cycle.

**b.** Use a reference rectangle ($2A = 2400$ by $P = 10$ units) and the rule of fourths to locate zeroes and max/min values, then sketch the unshifted graph $y = 1200\cos\left(\frac{\pi}{5}t\right)$. Since $P = 10$, these occur at $t = 0, 2.5, 5, 7.5$, and 10 (see Figure 6.66). Shift this graph upward 9000 units (using an appropriate scale) to obtain the graph of $P(t)$ shown in Figure 6.67.

**Figure 6.66**

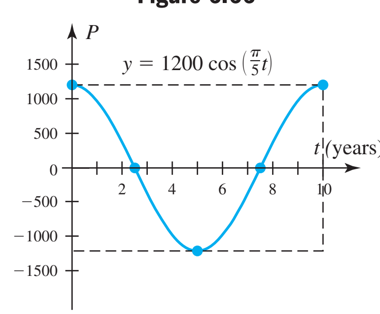
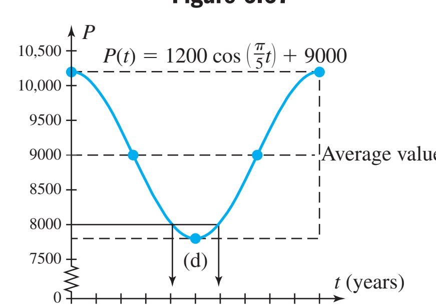



**Figure 6.67**




**c.** The maximum value is $9000 + 1200 = 10{,}200$ and the minimum value is $9000 - 1200 = 7800$.

**d.** As determined from the graph, the population drops below 8000 animals for approximately 2 yr.

NOW TRY EXERCISES 19 AND 20

## B. Horizontal Translations: $y = A\sin(Bt + C) + D$

In some cases, scientists would rather "benchmark" their study of sinusoidal phenomenon by placing the average value at $t = 0$ instead of a maximum value (as in Example 3), or by placing the maximum or minimum value at $t = 0$ instead of the average value (as in Example 1). Rather than make additional studies or recompute using available data, *we can simply shift these graphs using a horizontal translation*. To help understand how, consider the graph of $y = x^2$ from Section 2.4. The graph is a parabola, concave up, with a vertex at the origin. Comparing this function with $y_1 = (x - 3)^2$ and $y_2 = (x + 3)^2$, we note $y_1$ is simply the parent graph shifted 3 units right, and $y_2$ is the parent graph shifted 3 units left ("opposite the sign"). See Figures 6.68 through 6.70.

**Figure 6.68**

$y = x^2$

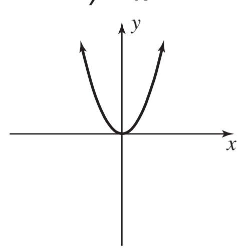



**Figure 6.69**

$y_1 = (x - 3)^2$

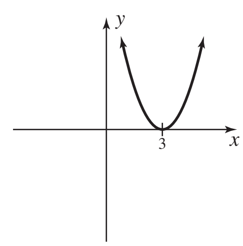
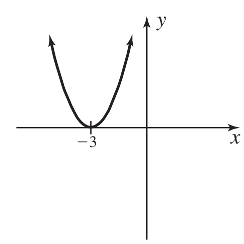



**Figure 6.70**

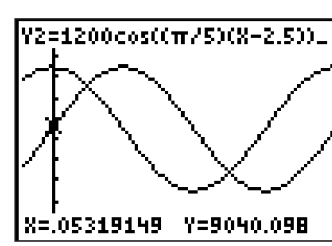
$y_2 = (x + 3)^2$




While ***quadratic functions have no maximum value if $A > 0$***, these graphs are a good reminder of how a basic graph can be horizontally shifted. We simply *replace the independent variable $x$ with $(x \pm h)$ or $t$ with $(t \pm h)$*, where $h$ is the desired shift and the sign is chosen depending on the direction of the shift.

**EXAMPLE 4** Use a horizontal translation to shift the graph from Example 3 so that the average population value begins at $t = 0$. Verify the result on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

**Solution:** For $P(t) = 1200\cos\left(\frac{\pi}{5}t\right) + 9000$ from Example 3, the average value first occurs at $t = 2.5$. For the average value to occur at $t = 0$, we can shift the graph to the right 2.5 units. Replacing $t$ with $(t - 2.5)$ gives $P(t) = 1200\cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$. A graphing calculator shows the desired result is obtained (see figure), and it appears to be a sine function with the same amplitude and period. The equation is $y = 1200\sin\left(\frac{\pi}{5}t\right) + 9000$.




NOW TRY EXERCISES 21 AND 22

Equations like $P(t) = 1200\cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$ from Example 4 are said to be written in **shifted form,** since we can easily tell the magnitude and direction of the shift in this form. To obtain the **standard form** we *distribute the value of B*: $P(t) = 1200\cos\left(\frac{\pi}{5}t - \frac{\pi}{2}\right) + 9000$. In general, the *standard form* of a sinusoidal equation (using *either* a cosine or sine function) is written $y = A\sin(Bt \pm C) + D$, with the *shifted form* found by factoring out $B$ from the argument:

$$y = A\sin(Bt \pm C) + D \rightarrow y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$$

In either case, C gives what is known as the **phase shift** of the function, and is used to discuss how far (as an angle measure) a given function is "out of phase" with a reference function. In the latter case, $\frac{C}{B}$ is simply the horizontal shift of the function and gives the magnitude and direction of the shift (opposite the sign).

**CHARACTERISTICS OF SINUSOIDAL MODELS**

Given the basic graph of $y = \sin t$, the graph can be transformed and written as $y = A\sin(Bt)$, where

1. $|A|$ gives the *amplitude* of the graph, or the maximum displacement from the average value.
2. $B$ is related to the *period* $P$ of the graph according to the ratio $P = \frac{2\pi}{B}$, giving the interval required for one complete cycle.

The graph of $y = A\sin(Bt)$ can be translated and written in the following forms:

| Standard form | Shifted form |
|---|---|
| $y = A\sin(Bt \pm C) + D$ | $y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$ |

3. In either case, $C$ is called the *phase shift* of the graph, while the ratio $\pm\frac{C}{B}$ gives the magnitude and direction of the *horizontal shift* (opposite the given sign).
4. $D$ gives the *vertical shift* of the graph, and the location of the average value. The shift will be in the same direction as the given sign.

It's important that you don't confuse the standard form with the shifted form. Each has a place and purpose, but the horizontal shift can be identified only by focusing on the change in an independent variable. Even though the equations $y = 2(x + 3)^2$ and $y = 2x^2 + 12x + 18$ are equivalent, only the first identifies that $y = 2x^2$ has been shifted three units left. Likewise $y = \sin[2(t + 3)]$ and $y = \sin(2t + 6)$ and are equivalent, but only the first explicitly gives the horizontal shift. Applications involving a horizontal shift come in an infinite variety, and the shifts are generally not uniform or standard. Knowing where each cycle begins and ends is a helpful part of sketching a graph of the equation model. The **primary interval** for a sinusoidal graph can be found by solving the inequality $0 \le Bt \pm C < 2\pi$, with the reference rectangle and rule of fourths giving the zeroes, max/min values, and a sketch of the graph in this interval. The graph can then be extended as needed in either direction, then shifted vertically $D$ units.

**EXAMPLE 5** For each function, identify the amplitude, period, horizontal shift, vertical shift (average value), and endpoints of the primary interval.

**a.** $y = 120 \sin\left[\frac{\pi}{8}(t - 6)\right] + 350$

**b.** $y = 2.5 \sin\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) + 6$

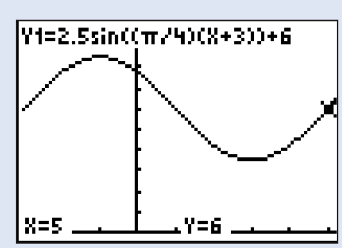
**Solution:** **a.** The equation shows amplitude $|A| = 120$, with an average value of $D = 350$. Since $B = \frac{\pi}{8}$, the period is $P = \frac{2\pi}{\pi/8} = 16$. With the argument in factored form, we note the horizontal shift is 6 units to the right (6 units opposite the sign). For the endpoints of the primary interval we solve $0 \le \frac{\pi}{8}(t - 6) < 2\pi$, giving $6 \le t < 22$.

**b.** The equation shows amplitude $|A| = 2.5$, with an average value of $D = 6$. With $B = \frac{\pi}{4}$, the period is $P = \frac{2\pi}{\pi/4} = 8$. To find the horizontal shift we factor out $\frac{\pi}{4}$ from the argument, to place the equation in shifted form: $\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) = \frac{\pi}{4}(t + 3)$. The horizontal shift is 3 units left (3 units opposite the sign). For the endpoints of the primary interval we solve $0 \le \frac{\pi}{4}(t + 3) < 2\pi$, which gives $-3 \le t < 5$.

NOW TRY EXERCISES 23 THROUGH 34

## GRAPHICAL SUPPORT

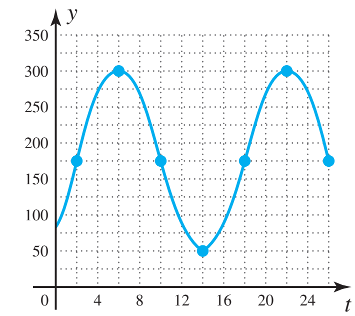
The analysis of $y = 2.5 \sin\left[\frac{\pi}{4}(t + 3)\right] + 6$ from Example 5(b) can be verified on a graphing calculator. Enter the function as $Y_1$ on the Y= screen and set the window size using the information gathered. Press the TRACE key and $-3$ ENTER and the calculator gives the average value $y = 6$ as output. Repeating this for $x = 5$ shows one complete cycle has been completed.




To help gain a better understanding of sinusoidal functions, their graphs, and the role the coefficients $A$, $B$, $C$, and $D$ play, it's often helpful to reconstruct the equation of a given graph.

**EXAMPLE 6** Determine the equation of the sinusoidal graph given.

**Solution:** From the graph it is apparent the maximum value is 300, with a minimum of 50. This gives a value of $\frac{300 + 50}{2} = 175$ for $D$ and $\frac{300 - 50}{2} = 125$ for $A$. The graph completes one cycle from $t = 2$ to $t = 18$, showing $P = 18 - 2 = 16$, with $B = \frac{\pi}{8}$. The average value first occurs at $t = 2$, so the graph has been shifted to the right 2 units. The equation is $y = 300 \sin\left[\frac{\pi}{8}(t - 2)\right] + 175$. Verify this using a graphing calculator.




**NOW TRY EXERCISES 35 THROUGH 44**

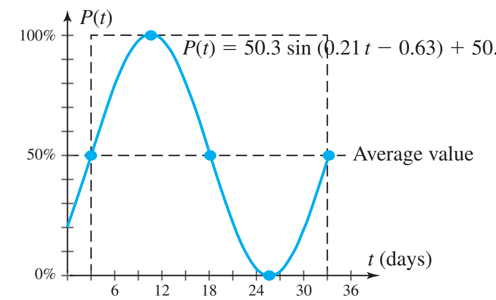
In Example 7, data gathered by an amateur astronomer living in Colorado were used to model the percent of the Moon that was illuminated during the month of July.

**EXAMPLE 7** The phases of the Moon (percent of illuminated surface area) follow a sinusoidal pattern. Using data she collected over a 31-day period, an amateur astronomer modeled this illumination using $P(t) = 50.3 \sin(0.21t - 0.63) + 50.1$, where $t$ is the time in days ($t = 1 \rightarrow$ July 1), and $P(t)$ is the illuminated portion of the Moon's surface expressed as a percent. (a) Graph the function and (b) determine the dates of the full Moon (100% illuminated) and new Moon (0% illuminated).

**Solution:** **a.** Taken directly from the equation, $A = 50.3$ and $D = 50.1$. As you might suspect, the period is about 30 days (one lunar month): $P = \frac{2\pi}{0.21} \approx 30$ days. To find the primary interval we solve the inequality given using $2\pi \approx 6.28$, which shows the graph will complete its primary cycle from $t = 3$ to 33 days (approximately):

$$0 \leq 0.21t - 0.63 < 6.28$$
$$0.63 \leq 0.21t < 6.91$$
$$3 \leq t < 32.9$$

Factoring $0.21t - 0.63$ to obtain the shifted form gives $0.21(t - 3)$, verifying the graph is shifted 3 units to the right. Using the period $P = 30$ gives the same endpoints as the inequality above for the primary interval: [3, 33). The graph was then extended left to $t = 0$, and shifted upward $D = 50.1$ units.




**b.** The period divided by 4 gives $\frac{30}{4} = 7.5$, which means 7.5 days after the cycle starts there will be a full Moon (since $P$ is increasing), 15 days later the Moon will be 50% illuminated, and 22.5 days later there will be a new Moon (0% illuminated). These dates are July 10, July 18, and July 25, respectively.

**NOW TRY EXERCISES 47 THROUGH 50**

## C. Simple Harmonic Motion: $y = A\sin(Bt)$

One of the least complicated applications of periodic functions involves a phenomenon known as **harmonic motion.** Objects in harmonic motion also tend to follow a sinusoidal pattern, one that can accurately be modeled by $y = A\sin(Bt)$ or $y = A\cos(Bt)$. While very simple, these **sinusoids** give us some fundamental insights into other forms of wave motion, including tides, sound, alternating electric current and others.

### Harmonic Models—Springs

Consider a spring hanging from a beam with a weight attached to one end. When the weight is at rest, we say it is in **equilibrium,** or has zero displacement from center. Stretching the spring and then releasing it causes the weight to "bounce up and down," with its displacement from center neatly modeled over time by a sine wave (see Figure 6.71).

**Figure 6.71**

At rest Stretched Released

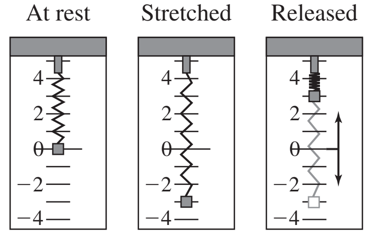
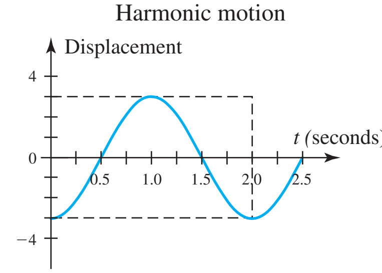



**Figure 6.72**

Harmonic motion




For objects in harmonic *motion* (there are other harmonic models), the input variable $t$ is always a time unit (seconds, minutes, days, etc.), so in addition to the period of the sinusoid, we are very interested in its **frequency**—the number of cycles it completes per unit time (see Figure 6.72). Since the period gives the time required to complete one cycle, the frequency $f$ is given by the reciprocal of the period: $f = \frac{1}{P} = \frac{B}{2\pi}$.

**EXAMPLE 8** For the harmonic motion modeled by the sinusoid in Figure 6.72, (a) find an equation of the form $y = A\sin(Bt)$; (b) determine the frequency; and (c) use the equation to find the position of the weight at $t = 1.8$ sec.

**Solution:**

**a.** By inspection the graph has an amplitude $|A| = 3$ and a period $P = 2$. After substitution into $P = \frac{2\pi}{B}$, we obtain $B = \pi$ and the equation $y = 3\sin(\pi t)$.

**b.** Frequency is the reciprocal of the period so $f = \frac{1}{2}$, showing one-half a cycle is completed each second (as the graph indicates).

**c.** Evaluating the model at $t = 1.8$ gives $y = 3\sin[\pi(1.8)] \approx -1.76$, meaning the weight is 1.76 in. below the equilibrium point at this time.

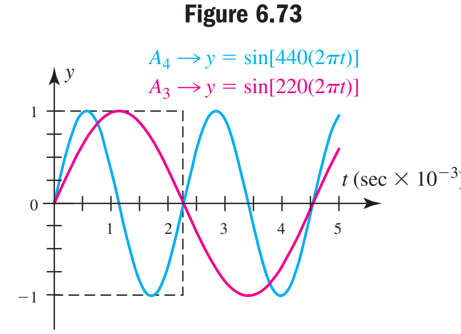
**NOW TRY EXERCISES 51 THROUGH 54**

### Harmonic Models—Sound Waves

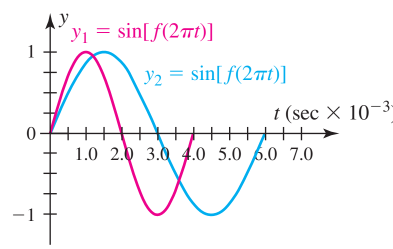
A second example of harmonic motion is the production of sound. For the purposes of this study, we'll look at musical notes. The vibration of matter produces a **pressure wave** or **sound energy,** which in turn vibrates the eardrum. Through the intricate structure of the middle ear, this sound energy is converted into mechanical energy and sent to the inner ear where it is converted to nerve impulses and transmitted to the brain. If the sound wave has a high frequency, the eardrum vibrates with greater frequency, which the brain interprets as a "high-pitched" sound. The *intensity* of the sound wave can also be transmitted to the brain via these mechanisms, and if the arriving sound wave has a high amplitude, the eardrum vibrates more forcefully and the sound is interpreted as "loud" by the brain. These characteristics are neatly modeled using $y = A\sin(Bt)$. For the moment we will focus on the frequency, keeping the amplitude constant at $A = 1$.

The musical note known as $A_4$ or "the A above middle C" is produced with a frequency of 440 vibrations per second, or 440 hertz (Hz) (this is the note most often used in the tuning of pianos and other musical instruments). For any given note, the same note one octave higher will have double the frequency, and the same note one octave lower will have one-half the frequency. In addition, with $f = \frac{1}{P}$ the value of $B = 2\pi\left(\frac{1}{P}\right)$ can always be expressed as $B = 2\pi f$, so $A_4$ has the equation $y = \sin[440(2\pi t)]$ (after rearranging the factors). The same note one octave lower is $A_3$ and has the equation $y = \sin[220(2\pi t)]$, with one-half the frequency. To draw the representative graphs, we must scale the $t$-axis in very small increments (seconds $\times 10^{-3}$) since $P = \frac{1}{440} \approx 0.0023$ for $A_4$, and $P = \frac{1}{220} \approx 0.0045$ for $A_3$. Both are graphed in Figure 6.73, where we see that the higher note completes two cycles in the same interval that the lower note completes one.

**Figure 6.73**



**EXAMPLE 9** The table here gives the frequencies by octaves of the 12 "chromatic" notes with frequencies between 110 Hz and 840 Hz. Two of the notes are graphed in the figure. Which two?




| | Frequency by Octave | | |
|---|---|---|---|
| Note | Octave 3 | Octave 4 | Octave 5 |
| A | 110.00 | 220.00 | 440.00 |
| A# | 116.54 | 233.08 | 466.16 |
| B | 123.48 | 246.96 | 493.92 |
| C | 130.82 | 261.64 | 523.28 |
| C# | 138.60 | 277.20 | 554.40 |
| D | 146.84 | 293.68 | 587.36 |
| D# | 155.56 | 311.12 | 622.24 |
| E | 164.82 | 329.24 | 659.28 |
| F | 174.62 | 349.24 | 698.48 |
| F# | 185.00 | 370.00 | 740.00 |
| G | 196.00 | 392.00 | 784.00 |
| G# | 207.66 | 415.32 | 830.64 |

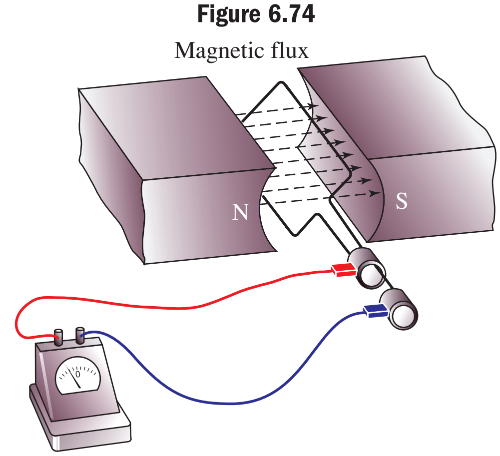
**Solution:** Since amplitudes are equal, the only difference is the frequency and period of the notes. It appears that $y_1$ has a period of about 0.004, giving a frequency of $\frac{1}{0.004} = 250$ Hz—very likely a $B_4$. $Y_2$ has a period of about 0.006, for a frequency of $\frac{1}{0.006} \approx 167$ Hz—probably an $E_3$.

NOW TRY EXERCISES 55 THROUGH 58

### Harmonic Models—Alternating Current

Perhaps a more practical example of harmonic motion comes from a study of electricity and alternating current. In simplistic terms, when an armature (molded wire) is rotated in a uniform magnetic field, a voltage $V$ is generated that depends on the strength of the field. As the armature is rotated, the voltage varies between a maximum and a minimum value, with the amount of voltage modeled by $V(\theta) = V_{\max}\sin(B\theta)$, with $\theta$ in degrees (the period formula in degrees is $P = \frac{360°}{B}$, so $B = 1$ in this case). Here, $V_{\max}$ represents the maximum voltage attained, and the input variable $\theta$ represents the angle the armature makes with the **magnetic flux**, indicated in Figure 6.74 by the dashed arrows between the magnets.

**Figure 6.74**



When the armature is perpendicular to the flux, we say $\theta = 0°$. At $\theta = 0°$ and $\theta = 180°$, no voltage is produced, while at $\theta = 90°$ and $\theta = 270°$, the voltage reaches its maximum and minimum values, respectively. Many electric dryers and other large appliances are labeled as 220 volt (V) appliances, but use an alternating current that varies from 311 V to −311 V (see the *Worthy of Note* that follows). This means when $\theta = 52°$, $V(52°) = 311\sin(52°) = 245$ V is being generated. In practical applications, we use time $t$ as the independent variable, rather the angle of the armature. These large appliances usually operate with a frequency of 60 cycles per second, or 1 cycle every $\frac{1}{60}$ of a second $\left(P = \frac{1}{60}\right)$. Using $B = \frac{2\pi}{P}$, we obtain $B = 120\pi$ and our equation model becomes $V(t) = 311\sin(120\pi t)$ with $t$ in radians. This variation in voltage is an excellent example of a simple harmonic model.

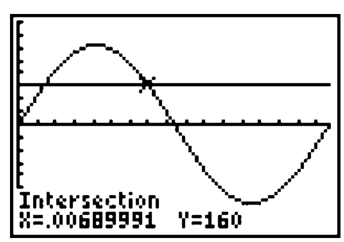
**EXAMPLE 10** Use the equation $V(t) = 311\sin(120\pi t)$ to:

**a.** Create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle $\left(t = \frac{1}{120} \approx 0.008\right)$.

**b.** Use a graphing calculator to find the times $t$ in this half-cycle when 160 V is being produced.

**Solution:** **a.** Starting at $t = 0$ and using increments of 0.001 sec produces the table shown.

**b.** From the table we note $V(t) = 160$ when $t \in (0.001, 0.002)$ and $t \in (0.006, 0.007)$. Using the intersection of graphs method places these values at $t \approx 0.0014$ and $t \approx 0.0069$ (see graph).




| Time $t$ | Voltage |
|---|---|
| 0 | 0 |
| 0.001 | 114.5 |
| 0.002 | 212.9 |
| 0.003 | 281.4 |
| 0.004 | 310.4 |
| 0.005 | 295.8 |
| 0.006 | 239.6 |
| 0.007 | 149.8 |
| 0.008 | 39.9 |

**NOW TRY EXERCISES 59 AND 60**

**WORTHY OF NOTE**

You may have wondered why we're using an amplitude of 311 for a 220-V appliance. Due to the nature of the sine wave, the average value of an alternating current is always zero and gives no useful information about the voltage generated. Instead, the root-mean-square (rms) of the voltage is given on most appliances. While the maximum voltage is 311 V, the rms voltage is $\frac{311}{\sqrt{2}} \approx 220$ V. See Exercise 67.

**TECHNOLOGY HIGHLIGHT**

## Locating Zeroes/Roots/x-intercepts on a Graphing Calculator

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

As you know, the zeroes of a function are *input* values that cause an *output* of zero, and are analogous to the roots of an equation. Graphically both show up as *x*-intercepts and once a function is graphed on the *TI-84 Plus,* they can be located (if they exist) using the 2nd CALC **2:zero** feature. This feature is similar to the **3:minimum** and **4:maximum** features from

Section 3.8, in that we have the calculator search a specified interval by giving a **left bound** and a **right bound**. To illustrate, enter $Y_1 = 3\sin\left(\frac{\pi}{2}x\right) - 1$ on the Y= screen and graph it using the ZOOM **7:ZTrig** option. The resulting graph shows there are six zeroes in this interval and we'll locate the first root to the left of zero. Knowing the ZOOM **7:Trig** option uses tic marks that are spaced every $\frac{\pi}{2}$ units, this root is in the interval $\left(-\pi, -\frac{\pi}{2}\right)$. After pressing 2nd CALC **2:zero** the calculator returns you to the graph, and requests a "Left Bound," asking you to narrow down the interval it has to search (see Figure 6.75). We enter $-\pi$ (press ENTER) as already discussed, and the calculator marks this choice at the top of the screen with a "▶" marker (pointing to the right), then asks you to enter a "Right Bound." After entering $-\frac{\pi}{2}$, the calculator marks this with a "◀" marker and asks for a "Guess." This option is primarily used when the interval selected has more than one zero and you want to guide the calculator toward a particular zero. Often we will simply bypass it by pressing ENTER once again (see Figure 6.76). The calculator searches the specified interval until it locates a zero (Figure 6.77) or displays an error message indicating it was unable to comply (no zeroes in the interval). It's important to note that the "solution" displayed is actually just a very accurate estimate. Use these ideas to locate the zeroes of the following functions in $[0, \pi]$.

**Figure 6.75**

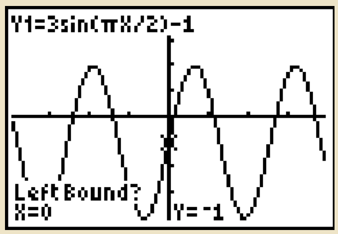



**Figure 6.76**

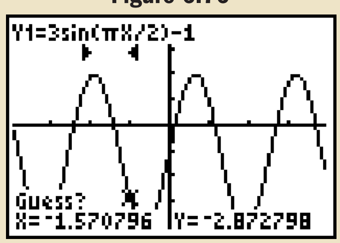
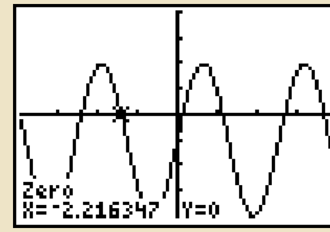



**Figure 6.77**




Exercise 1: $y = -2\cos(\pi t) + 1$

Exercise 2: $y = 0.5\sin[\pi(t - 2)]$

Exercise 3: $y = \frac{3}{2}\tan(2x) - 1$

Exercise 4: $y = x^3 - \cos x$

## 6.5 EXERCISES

### CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** A sinusoidal wave or pattern is one that can be modeled by functions of the form ____________ or ____________.

**2.** The graph of $y = \sin x + k$ is the graph of $y = \sin x$ shifted __________ $k$ units. The graph of $y = \sin(x - h)$ is the graph of $y = \sin x$ shifted __________ $h$ units.

**3.** To find the primary interval of a sinusoidal graph, solve the inequality ____________.

**4.** Given the period $P$, the frequency is ________, and given the frequency $f$, the value of $B$ is ________.

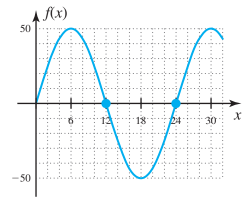
**5.** Explain/discuss the difference between the *standard form* of a sinusoidal equation, and the *shifted form*. How do you obtain one from the other? For what benefit?

**6.** Write out a step-by-step procedure for sketching the graph of $y = 30\sin\left(\frac{\pi}{2}t - \frac{1}{2}\right) + 10$. Include use of the reference rectangle, primary interval, zeroes, max/mins, and so on. Be complete and thorough.

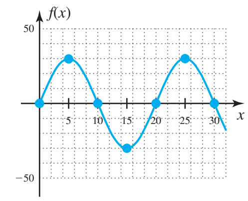
## DEVELOPING YOUR SKILLS

Use the graphs given to (a) state the amplitude $A$ and period $P$ of the function; (b) estimate the value at $x = 14$; and (c) estimate the interval in $[0, P]$, where $f(x) \geq 20$.

**7.**

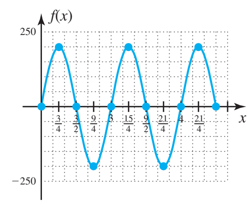


**8.**

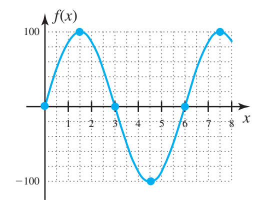


Use the graphs given to (a) state the amplitude $A$ and period $P$ of the function; (b) estimate the value at $x = 2$; and (c) estimate the interval in $[0, P]$, where $f(x) \leq -100$.

**9.**



**10.**



Use the information given write a sinusoidal equation and sketch its graph. Recall $B = \dfrac{2\pi}{P}$.

**11.** Max: 100, min: 20, $P = 30$

**12.** Max: 95, min: 40, $P = 24$

**13.** Max: 20, min: 4, $P = 360$

**14.** Max: 12,000, min: 6500, $P = 10$

Use the information given to write a sinusoidal equation, sketch its graph, and answer the question posed.

**15.** In Geneva, Switzerland, the daily temperature in January ranges from an average high of 39°F to an average low of 29°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches the freezing point (32°F). Assume $t = 0$ corresponds to noon.

Source: *2004 Statistical Abstract of the United States,* Table 1331.

**16.** In Nairobi, Kenya, the daily temperature in January ranges from an average high of 77°F to an average low of 58°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches a comfortable 72°F? Assume $t = 0$ corresponds to noon.

Source: *2004 Statistical Abstract of the United States,* Table 1331.

**17.** In Oslo, Norway, the number of hours of daylight reaches a low of 6 hr in January, and a high of nearly 18.8 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month ≈ 30.5 days. Assume $t = 0$ corresponds to January 1.

Source: www.visitnorway.com/templates

**18.** In Vancouver, British Columbia, the number of hours of daylight reaches a low of 8.3 hr in January, and a high of nearly 16.2 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month ≈ 30.5 days. Assume $t = 0$ corresponds to January 1.

Source: www.bcpassport.com/vital/temp

**19.** Recent studies seem to indicate the population of North American porcupine (*Erethizon dorsatum*) varies sinusoidally with the solar (sunspot) cycle due to its effects on Earth's ecosystems. Suppose the population of this species in a certain locality is modeled by the function $P(t) = 250\cos\left(\frac{2\pi}{11}t\right) + 950$, where $P(t)$ represents the population of porcupines in year $t$. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum values; and (d) estimate the number of years the population is less than 740 animals.

Source: Ilya Klvana, McGill University (Montreal), Master of Science thesis paper, November 2002.

**20.** The population of mosquitoes in a given area is primarily influenced by precipitation, humidity, and temperature. In tropical regions, these tend to fluctuate sinusoidally in the course of a year. Using trap counts and statistical projections, fairly accurate estimates of a mosquito population can be obtained. Suppose the population in a certain region was modeled by the function $P(t) = 50\cos\left(\frac{\pi}{26}t\right) + 950$, where $P(t)$ was the mosquito population (in thousands) in week $t$ of the year. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum population values; and (d) estimate the number of weeks the population is less than 915,000.

**21.** Use a horizontal translation to shift the graph from Exercise 19 so that the average population of the North American porcupine begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

**22.** Use a horizontal translation to shift the graph from Exercise 20 so that the average population of mosquitoes begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

Identify the amplitude (A), period (P), horizontal shift (HS), vertical shift (VS), and endpoints of the primary interval (PI) for each function given.

**23.** $y = 120\sin\left[\frac{\pi}{12}(t - 6)\right]$

**24.** $y = 560\sin\left[\frac{\pi}{4}(t + 4)\right]$

**25.** $h(t) = \sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$

**26.** $r(t) = \sin\left(\frac{\pi}{10}t - \frac{2\pi}{5}\right)$

**27.** $y = \sin\left(\frac{\pi}{4}t - \frac{\pi}{6}\right)$

**28.** $y = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{12}\right)$

**29.** $f(t) = 24.5\sin\left[\frac{\pi}{10}(t - 2.5)\right] + 15.5$

**30.** $g(t) = 40.6\sin\left[\frac{\pi}{6}(t - 4)\right] + 13.4$

**31.** $g(t) = 28\sin\left(\frac{\pi}{6}t - \frac{5\pi}{12}\right) + 92$

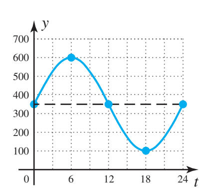
**32.** $f(t) = 90\sin\left(\frac{\pi}{10}t - \frac{\pi}{5}\right) + 120$

**33.** $y = 2500\sin\left(\frac{\pi}{4}t + \frac{\pi}{12}\right) + 3150$

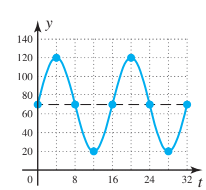
**34.** $y = 1450\sin\left(\frac{3\pi}{4}t + \frac{\pi}{8}\right) + 2050$

Find the equation of the graph given. Write your answer in the form $y = A\sin(Bt + C) + D$.

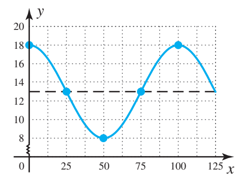
**35.**



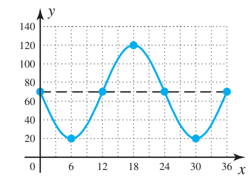

**36.**



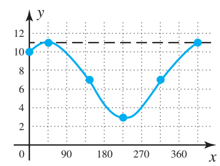
**37.**



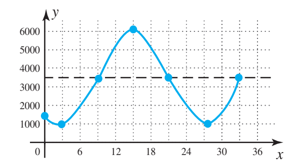
**38.**



**39.**



**40.**



Sketch one complete period of each function.

**41.** $f(t) = 25\sin\left[\frac{\pi}{4}(t-2)\right] + 55$

**42.** $g(t) = 24.5\sin\left[\frac{\pi}{10}(t-2.5)\right] + 15.5$

**43.** $h(t) = 1500\sin\left(\frac{\pi}{8}t + \frac{\pi}{4}\right) + 7000$

**44.** $p(t) = 350\sin\left(\frac{\pi}{6}t + \frac{\pi}{3}\right) + 420$

## WORKING WITH FORMULAS

**45. The relationship between the coefficient $B$, the frequency $f$, and the period $P$**

In many applications of trigonometric functions, the equation $y = A\sin(Bt)$ is written as $y = A\sin[(2\pi f)t]$, where $B = 2\pi f$. Justify the new equation using $f = \frac{1}{P}$ and $P = \frac{2\pi}{B}$. In other words, explain how $A\sin(Bt)$ becomes $A\sin[(2\pi f)t]$, as though you were trying to help another student with the ideas involved.

**46. Number of daylight hours: $D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t-79)\right] + 12$**

The number of daylight hours for a particular day of the year is modeled by the formula given, where $D(t)$ is the number of daylight hours on day $t$ of the year and $K$ is a constant related to the total variation of daylight hours, latitude of the location, and other factors. For the city of Reykjavik, Iceland, $K \approx 17$, while for Detroit, Michigan, $K \approx 6$. How many hours of daylight will each city receive on June 30 (the 182nd day of the year)?

## APPLICATIONS

**Modeling phases of the Moon:** At different locations on the Earth and at different times of the year, the phases of the Moon (percent of illuminated surface area) are seen in a different perspective, resulting in a different equation model.

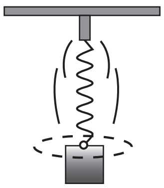
**47.** Data from the U.S. Naval Observatory in Washington, D.C., were used to obtain the equation $P(t) = 0.5\sin(0.21t + 0.05) + 0.51$ as a model for the percent of the Moon that was illuminated in the month of January of 2004 ($t = 1 \rightarrow$ Jan.1). (a) Graph the function; (b) determine the dates of the full Moon and new Moon; and (c) during what dates was the illumination of the Moon less than 20%?

**48.** The phases of the Moon are so predictable that data is available from the U.S. Naval Observatory to predict the Moon's phases at anytime, past or future. The equation $P(t) = 0.5\sin(0.21t - 0.36) + 0.47$ gives the percent of the Moon that will be illuminated in the month of August 2006 for a particular location ($t = 1 \rightarrow$ Aug.1). (a) Graph the function; (b) determine the dates of the full Moon and new Moon; and (c) during what dates will the illumination of the Moon be greater than 80%?

**49.** Find an equation modeling the illumination of the Moon, given that a full Moon (100% illuminated) occurs on the 12th day of the month. Use $P = 30$ for the period (the max/min values are known).

**50.** Find an equation modeling the illumination of the Moon, given that a new Moon (0% illuminated) occurs on the 3rd day of the month. Use $P = 30$ for the period (the max/min values are known).

**51.** **Harmonic motion:** A weight on the end of a spring is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 6\sin\left(\frac{\pi}{2}t\right)$, where $d$ is the distance (in centimeters) from the equilibrium point in $t$ sec.



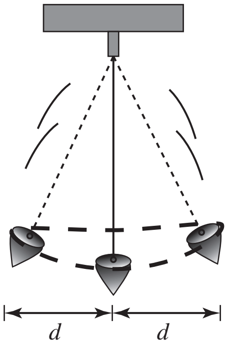
**a.** What is the period of the motion? What is the frequency of the motion?

**b.** What is the displacement from equilibrium at $t = 2.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

**c.** What is the displacement from equilibrium at $t = 3.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

**d.** How far does the weight move between $t = 1$ and $t = 1.5$ sec? What is the average velocity for this interval? Do you expect a greater or lesser velocity for $t = 1.75$ to $t = 2$? Explain why.

**52.** **Harmonic motion:** The bob on the end of a 24-in. pendulum is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 20\cos(4t)$, where $d$ is the distance (in inches) from the equilibrium point, $t$ sec after being released from one side.




**a.** What is the period of the motion? What is the frequency of the motion?

**b.** What is the displacement from equilibrium at $t = 0.25$ sec? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

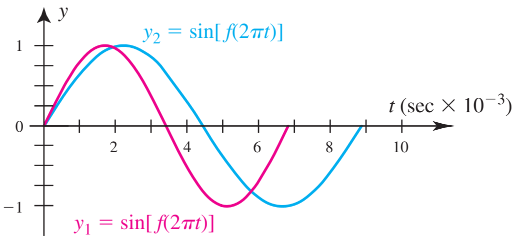
**c.** What is the displacement from equilibrium at $t = 1.3$ seconds? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

**d.** How far does the bob move between $t = 0.25$ and $t = 0.35$ sec? What is it's average velocity for this interval? Do you expect a greater velocity for the interval $t = 0.55$ to $t = 0.6$? Explain why.

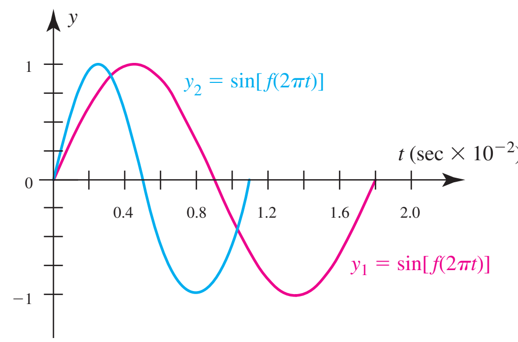
**53.** **Harmonic motion:** A simple pendulum 36 in. in length is oscillating in harmonic motion. The bob at the end of the pendulum swings through an arc of 30 in. (from the far left to the far right or one-half cycle) in about 0.8 sec. What is the equation model for this harmonic motion?

**54.** **Harmonic motion:** As part of a study of wave motion, the motion of a floater is observed as a series of uniform ripples of water move beneath it. By careful observation, it is noted that the floater bobs up-and-down through a distance of 2.5 cm every $\frac{1}{3}$ sec. What is the equation model for this harmonic motion?

**55.** **Sound waves:** Two of the musical notes from the chart on page 647 are graphed in the figure. Use the graphs given to determine which two.



**56. Sound waves:** Two chromatic notes *not on the chart from page* 647 are graphed in the figure. Use the graphs and the discussion regarding octaves to determine which two. Note the scale of the $t$-axis *has been* changed to hundredths of a second.




**Sound waves:** Use the chart on page 647 to draw graphs representing each pair of these notes. Write the equation for each note in the form $y = \sin[f(2\pi t)]$ and clearly state the period of each note.

**57.** notes $D_3$ and $G_4$

**58.** the notes $A_5$ and $C\#_3$

**59. Electric current:** In the United States, electric power is supplied to homes and offices via a "120 V circuit," using an alternating current that varies from 170 V to −170 V, at a frequency of 60 cycles/sec. (a) Write the voltage equation for U.S. households, (b) create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle, and (c) find the first time $t$ in this half-cycle when exactly 140 V is being produced.

**60. Electric current:** While the electricity supplied in Europe is still not quite uniform, most countries employ 230-V circuits, using an alternating current that varies from 325 V to −325 V. However, the frequency is only *50 cycles per second.* (a) Write the voltage equation for these European countries, (b) create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle, and (c) find the first time $t$ in this half-cycle when exactly 215 V is being produced.

**Temperature models:** Use the information given to determine the amplitude, period, average value (AV), and horizontal shift. Then write the equation and sketch the graph. Assume each context is sinusoidal.

**61.** During a typical January (summer) day in Buenos Aires, Argentina, the daily high temperature is 84°F and the daily low is 64°F. Assume the low temperature occurs around 6:00 A.M. Assume $t = 0$ corresponds to midnight.

**62.** In Moscow, Russia, a typical January day brings a high temperature of 21°F and a low of 11°F. Assume the high temperature occurs around 4:00 P.M. Assume $t = 0$ corresponds to midnight.

**Daylight hours model:** Solve using a graphing calculator and the formula given in Exercise 46.

**63.** For the city of Caracas, Venezuela, $K \approx 1.3$, while for Tokyo, Japan, $K \approx 4.8$.

**a.** How many hours of daylight will each city receive on January 15th (the 15th day of the year)?

**b.** Graph the equations modeling the hours of daylight on the same screen. Then determine (i) what days of the year these two cities will have the same number of hours of daylight and (ii) the number of days each year that each city receives 11.5 hr of daylight or less.

**64.** For the city of Houston, Texas, $K \approx 3.8$, while for Pocatello, Idaho, $K \approx 6.2$.

**a.** How many hours of daylight will each city receive on December 15 (the 349th day of the year)?

**b.** Graph the equations modeling the hours of daylight on the same screen. Then determine (i) how many days each year Pocatello receives more daylight than Houston and (ii) the number of days each year that each city receives 13.5 hr of daylight or more.

## WRITING, RESEARCH, AND DECISION MAKING

**65.** Some applications of sinusoidal graphs use variable amplitudes and variable frequencies, rather than constant values of $f$ and $P$ as illustrated in this section. The classic example is AM (amplitude modulated) and FM (frequency modulated) radio waves. Use the Internet, an encyclopedia, or the resources of a local library to research the similarities and difference between AM and FM radio waves, and include a sketch of each type of wave. What are the advantages and disadvantages of each? Prepare a short summary on what you find. For other applications involving variable amplitudes, see the *Calculator Exploration and Discovery* feature at the end of this chapter.

**66.** A laser beam is actually a very narrow beam of light with a constant frequency and very high intensity (LASER is an acronym for *light amplified* by *stimulated emission* of *radiation*). Lasers have a large and growing number of practical applications, including repair of the eye's retina, sealing of blood vessels during surgery, treatment of skin cancer, drilling small holes, making precise measurements, and many others. In the study of various kinds of laser light, sinusoidal equations are used to model certain characteristics of the light emitted. Do some further reading and research on lasers, and see if you can locate and discuss some of these sinusoidal models.

**67.** Using the Internet, a trade manual, or some other resource, find the voltage and frequency at which electricity is supplied to most of Japan (oddly enough—two different frequencies are in common use). As in Example 10, the voltage given will likely be the root-mean-square (rms) voltage. Use the information to find the true voltage and the equation model for voltage in most of Japan.

## EXTENDING THE CONCEPT

**68.** The formulas we use in mathematics can sometimes seem very mysterious. We know they "work," and we can graph and evaluate them—but where did they come from? Consider the formula for the number of daylight hours from Exercise 46: $D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12$. Use the knowledge gained from this section.

**a.** We know that the addition of 12 represents a vertical shift, but what does a vertical shift of 12 mean *in this context?* Why is it applied as part of the formula?

**b.** We also know the factor $(t - 79)$ represents a phase shift of 79 to the right. But what does a horizontal (phase) shift of 79 mean *in this context?* Why is it applied as part of the formula?

**c.** Finally, the coefficient $\frac{K}{2}$ represents a change in amplitude, but what does a change of amplitude mean *in this context?* Why is the coefficient bigger for the northern latitudes?

**69.** Use a graphing calculator to graph the equation $f(x) = \frac{3x}{2} - 2\sin(2x) - 1.5$.

**a.** Determine the interval between each peak of the graph. What do you notice?

**b.** Graph $g(x) = \frac{3x}{2} - 1.5$ on the same screen and comment on what you observe.

**c.** What would the graph of $f(x) = -\frac{3x}{2} + 2\sin(2x) + 1.5$ look like? What is the $x$-intercept?

## MAINTAINING YOUR SKILLS

**70.** (6.1) In what quadrant does the angle $t = 3.7$ terminate? What is the reference angle?

**71.** (2.5) Given $f(x) = -3(x + 1)^2 - 4$, name the vertex and solve the inequality $f(x) > 0$.

**72.** (6.2) Given $\sin\theta = -\frac{5}{12}$ with $\tan\theta < 0$, find the value of all six trig functions of $\theta$.

**73.** (1.4) Compute the sum, difference, product, and quotient of $-1 + i\sqrt{5}$ and $-1 - i\sqrt{5}$.

**74.** (6.1) The vertices of a triangle have coordinates $(-2, 6)$, $(1, 10)$, and $(2, 3)$. Verify that a right triangle is formed and find the measure of the two acute angles

**75.** (6.3/6.4) Draw a quick sketch of $y = 2\sin x$, $y = 2\cos x$, and $y = 2\tan x$ for $x \in \left[-\frac{\pi}{2}, 2\pi\right]$.

# 6.6 The Trigonometry of Right Triangles

### LEARNING OBJECTIVES

***In Section 6.6 you will learn how to:***

**A. Find values of the six trigonometric functions from their ratio definition**

**B. Solve right triangles given one angle and one side**

**C. Solve a right triangle given two sides**

**D. Use cofunctions and complements to write equivalent expressions**

**E. Solve applications involving angles of elevation and depression**

**F. Solve general applications of right triangles**

### INTRODUCTION

Over a long period of time, what began as a study of chord lengths by Hipparchus, Ptolemy, Aryabhata, and others, became a systematic application of the ratios of the sides of a right triangle. In this section we develop the sine, cosine, and tangent functions from a right triangle perspective, and further explore relationships that exist between them. As with the unit circle approach, this view of the trig functions also leads to a number of significant applications.

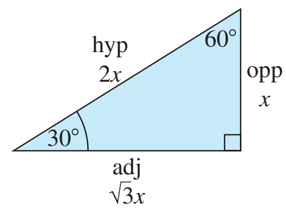
### POINT OF INTEREST

The study of chords and half-chords is closely linked to the modern concept of the sine ratio: $\frac{\text{opp}}{\text{hyp}}$. In the centuries following Ptolemy (~85–165 A.D.) numerous tables of these chord and half-chord lengths were published in various parts of the world, and with greater and greater accuracy. Through a mistranslation, the Hindu word for half-chord became the Latin word *sinus*, from which we have the *sine* function. The word *cosine* is actually a shortened form of the words "*complement of sine,*" a designation suggested by Edmund Gunter around 1620 since the sine of an angle is equal to the cosine of its complement $[\text{sine}\,\theta = \text{cosine}(90° - \theta)]$.

## A. Trigonometric Ratios and Their Values

In Section 6.1, we looked at applications involving 45-45-90 and 30-60-90 triangles, using the fixed ratios that exist between their sides. To apply this concept more generally using other right triangles, each side is given a specific name using its relationship to a specified angle. For the 30-60-90 triangle in Figure 6.78, the side **opposite (opp)** and the side **adjacent (adj)** are named with respect to the 30° angle, with the **hypotenuse (hyp)** always across from the right angle. Likewise for the 45-45-90 triangle in Figure 6.79. Using these designations to name the various ratios, we can develop a systematic method for applying the concept. Note that the $x$'s "cancel" in each ratio, reminding us the ratios are independent of the triangle's size.

**Figure 6.78**



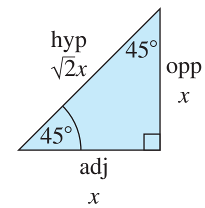

*for 30°*

$$\frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

$$\frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

**Figure 6.79**




*for 45°*

$$\frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{\text{opp}}{\text{adj}} = \frac{1}{1}$$

As hinted at in the *Point of Interest,* ancient mathematicians were able to find values for the ratios corresponding to *any acute angle* in a right triangle, and it soon became apparent that *naming* each ratio would be helpful. These ratios are **sine** $\rightarrow \frac{\text{opp}}{\text{hyp}}$, **cosine** $\rightarrow \frac{\text{adj}}{\text{hyp}}$, and **tangent** $\rightarrow \frac{\text{opp}}{\text{adj}}$. Since each ratio depends on the measure of an acute angle $\theta$, they are often referred to as **functions of an acute angle** and written in function form.

$$\text{sine}\,\theta = \frac{\text{opp}}{\text{hyp}} \qquad \text{cosine}\,\theta = \frac{\text{adj}}{\text{hyp}} \qquad \text{tangent}\,\theta = \frac{\text{opp}}{\text{adj}}$$

The reciprocal ratios, for example, $\frac{\text{hyp}}{\text{opp}}$ instead of $\frac{\text{opp}}{\text{hyp}}$, also play a significant role in this view of trigonometry, and are likewise given a name:

$$\text{cosecant}\,\theta = \frac{\text{hyp}}{\text{opp}} \qquad \text{secant}\,\theta = \frac{\text{hyp}}{\text{adj}} \qquad \text{cotangent}\,\theta = \frac{\text{adj}}{\text{opp}}$$

The definitions hold regardless of the triangle's orientation (how it is drawn) or which of the acute angles is used.

In actual use, each function name is written in the abbreviated form seen previously. Over the course of this study, you will see many connections between this view of trigonometry and the unit circle approach. In particular, note that based on the designations above, we have:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

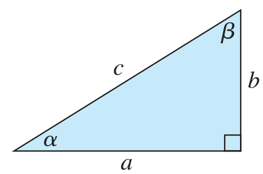
In general, we have:

**WORTHY OF NOTE**

Over the years, a number of memory tools have been invented to help students recall these ratios correctly. One such tool is the acronym SOH CAH TOA, from the first letter of the function and the corresponding ratio. It is often recited as, "Sit On a Horse, Canter Away Hurriedly, To Other Adventures." Try making up a memory tool of your own.

**TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE**

$$\sin\alpha = \frac{b}{c} \qquad \sin\beta = \frac{a}{c}$$
$$\cos\alpha = \frac{a}{c} \qquad \cos\beta = \frac{b}{c}$$
$$\tan\alpha = \frac{b}{a} \qquad \tan\beta = \frac{a}{b}$$

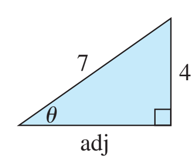



Now that these ratios have been formally named, we can state the value of all six functions given sufficient information about a right triangle.

**EXAMPLE 1** Given that $\sin\theta = \frac{4}{7}$, find the value of the remaining trig functions of $\theta$.

**Solution:** For $\sin\theta = \frac{4}{7} = \frac{\text{opp}}{\text{hyp}}$, we draw a triangle with a side of 4 units opposite a designated angle $\theta$, and label a hypotenuse of 7 (see the figure). Using the Pythagorean theorem we find the length of the adjacent side: adj $= \sqrt{7^2 - 4^2} = \sqrt{33}$. The ratios are:

$$\sin\theta = \frac{4}{7} \qquad \cos\theta = \frac{\sqrt{33}}{7} \qquad \tan\theta = \frac{4}{\sqrt{33}}$$
$$\csc\theta = \frac{7}{4} \qquad \sec\theta = \frac{7}{\sqrt{33}} \qquad \cot\theta = \frac{\sqrt{33}}{4}$$




NOW TRY EXERCISES 7 THROUGH 12

It's worth noting that due to similar triangles, identical results would be obtained using any ratio of sides that is equivalent to $\frac{4}{7}$. In other words, $\frac{2}{3.5} = \frac{4}{7} = \frac{8}{14} = \frac{16}{28}$ and so on, will all give the same end result.

## B. Solving Right Triangles Given One Angle and One Side

Example 1 gave values of the trig functions for an *unknown angle* $\theta$. Using standard triangles, we can state the value of each trig function for 30°, 45°, and 60° based on the related ratio. Note they're the same as those from our unit circle study of trigonometry (see Table 6.10).

**Table 6.10**

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|
| 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ | 2 | $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ | $\sqrt{3}$ |
| 45° | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ | 1 |
| 60° | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ | 2 | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ |
| 90° | 1 | 0 | undefined | 1 | undefined | 0 |

To **solve a right triangle** means to find the measure of all three angles and all three sides. This is accomplished using any combination of the Pythagorean theorem, the properties of triangles, and the trigonometric ratios. We will adopt the convention of naming the angles with a capital letter at the vertex or using a Greek letter on the interior (see Figure 6.80). Each side is labeled using the related small case letter from the angle opposite. The complete solution should be organized in table form as in Table 6.11. In this illustration and the examples that follow, the quantities shown in **bold** were given, the remaining values were found using the techniques mentioned above.

**Figure 6.80**

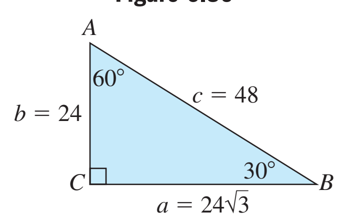
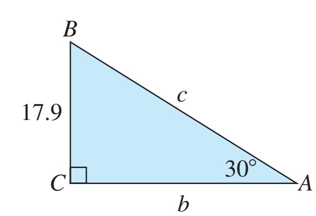



**Table 6.11**

| Angles | Sides |
|---|---|
| **A = 60°** | $a = 24\sqrt{3}$ |
| $B = 30°$ | **b = 24** |
| **C = 90°** | $c = 48$ |

**EXAMPLE 2** Solve the triangle given.




**Solution:** Applying the ratio $\sin 30° = \frac{\text{opp}}{\text{hyp}}$ we have

For side $c$:

$$\sin 30° = \frac{17.9}{c} \qquad \text{sin } 30° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$c \sin 30° = 17.9 \qquad \text{cross multiply}$$

$$c = \frac{17.9}{\sin 30°} \qquad \text{divide by } \sin 30° = \frac{1}{2}$$

$$= 35.8 \qquad \text{result}$$

Using the Pythagorean theorem we find side $b \approx 31$, and since $\angle A$ and $\angle B$ are complements, $B = 60°$. The complete solution is shown. Note the results would have been identical if the standard ratios from the 30-60-90 triangle were applied. The hypotenuse is twice the shorter side: $c = 2(17.9) = 35.8$, and the longer side is $\sqrt{3}$ times the shorter, giving $b = 17.9(\sqrt{3}) \approx 31$.

| Angles | Sides |
|---|---|
| **A = 30°** | $a =$ **17.9** |
| $B = 60°$ | $b \approx 31$ |
| **C = 90°** | $c = 35.8$ |

NOW TRY EXERCISES 13 THROUGH 16

As mentioned in the *Point of Interest,* what was begun by Hipparchus and Ptolemy in their study of chords, gradually became a complete table of values for the sine, cosine, and tangent of acute angles (and half-angles). Prior to the widespread availability of hand-held calculators, these tables were used to find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for nonstandard angles. A small portion of the table for $\sin\theta$ is shown in Table 6.12, where we note $\sin 45° \approx 0.7071$ and $\sin 47°50' \approx 0.7412$ (in blue). Today these trig values are programmed into your calculator and we can retrieve them with the push of a button (or two). The calculator gives a much better approximation for $\sin 45° \to 0.7071067812$, while Figure 6.79 gives the exact value: $\sin 45° = \frac{\sqrt{2}}{2}$.

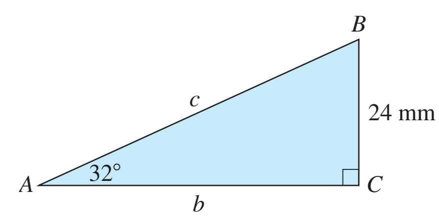
**Table 6.12**

$\sin\theta$

| $\theta$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 0.7071 | 0.7092 | 0.7112 | 0.7133 | 0.7153 | 0.7173 |
| 46 | 0.7193 | 0.7214 | 0.7234 | 0.7254 | 0.7274 | 0.7294 |
| 47 | 0.7314 | 0.7333 | 0.7353 | 0.7373 | 0.7392 | 0.7412 |
| 48 | 0.7431 | 0.7451 | 0.7470 | 0.7490 | 0.7509 | 0.7528 |
| (49) ← | ~~0.7547~~ | ~~0.7566~~ | ~~0.7585~~ | 0.7604 | 0.7623 | 0.7642 |

**EXAMPLE 3** Solve the triangle shown in the figure.

**Solution:** We know $\angle B = 58°$ since $A + B = 90°$. We can find length $b$ using the tangent function:




$$\tan 32° = \frac{24}{b} \qquad \tan 32° = \frac{\text{opposite}}{\text{adjacent}}$$

$$b \tan 32° = 24 \qquad \text{cross multiply}$$

$$b = \frac{24}{\tan 32°} \qquad \text{divide by } \tan 32° \approx 0.6249$$

$$\approx 38.41 \text{ mm} \qquad \text{result}$$

We can find the length $c$ by simply applying the Pythagorean theorem, or using another trig ratio and a known angle.

For side $c$:

$$\sin 32° = \frac{24}{c} \qquad \sin 32° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$c \sin 32° = 24 \qquad \text{cross multiply}$$

$$c = \frac{24}{\sin 32°} \qquad \text{divide by } \sin 32° \approx 0.5299$$

$$\approx 45.29 \text{ mm} \qquad \text{result}$$

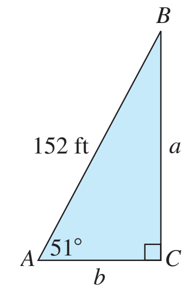
| Angles | Sides |
|---|---|
| $A = $ **32°** | $a = $ **24** |
| $B = 58°$ | $b \approx 38.41$ |
| $C = $ **90°** | $c \approx 45.29$ |

The complete solution is shown in the table.

NOW TRY EXERCISES 17 THROUGH 22

When the trig functions are applied to solve a triangle, any of the relationships available can be used: (1) angles must sum to 180°, (2) Pythagorean theorem, (3) standard triangles, and (4) the trigonometric functions of an acute angle. However the resulting equation must have only one unknown or it cannot be used. For the triangle shown in Figure 6.81, we cannot begin with the Pythagorean theorem since sides $a$ and $b$ are unknown, and $\tan 51°$ is unusable for the same reason. Since the hypotenuse is given, we could begin with $\cos 51° = \frac{b}{152}$ and first solve for $b$, or with $\sin 51° = \frac{a}{152}$ and solve for $a$, then work out a complete solution. Verify that $a \approx 118.13$ ft and $b \approx 95.66$ ft.

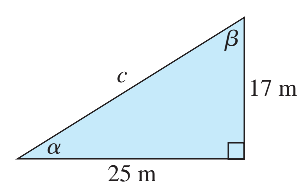
**Figure 6.81**




## C. Solving Right Triangles Given Two Sides

The partial table for $\sin\theta$ given in Table 6.12 was also used in times past to find an angle whose sine was known, meaning if $\sin\theta \approx 0.7604$, then $\theta$ must be 49.5° (see Table 6.12) This is shown on the last line of the table. As we noted in Section 6.2, the modern notation for "an angle whose sine is known" is $\theta = \mathbf{sin^{-1}}\boldsymbol{x}$ or $\theta = \mathbf{arcsin}\,\boldsymbol{x}$, where $x$ is the known value for $\sin\theta$. The values for the acute angles $\theta = \sin^{-1}x$, $\theta = \cos^{-1}x$, and $\theta = \tan^{-1}x$ are also programmed into your calculator and are generally accessed using the INV or 2nd keys with the related SIN, COS, or TAN key. With these we are completely equipped to find all six measures of a right triangle, given any three. As an alternative to naming the angles with a capital letter, we sometimes use a Greek letter at the interior of the vertex.

**EXAMPLE 4** Solve the triangle given in the figure.




**Solution:** Since the hypotenuse is unknown, we cannot begin with the sine or cosine ratios. The opposite and adjacent sides *are* known, so we use $\tan\alpha$. For $\tan\alpha = \frac{17}{25}$ we find $\alpha = \tan^{-1}\left(\frac{17}{25}\right) \approx 34.2°$. Since $\alpha$ and $\beta$ are complements, $\beta \approx 90 - 34.2 = 55.8°$. The Pythagorean theorem shows the hypotenuse is about 30.23 m.

| Angles | Sides |
|---|---|
| $\alpha \approx 34.2°$ | $a = \mathbf{17}$ |
| $\beta \approx 55.8°$ | $b = \mathbf{25}$ |
| $\gamma = \mathbf{90°}$ | $c = 30.23$ |

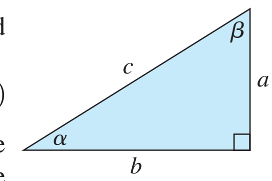
NOW TRY EXERCISES 23 THROUGH 54

We could have started Example 4 by using the Pythagorean theorem to find the hypotenuse, then $\sin^{-1}\left(\frac{17}{30.23}\right)$ or $\cos^{-1}\left(\frac{25}{30.23}\right)$ to find $\alpha$. Either expression gives a value very near 34.2°.

## D. Using Cofunctions and Complements to Write Equivalent Expressions

In the Figure 6.82, $\angle\alpha$ and $\angle\beta$ must be complements since we have a right triangle, and the sum of the angles must be 180°. The complementary angles in a right triangle have a unique relationship that is often used. Specifically $\alpha + \beta = 90°$ means $\beta = 90° - \alpha$. Note that $\sin\alpha = \frac{a}{c}$ and $\cos\beta = \frac{a}{c}$. This means $\sin\alpha = \cos\beta$ or $\sin\alpha = \cos(90° - \alpha)$ by substitution. In words, "The sine of an angle is equal to the cosine of its complement." For this reason sine and cosine are called **cofunctions** (hence the name cosine), as are secant/cosecant, along with tangent/cotangent. As a test, we use a calculator to test the statement $\sin 52.3° = \cos(90 - 52.3)°$

**Figure 6.82**




$$\sin 52.3 = \cos 37.7$$
$$0.791223533 = 0.791223533 ✓$$

**SUMMARY OF COFUNCTIONS**

| sine and cosine | tangent and cotangent | secant and cosecant |
|---|---|---|
| $\sin\theta = \cos(90 - \theta)$ | $\tan\theta = \cot(90 - \theta)$ | $\sec\theta = \csc(90 - \theta)$ |
| $\cos\theta = \sin(90 - \theta)$ | $\cot\theta = \tan(90 - \theta)$ | $\csc\theta = \sec(90 - \theta)$ |

For use in Example 5 and elsewhere in the text, note the expression $\cos^2 75°$ is simply a more convenient way of writing $(\cos 75°)^2$.

**EXAMPLE 5** ▷ Given $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$ in exact form, find the exact value of $4\cos^2 75°$. Check the result using a calculator.

**Solution:** ▷ Since $\sin 15°$ is given and $\cos 75°$ is unknown, we use the cofunction relationship $\cos 75° = \sin(90° - 75°) = \sin 15°$. This leads to

$$
\begin{aligned}
4\cos^2 75° &= 4\sin^2 15° && \text{cofunctions} \\
&= 4\left(\frac{1}{2}\sqrt{2 - \sqrt{3}}\right)^2 && \text{substitute known value} \\
&= 4\left[\frac{1}{4}(2 - \sqrt{3})\right] && \text{square as indicated} \\
&= 2 - \sqrt{3} && \text{result}
\end{aligned}
$$

Using a calculator we find $4\cos^2 75° \approx 0.2679491924 \approx 2 - \sqrt{3}$.

NOW TRY EXERCISES 55 THROUGH 68 ▷

## E. Applications Using Angles of Elevation/Depression

While we've used the basic concept previously, in more formal terms an **angle of elevation** is defined to be the acute angle formed by a horizontal **line of orientation** (parallel to level ground) and the light of sight (see Figure 6.83). An **angle of depression** is likewise defined but involves a line of sight that is below the line of orientation. (Figure 6.84).

**Figure 6.83**

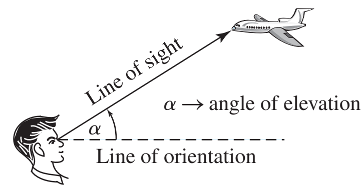
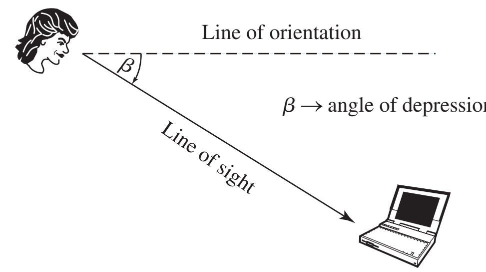
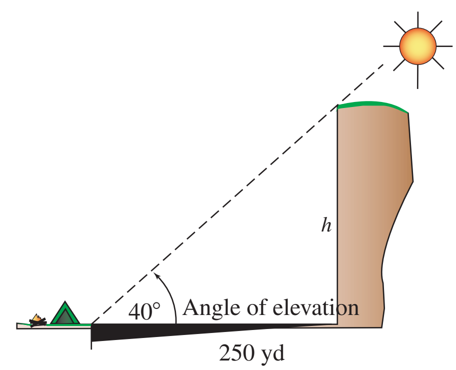



**Figure 6.84**




Angles of elevation/depression make distance and length computations of all magnitudes a relatively easy matter and are extensively used by surveyors, engineers, astronomers, and even the casual observer who is familiar with the basics of trigonometry.

**EXAMPLE 6** ▶ In Example 4 from Section 6.1, a group of campers used a 45-45-90 triangle to estimate the height of a cliff. It was a time consuming process as they had to wait until mid-morning for the shadow of the cliff to make the needed 45° angle. If the campsite was 250 yd from the base of the cliff and the angle of elevation was 40° at that point, how tall is the cliff?




**Solution:** ▶ As described we want to know the height of the opposite side, given the adjacent side, so we use the tangent function.

For height $h$: $\tan 40° = \dfrac{h}{250}$ $\quad$ $\tan 40° = \dfrac{\text{opposite}}{\text{adjacent}}$

$250 \tan 40° = h$ $\quad$ cross multiply

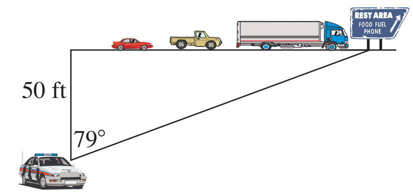
$209.8 \approx h$ $\quad$ result ($\tan 40° \approx 0.8391$)

The cliff is approximately 209.8 yd high (about 629 ft).

NOW TRY EXERCISES 71 THROUGH 74 ▶

Closely related to angles of depression/elevation, are acute angles of rotation from a fixed orientation to a fixed line of sight. In this case, the movement is simply horizontal rather than vertical.

**EXAMPLE 7** ▶ To thwart drivers who have radar detection equipment, a state trooper takes up a hidden position 50 ft from the roadway. Using a sighting device she finds the angle between her position and a road sign in the distance is 79°. She then uses a stop watch to determine how long it takes a vehicle to pass her location and reach the road sign. In quick succession—an 18-wheeler, a truck, and a car pass her position, with the time each takes to travel this distance noted. Find the speed of each vehicle in miles per hour if (a) the 18-wheeler takes 2.7 sec, (b) the truck takes 2.3 sec, and (c) the car takes 1.9 sec.




**Solution:** We begin by finding the distance traveled by each vehicle. Using $\tan 79° = \frac{d}{50}$, gives $d = 50 \tan 79° \approx 257$ ft. To convert a rate given in feet per second to miles per hour, recall there are 5280 feet in 1 mi and 3600 sec in 1 hr.

**a.** 18-wheeler: $\left(\frac{257 \text{ ft}}{2.7 \text{ sec}}\right)\left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right)\left(\frac{3600 \text{ sec}}{1 \text{ hr}}\right) \approx \left(\frac{65 \text{ mi}}{1 \text{ hr}}\right)$
The 18-wheeler is traveling approximately 65 mph.

**b.** Using the same calculation with 2.3 sec shows the truck was going about 76 mph.

**c.** At 1.9 sec, the car was traveling about 92 mph.

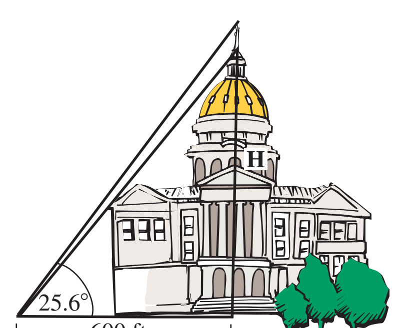
NOW TRY EXERCISES 75 AND 76

## F. General Applications of Right Triangles

In their widest and most beneficial use, the trig functions of acute angles are used in the context of other problem-solving skills, such as drawing a diagram, labeling unknowns, working the solution out in stages, and so on. Examples 8 and 9 serve to illustrate some of these combinations.

**EXAMPLE 8** Virtually everyone is familiar with the Statue of Liberty in New York Bay, but fewer know that America is home to a second "Statue of Liberty" standing proudly atop the iron dome of the Capitol Building. From a distance of 600 ft, the angle of elevation from ground level to the top of the statue (from the east side) is 25.60°. The angle of elevation to the base of the statue is 24.07°. How tall is *Freedom,* the name sculptor Thomas Crawford gave this statue?

**Solution:** Begin by drawing a diagram that relates all of the information given. Using tan 25.60° we can find the height $H$ from ground level to the top of the statue.




For height $H$:

$$\tan 25.6° = \frac{H}{600} \qquad \text{tan } 25.6° = \frac{\text{opposite}}{\text{adjacent}}$$

$$600 \tan 25.6° = H \qquad \text{solve for } H$$

$$287.5 \approx H \qquad \text{result } (\tan 25.60° \approx 0.4791)$$

The height $H$ is approximately 287.5 ft. Using tan 24.07° we can find the height $h$ from ground level to the top of the dome (the foot of the statue).

For height $h$:

$$\tan 24.07° = \frac{h}{600} \qquad \text{tan } 24.07° = \frac{\text{opposite}}{\text{adjacent}}$$

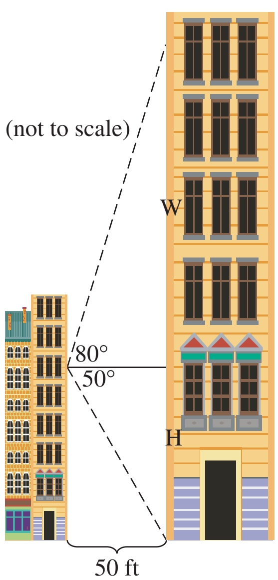
$$600 \tan 24.07° = h \qquad \text{cross multiply}$$

$$268.0 \approx h \qquad \text{result } (\tan 24.07° \approx 0.4467)$$

The height of *Freedom* is $H - h$, or approximately 19.5 ft.

**NOW TRY EXERCISES 77 AND 78**

**EXAMPLE 9** From his hotel room window on the fifth floor, Singh notices some window washers high above him on the hotel across the street. Curious as to their height above ground, he quickly estimates the buildings are 50 ft apart, the angle of elevation to the workers is about 80° and the angle of depression to the base of the hotel is about 50°. (a) How high above ground is the window of Singh's hotel room? (b) How high above ground are the workers?




**Solution:** **a.** To find the height of the window we'll use the tangent ratio, since the adjacent side of the angle is known, and the opposite side is the height we desire.

For the height $h$:

$$\tan 50° = \frac{h}{50} \qquad \tan 50° = \frac{\text{opposite}}{\text{adjacent}}$$

$$50 \tan 50° = h \qquad \text{solve for } h$$

$$59.6 \approx h \qquad \text{result } (\tan 50° \approx 1.1918)$$

The window is approximately 59.6 ft above ground.

**b.** For the workers $w$:

$$\tan 80° = \frac{w}{50} \qquad \tan 80° = \frac{\text{opposite}}{\text{adjacent}}$$

$$50 \tan 80° = h \qquad \text{solve for } h$$

$$283.6 \approx h \qquad \text{result } (\tan 80° \approx 5.6713)$$

The workers are approximately $283.6 + 59.6 = 343.2$ ft above ground.

**NOW TRY EXERCISES 79 AND 80**

There are a number of additional, interesting applications in the exercise set.

## TECHNOLOGY HIGHLIGHT

## Using the Storage and Recall Features of a Graphing Calculator

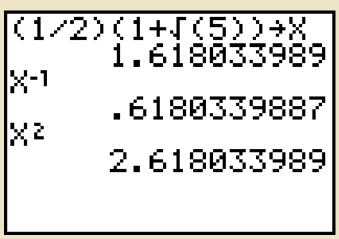
*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

Computations involving the trig ratios often produce irrational numbers. Sometimes the number is used numerous times in an application, and it helps to store the value in a memory location so it can instantly be recalled without having to look it up, recompute its value or enter it digit by digit. Storage locations are also used when writing programs for your graphing calculator. Suppose the value $\frac{1+\sqrt{5}}{2} \approx 1.6180339887$ were to be used repeatedly as you explored certain relationships. You could store this value in the *temporary memory location* X,T,θ,n using the keystrokes $\frac{1+\sqrt{5}}{2}$ STO▸ X,T,θ,n, or in a permanent memory location using the ALPHA locations A through Z. In temporary storage the value could potentially be overwritten once you leave the home screen. To use a stored value, we simply bring up the location name to the home screen.

EXAMPLE 1 Save the value of $\frac{1+\sqrt{5}}{2}$ in location X,T,θ,n, then investigate the relationship between (a) this number and its reciprocal and (b) this number and its square. What do you notice? Why does the value of $x^{-1}$ seem to be off by one decimal place?

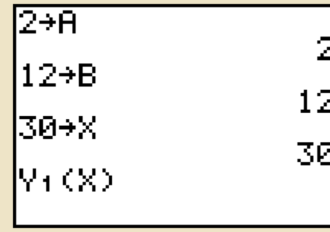
Solution: After storing the number as shown in Figure 6.85, we find its reciprocal is equal to the original number minus 1, while its square is equal to the original number plus 1. The value of $x^{-1}$ appears off due to rounding.

**Figure 6.85**




Strangely enough, since the discovery of this number had nothing to do with trig, this is also the value of 2 sin 54°. Actually $\frac{1+\sqrt{5}}{2}$ is closely related to the ratio $\frac{\sqrt{5}-1}{2}$, called the **golden ratio,** which has a number of interesting properties itself.

**Figure 6.86**




Now suppose we wanted to investigate the trigonometric formula for a triangle's area: $A = \frac{1}{2}ab\sin\theta$, where $b$ is the base, $a$ is the length of one side, and $\theta$ is the angle between them. The formula has four unknowns, $A$, $a$, $b$, and $\theta$. The idea here is to let $\theta$ represent the independent variable $x$, and $A$ the dependent variable $y$, then evaluate the function for different values of $a$ and $b$. On the Y= screen, enter the formula as $Y_1 = 0.5\text{AB}\sin \text{X}$. On the home screen, store a value of 2 in location A and 12 in location B using 2 STO▸ ALPHA MATH and 12 STO▸ ALPHA APPS, respectively, as shown in Figure 6.86. The final result can be computed on the home screen (or using the TABLE feature) by supplying a value to memory location X,T,θ,n, then calling up $Y_1$ ( VARS ▶ **(Y-VARS)** 1:Function ENTER ) and evaluating $Y_1(X)$. As Figure 6.86 shows, we used for 30 STO▸ X,T,θ,n, and the area of a triangle with $a = 2$, $b = 12$, and $\theta = 30°$ is 6 units$^2$ (although the display couldn't hold all of the information without scrolling).

Exercise 1: Evaluate the area formula once again, using $A = 5$ and $B = 5\sqrt{3}$. What values of X will give an area greater than 10 units$^2$?

## 6.6 EXERCISES

### CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** The phrase, "an angle whose tangent is known," is written notationally as ___________.

**2.** Given $\sin\theta = \frac{7}{24}$, $\csc\theta =$ ___________ because they are ___________.

**3.** The sine of an angle involves the ratio of the ___________ side to the ___________.

**4.** The cosine of an angle involves the ratio of the ___________ side to the ___________.

5. Discuss/explain exactly what is meant when you are asked to "solve a triangle." Include an illustrative example.

6. Given an acute angle and the length of the adjacent leg, which four (of the six) trig functions could be used to begin solving the triangle?

## DEVELOPING YOUR SKILLS

Use the function value given to determine the value of the other five trig functions of the acute angle $\theta$. Answer in exact form (a diagram will help).

7. $\cos\theta = \frac{5}{13}$

8. $\sin\theta = \frac{20}{29}$

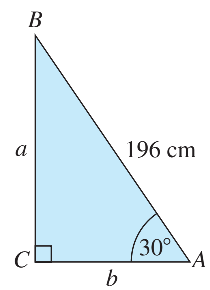
9. $\tan\theta = \frac{84}{13}$

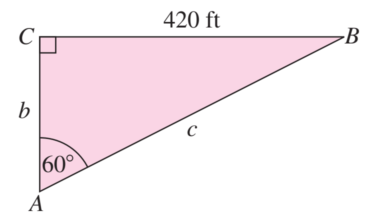
10. $\sec\theta = \frac{53}{45}$

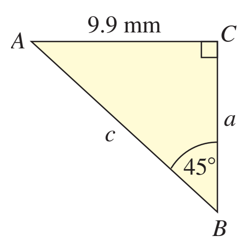
11. $\cot\theta = \frac{2}{11}$

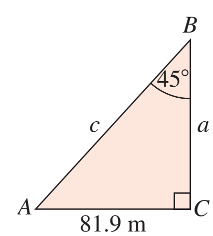
12. $\cos\theta = \frac{2}{3}$

Solve each triangle using trig functions of an acute angle $\theta$. Give a complete answer (in table form) using exact values.

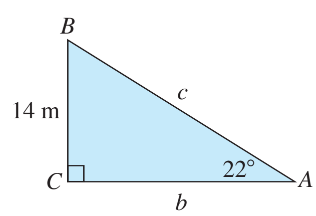
13.


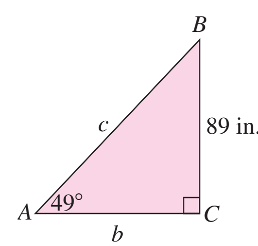

14.


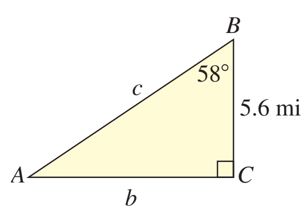

15.


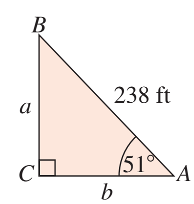

16.



Solve the triangles shown and write answers in table form. Round sides to the nearest 100th of a unit. Double check that angles sum to 180° and that the three sides satisfy (approximately) the Pythagorean theorem.

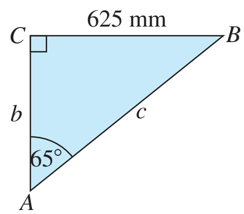
17.



18.


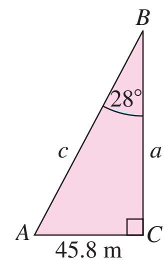

19.



20.


**21.**




**22.**




Use a calculator to find the indicated ratios for each angle, rounded to four decimal places.

**23.** sin 27° **24.** cos 72° **25.** tan 40° **26.** cot 57.3°

**27.** sec 40.9° **28.** csc 39° **29.** sin 65° **30.** tan 84.1°

Use a calculator to find the angle whose corresponding ratio given. Round to the nearest 10th of a degree. For Exercises 31 through 38, note the relationship to Exercises 23 through 30.

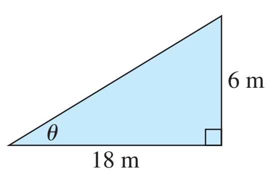
**31.** $\sin A = 0.4540$ **32.** $\cos B = 0.3090$ **33.** $\tan\theta = 0.8391$ **34.** $\cot A = 0.6420$

**35.** $\sec B = 1.3230$ **36.** $\csc\beta = 1.5890$ **37.** $\sin A = 0.9063$ **38.** $\tan B = 9.6768$

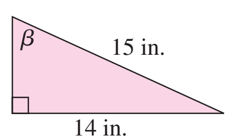
**39.** $\tan\alpha = 0.9903$ **40.** $\cot\alpha = 0.9903$ **41.** $\sin\alpha = 0.9903$ **42.** $\tan\alpha = 3.1245$

Select an appropriate ratio to find the angle indicated (round to 10ths of a degree).

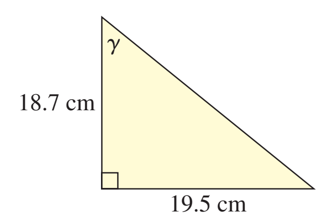
**43.**



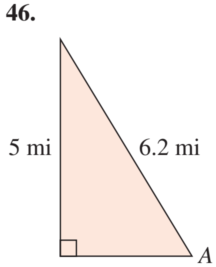

**44.**



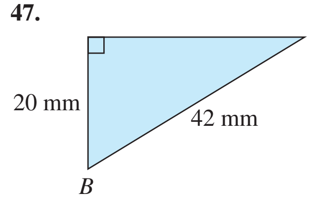

**45.**



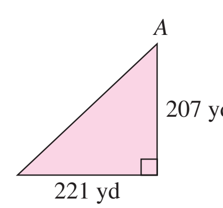

**46.**




**47.**

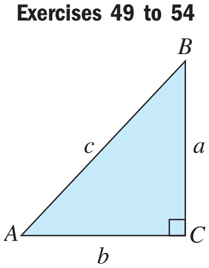



**48.**




Draw a right triangle *ABC* as shown, using the information given. Then select an appropriate ratio to find the side indicated. Round to the nearest 100th.

**Exercises 49 to 54**




**49.** $\angle A = 25°$
$c = 52$ mm
find side $a$

**50.** $\angle B = 55°$
$b = 31$ ft
find side $c$

**51.** $\angle A = 32°$
$a = 1.9$ mi
find side $b$

**52.** $\angle B = 29.6°$
$c = 9.5$ yd
find side $a$

**53.** $\angle A = 62.3°$
$b = 82.5$ furlongs
find side $c$

**54.** $\angle B = 12.5°$
$a = 32.8$ km
find side $b$

Use a calculator to evaluate each pair of functions and comment on what you notice.

**55.** sin 25°, cos 65° **56.** sin 57°, cos 33° **57.** tan 5°, cot 85° **58.** sec 40°, csc 50°

Based on your observations in Exercises 55 to 58, fill in the blank so that the ratios given are equal.

**59.** sin 47°, cos ____ **60.** cos ____, sin 12° **61.** cot 69°, tan ____ **62.** csc 17°, sec ____

Complete the following tables without referring to the text or using a calculator.

**63.**

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\sin(90-\theta)$ | $\cos(90-\theta)$ | $\tan(90-\theta)$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| 30° | | | | | | | | | |

**64.**

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\sin(90-\theta)$ | $\cos(90-\theta)$ | $\tan(90-\theta)$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| 45° | | | | | | | | | |

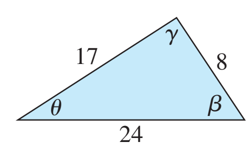
Evaluate the expressions below without a calculator, using the cofunction relationship and the following exact forms: $\sin 18° = \frac{1}{4}(\sqrt{5}-1)$; $\tan 75° = 2+\sqrt{3}$; and $\sin 22.5 = \frac{1}{2}\sqrt{2-\sqrt{2}}$

**65.** $4\cos 72°$ **66.** $\cot^2 15°$ **67.** $\cos 67.5°$ **68.** $4\cos^2 67.5°$

## WORKING WITH FORMULAS

**69. The sine of an angle between two sides of a triangle:** $\sin\theta = \dfrac{2A}{ab}$

If the area $A$ and two sides $a$ and $b$ of a triangle are known, the sine of the angle between the two sides is given by the formula shown. Find the angle $\theta$ for the triangle to the right given $A \approx 38.9$, and use it to solve the triangle. (*Hint:* Apply the same concept to angle $\gamma$ or $\beta$.)




**70. Illumination of a surface:** $E = \dfrac{I\cos\theta}{d^2}$

The illumination $E$ of a surface by a light source is a measure of the luminous flux per unit area that reaches the surface. The value of $E$ [in lumens (lm) per square foot] is given by the formula shown, where $d$ is the distance from the light source (in feet), $I$ is the intensity of the light [in candelas (cd)], and $\theta$ is the angle the light flux makes with the vertical. For reading a book, an illumination $E$ of at least 18 lm/ft$^2$ is recommended. Assuming the open book is lying on a horizontal surface, how far away should a light source be placed if it has an intensity of 90 cd (about 75 W) and the light flux makes an angle of 65° with the book's surface (i.e., $\theta = 25°$)?

## APPLICATIONS

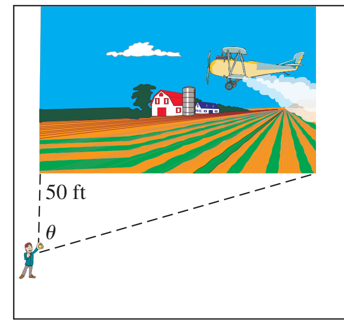
**71. Angle of elevation:** For a person standing 100 m from the center of the base of the Eiffel Tower, the angle of elevation to the top of the tower is 71.6°. How tall is the Eiffel Tower?

**72. Angle of depression:** A person standing near the top of the Eiffel Tower notices a car wreck some distance from the tower. If the angle of depression from the person's eyes to the wreck is 32°, how far away is the accident? See Exercise 71.

**73. Angle of elevation:** In 2001, the tallest building in the world was the Petronas Tower I in Kuala Lumpur, Malaysia. For a person standing 25.9 ft from the base of the tower, the angle of elevation to the top of the tower is 89°. How tall is the Petronas tower?

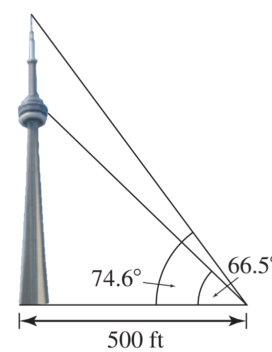
**74.** **Angle of depression:** A person standing on the top of the Petronas Tower I looks out across the city and pinpoints her residence. If the angle of depression from the person's eyes to her home is 5°, how far away (in feet and in miles) is the residence? See Exercise 7.3.

**75.** **Acute angle of rotation:** While standing near the edge of a farmer's field, Johnny watches a crop-duster dust the farmer's field for insect control. Curious as to the plane's speed during each drop, Johnny attempts an estimate using the angle of rotation from one end of the field to the other, while standing 50 ft from one corner. Using a stopwatch he finds the plane makes each pass in 2.35 sec. If the angle of rotation was 83°, how fast (in miles per hour) is the plane flying as it applies the insecticide?




**76.** **Acute angle of rotation:** While driving to their next gig, Josh and the boys get stuck in a line of cars at a railroad crossing as the gates go down. As the sleek, speedy express train approaches, Josh decides to pass the time estimating its speed. He spots a large oak tree beside the track some distance away, and figures the angle of rotation from the crossing to the tree is about 80°. If their car is 60 ft from the crossing and it takes the train 3 sec to reach the tree, how fast is the train moving in miles per hour?

**77.** **Angle of elevation:** The tallest free-standing tower in the world is the CNN Tower in Toronto, Canada. The tower includes a rotating restaurant high above the ground. From a distance of 500 ft the angle of elevation to the pinnacle of the tower is 74.6°. The angle of elevation to the restaurant from the same vantage point is 66.5°. How tall is the CNN Tower? How far below the pinnacle of the tower is the restaurant located?



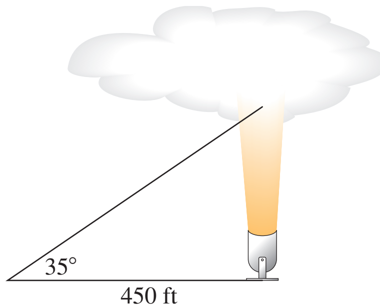

**78.** **Angle of elevation:** In August 2004, Taipei 101 captured the record as the world's tallest building, according to the Council on Tall Buildings and Urban Habitat [Source: www.ctbuh.org]. Measured at a point 108 m from its base, the angle of elevation to the top of the spire is 78°. From a distance of about 95 m, the angle of elevation to the top of the roof is also 78°. How tall is Taipei 101 from street level to the top of the spire? How tall is the spire itself?

**79.** A local Outdoors Club has just hiked to the south rim of a large canyon, when they spot some climbers attempting to scale the northern face. Knowing the distance between the northern and southern faces of the canyon is approximately 175 yd, they attempt to compute the distance remaining for the climbers to reach the top of the northern rim. Using a homemade transit, they sight an angle of depression of 55° to the bottom of the north face, and angles of elevation of 24° and 30° to the climbers and top of the northern rim respectively. (a) How high is the southern rim of the canyon? (b) How high is the northern rim? (c) How much further until the climbers reach the top?

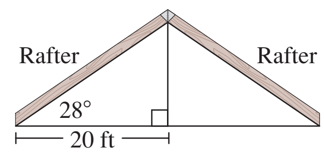
**80.** From her elevated observation post 300 ft away, a naturalist spots a troop of baboons high up in a tree. Using the small transit attached to her telescope, she finds the angle of depression to the bottom of this tree is 14°, while the angle of elevation to the top of the tree is 25°. The angle of elevation to the troop of baboons is 21°. Use this information to find (a) the height of the observation post, (b) the height of the baboon's tree, and (c) the height of the baboons above ground.

**81.** **Cloud cover:** To find the height of cloud cover, a search light is beamed directly upward and the measurements shown in the figure are taken. Determine the height of the clouds.

**Exercise 81**



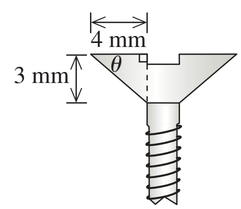

**Exercise 82**



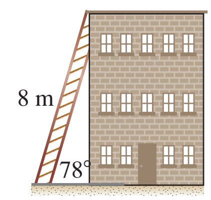

**82.** **Rafter length:** Find the length of the rafter for a roof with the dimensions shown.

**83.** **Angle of taper:** Find the *angle of taper* $\theta$ for the screw head with the dimensions shown.

**Exercise 83**

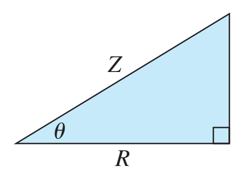



**Exercise 84**




**84.** **Ladder safety:** The recommended safety angle for a ladder is 78°. If a 8-m ladder is used, how far up a building will the ladder safely reach?

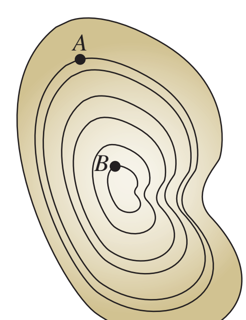
**Alternating current:** In AC (alternating current) applications, the relationship between measures known as the impedance ($Z$), resistance ($R$), and the phase angle ($\theta$) can be demonstrated using a right triangle. Both the resistance and the impedance are measured in ohms, with the symbol $\Omega$.



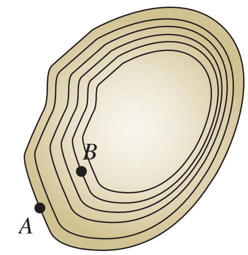

**85.** Find the impedance $Z$ if the phase angle $\theta$ is 34°, and the resistance $R$ is 320 $\Omega$.

**86.** Find the phase angle $\theta$ if the impedance $Z$ is 420 $\Omega$, and the resistance $R$ is 290 $\Omega$.

**87.** **Contour maps:** In the figure shown, the *scale of elevation* is 1:175 (each concentric line represents an increase of 175 m in elevation), and the scale of horizontal distances is 1 cm = 500 m. (a) Find the vertical change from $A$ to $B$ (the increase in elevation); (b) use a proportion to find the horizontal change between points $A$ and $B$ if the measured distance on the map is 2.4 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects $A$ and $B$, then use trig to compute the approximate angle of incline as the hiker climbs from point $A$ to point $B$.



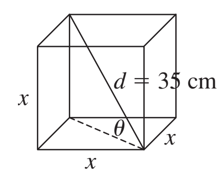
**88. Contour maps:** In the figure shown the *scale of elevation* is 1:150 (each concentric line represents an increase of 150 m in elevation), and the scale of horizontal distances is 1 cm = 250 m. (a) Find the vertical change from $A$ to $B$ (the increase in elevation); (b) use a proportion to find the horizontal change between points $A$ and $B$ if the measured distance on the map is 4.5 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects $A$ and $B$, then use trig to compute the approximate angle of incline as the hiker climbs from point $A$ to point $B$.




**89. Height of a rainbow:** While visiting the Lapahoehoe Memorial on the island of Hawaii, Bruce and Carma see a spectacularly vivid rainbow arching over the bay. Bruce speculates the rainbow is 500 ft away, while Carma estimates the angle of elevation to the highest point of the rainbow is about 27°. What was the approximate height of the rainbow?

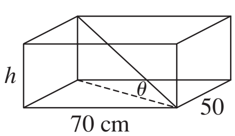
**90. High-wire walking:** As part of a circus act, a high-wire walker not only "walks the wire," she walks a wire that is *set at an incline of* 10° to the horizontal! If the length of the (inclined) wire is 25.39 m, (a) how much higher is the wire set at the destination pole than at the departure pole? (b) How far apart are the poles?

**91. Diagonal of a cube:** A cubical box has a diagonal measure of 35 cm. (a) Find the dimensions of the box and (b) the angle $\theta$ that the diagonal makes at the lower corner of the box.




**92. Diagonal of a rectangular parallelepiped:** A rectangular box has a width of 50 cm and a length of 70 cm. (a) Find the height $h$ that ensures the diagonal across the middle of the box will be 90 cm and (b) the angle $\theta$ that the diagonal makes at the lower corner of the box.

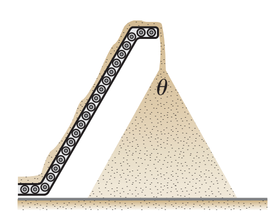
**Exercise 92**



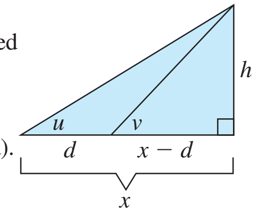

## WRITING, RESEARCH, AND DECISION MAKING

**93.** As you can see from the preceding collection of exercises, trigonometry has many intriguing, useful, and sometimes fun applications. Create two application exercises of your own that are modeled on those here, or better yet come up with an original! Ask a fellow student to solve them and if you both agree they are "*good*" exercises, see if your instructor will consider using one of them on the next quiz or test (you never know).

**94.** An *angle of repose* is the angle at which the very top layer of elements begins sliding down the sides of a conical pile as more material is added to the top. In other words, the height and width of the conical pile may change, but the angle of repose will remain fairly constant and depends on the type of material being dumped. Angles of repose are important to the study and prevention of avalanches. Using the library, Internet, or other available resources, do some further research on angles of repose and their applications. Write up a short summary of what you find and include a few illustrative examples.



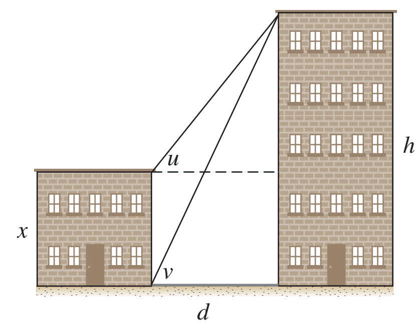
## EXTENDING THE CONCEPT



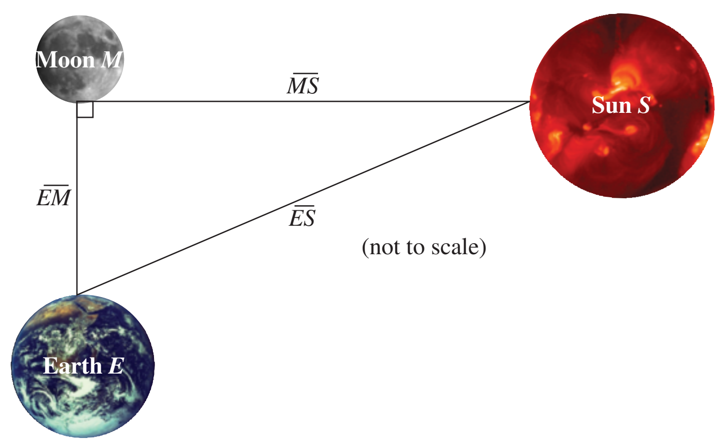

**95.** In Exercise 48 of Section 6.4 the formula $h = \dfrac{d}{\cot u - \cot v}$ was used to calculate the height $h$ of a building when distance $x$ is unknown but distance $d$ is known (see the diagram). Use the ratios for $\cot u$ and $\cot v$ to derive the formula (note $x$ is "absent" from the formula).

**96.** Use the diagram given to derive a formula for the height $h$ of the taller building in terms of the height $x$ of the shorter building and the ratios for $\tan u$ and $\tan v$. Then use the formula to find $h$ given the shorter building is 75 m tall with $u = 40°$ and $v = 50°$.

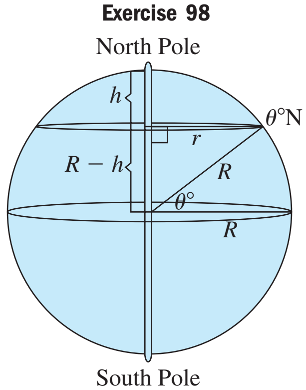
**Exercise 96**




**97.** Aristarchus of Samos (~310–230 B.C.) was a Greek astronomer/mathematician. He appears to be among the first to realize that when the moon is in its first quarter, the triangle formed by the Sun, the Earth, and the Moon ($\Delta EMS$ in the figure) must be a right triangle. Although he did not have trigonometry or even degrees at his disposal, in effect he estimated $\angle MES$ to be 87° and used this right triangle to reckon how many times further the Sun was from the Earth, than the Moon was from the Earth (the true angle is much closer to 89.85°). Using 240,000 mi as the distance $\overline{EM}$ from the Earth to the Moon, (a) find Aristarchus' original estimate of the Sun's distance from the Earth, (b) compute the difference between Aristarchus' estimate and the improved estimate using 89.85°; and (c) determine why the error is so large for a mere 2.85° difference.




**Exercise 98**

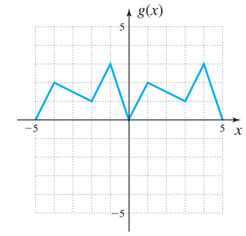



**98.** The radius of the Earth at the equator (0° N latitude) is approximately 3960 mi. Beijing, China, is located at 39.5° N latitude, 116° E longitude. Philadelphia, Pennsylvania, is located at the same latitude, but at 75° W longitude. (a) Use the diagram given and a cofunction relationship to find the radius $r$ of the Earth (parallel the equator) at this latitude; (b) use the arc length formula to compute the *shortest distance* between these two cities along this latitude; and (c) if the supersonic Concorde flew a direct flight between Beijing and Philadelphia along this latitude, approximate the flight time assuming a cruising speed of 1250 mph. (Note: The shortest distance is actually found by going over the North Pole.)

## MAINTAINING YOUR SKILLS

**99.** (2.2) Evaluate the function $f(x) = x^2 - 6x$ at $f(-1)$, $f(\frac{1}{2})$, and $f(x + h)$.

**100.** (2.4) Name and sketch the graph of the six toolbox functions, and classify each as even, odd, or neither.




**101.** (3.8/6.3) State the period and relative max/min values of the function whose graph is shown, then use the graph to state all real solutions to $g(x) = 3$.

**102.** (2.1) Regarding Exercise 101, use right triangles and the Pythagorean theorem to find the *length* of the graph from $x = -5$ to $x = 0$.

**103.** (4.2) Use the remainder theorem and synthetic division to determine if $x = \frac{2}{3}$ is a zero of $f(x) = 3x^3 + 4x^2 - 13x + 6$.

**104.** (6.5) Given $y = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{8}\right) + 4$, state the amplitude, period, average value, horizontal shift, and endpoints of the primary interval. Then sketch its graph.

# 6.7 Trigonometry and the Coordinate Plane

## LEARNING OBJECTIVES

***In Section 6.7 you will learn how to:***

**A.** **Define the trigonometric functions using the coordinates of a point in QI**

**B.** **Use references angles to evaluate the trig functions for any angle**

**C.** **Solve applications using the trig functions of any angle**

## INTRODUCTION

This section tends to bridge the study of *dynamic trigonometry* and the unit circle, with the study of *static trigonometry* and the angles of a right triangle. This is accomplished by noting the domain of the trig functions from a triangle point of view *need not be restricted to acute angles.* We'll soon see that the domain can be extended to include trig functions of any angle, a view that greatly facilitates our work in Chapter 8, where many applications involve angles greater than 90°.

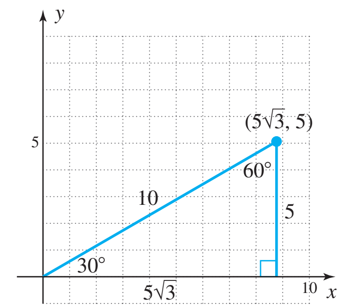
### POINT OF INTEREST

Science and mathematics flowered brilliantly in India from 500 to 1200 A.D. This period saw publication of *Surya Siddhanta* (Knowledge from the Sun), a remarkable treatise on astronomy. Significant advances were also made in mathematics, as evidenced by the works of Aryabhata (476–551 A.D.), Brahmagupta (~630 A.D.), Mahavira (~850 A.D.), and Bhaskara (1114–1185 A.D.). Of interest to us here is Bhaskara's recognition, although he did not state it expressly, that the sides of all right triangles are in proportion $2x:x^2 - 1:x^2 + 1$ for some real number $x$. For the Pythagorean triple (3, 4, 5), $x = 2$, while for (28, 45, 53), $x = \frac{7}{2}$. For more on Pythagorean triples, see the *Technology Extension: Generating Pythagorean Triples at* www.mhhe.com/coburn.

## A. Trigonometric Ratios and the Point $P(x, y)$

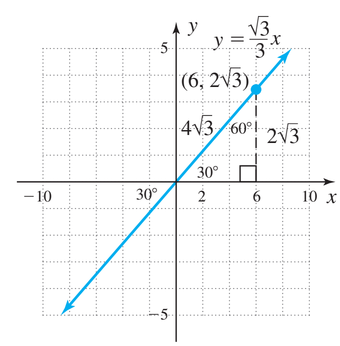
Regardless of where a right triangle is situated or how it is oriented, the definition of each trig function is defined as a given ratio of sides with respect to a given angle. In this light, consider a 30-60-90 triangle placed in the first quadrant with the 30° angle at the origin and the longer side along the $x$-axis. From our previous study of similar triangles, the trig ratios will have the same value regardless of the triangle's size so for convenience, we'll use a hypotenuse of 10. This gives sides of 5, $5\sqrt{3}$, and 10, and from the diagram in Figure 6.87 we note the point $(x, y)$ marking the vertex of the 60° angle has coordinates $(5\sqrt{3}, 5)$. Further, the diagram shows that sin 30°, cos 30°, and tan 30° can all be expressed in terms of these coordinates since $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{10} = \frac{y}{r}$, $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5\sqrt{3}}{10} = \frac{x}{r}$, and $\frac{\text{opposite}}{\text{adjacent}} = \frac{5}{5\sqrt{3}} = \frac{y}{x}$, where $r$ is the length of the hypotenuse. Each result reduces to the more familiar values seen earlier: $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, and $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. This suggests we might be able to define the six trig functions in terms of $x$, $y$, and $r$, where $r = \sqrt{x^2 + y^2}$ (the distance from the point to the origin). The slope of the line coincident with the hypotenuse is $\frac{\text{rise}}{\text{run}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, and since the line goes through the origin its equation must be $y = \frac{\sqrt{3}}{3}x$. Any point $(x, y)$ on this line will be at the 60° vertex of a right triangle formed by drawing a perpendicular line from the point $(x, y)$ to the $x$-axis. As Example 1 shows, we obtain the standard values for sin 30°, cos 30°, and tan 30° regardless of the point chosen.

**Figure 6.87**




**EXAMPLE 1** Pick an arbitrary point in QI that satisfies $y = \frac{\sqrt{3}}{3}x$, then draw the corresponding right triangle and evaluate sin 30°, cos 30°, and tan 30°.

**Solution:** The coefficient of $x$ has a denominator of 3, so we choose a multiple of 3 for convenience. For $x = 6$ we have $y = \frac{\sqrt{3}}{3}(6) = 2\sqrt{3}$. As seen in the figure, the point $(6, 2\sqrt{3})$ is on the line and at the vertex of the 60° angle. Evaluating the trig functions at 30°, we obtain:




$$\sin 30° = \frac{y}{r} = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2} \qquad \cos 30° = \frac{x}{r} = \frac{6}{4\sqrt{3}} = \frac{6\sqrt{3}}{4\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{y}{x} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

NOW TRY EXERCISES 7 AND 8

In general, consider *any* two points $(x, y)$ and $(X, Y)$ on an arbitrary line $y = kx$, at corresponding distances $r$ and $R$ from the origin (Figure 6.88). Because the triangles formed are similar, we have $\frac{y}{x} = \frac{Y}{X}, \frac{x}{r} = \frac{X}{R}$, and so on, and we conclude that the value of the trig functions are indeed independent of the point chosen.

**Figure 6.88**

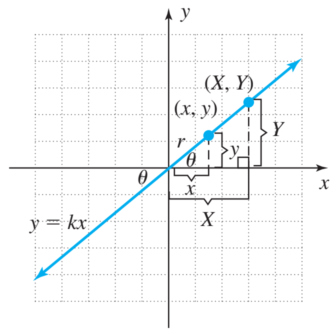
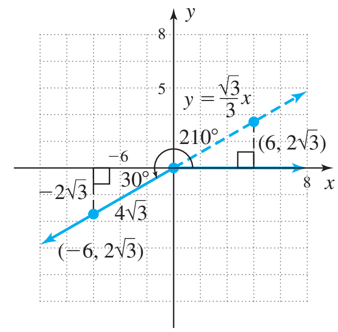


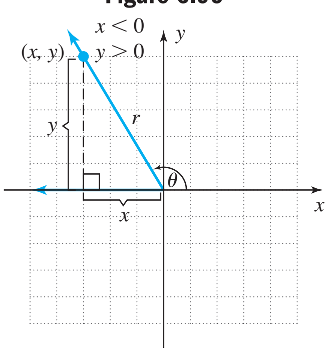

**Figure 6.89**




Viewing the trig functions in terms of $x$, $y$, and $r$ produces significant and powerful results. In Figure 6.89, we note the line $y = \frac{\sqrt{3}}{3}x$ from Example 1 also extends into QIII, and *creates another* 30° *angle whose vertex is at the origin* (since vertical angles are equal). The sine, cosine, and tangent functions can still be evaluated for this angle, but in QIII both $x$ and $y$ are negative. It's here that our view of angles as a rotation bridges the unit circle view of trigonometry with the right triangle view. If we consider the angle in QIII to be a positive rotation of 210° (180° + 30°), we can evaluate the trig functions using the values of $x$, $y$, and $r$ from any point on the terminal side, since these are fixed by the 30° angle created, and they're *equivalent to those in QI except for their sign:*

$$\sin 210° = \frac{y}{r} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2} \qquad \cos 210° = \frac{x}{r} = \frac{-6}{4\sqrt{3}} = -\frac{\sqrt{3}}{2} \qquad \tan 210° = \frac{y}{x} = \frac{-2\sqrt{3}}{-6} = \frac{\sqrt{3}}{3}$$

**Figure 6.90**




For *any* rotation $\theta$ and a point $(x, y)$ on the terminal side, the distance $r$ can be found using $r = \sqrt{x^2 + y^2}$ and the six trig functions defined in terms of $x$, $y$, and $r$. Figure 6.90 shows a rotation that terminates in QII, where $x$ is negative and $y$ is positive. Correctly evaluating the trig functions of any angle depends heavily on the quadrant of the terminal side, since this will dictate the signs for $x$ and $y$. Students are strongly encouraged to make this determination the *first step* in any solution process. In summary, we have:

**TRIGONOMETRIC FUNCTIONS OF ANY ANGLE**

Given $P(x, y)$ is any point on the terminal side of angle $\theta$ in standard position, with $r = \sqrt{x^2 + y^2}$ the distance from the origin to $(x, y)$, the six trigonometric functions of $\theta$ are

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}, \; x \neq 0$$

$$\csc\theta = \frac{r}{y}, \; y \neq 0 \qquad \sec\theta = \frac{r}{x}, \; x \neq 0 \qquad \cot\theta = \frac{x}{y}, \; y \neq 0$$

**EXAMPLE 2** ▶ Find the value of the six trigonometric functions given $P(-5, 5)$ is on the terminal side of angle $\theta$ in standard position.

**Solution:** ▶ For $P(-5, 5)$ we have $x < 0$ and $y > 0$ so the terminal side is in QII. Solving for $r$ yields $r = \sqrt{(-5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$. For $x = -5$, $y = 5$, and $r = 5\sqrt{2}$, we obtain

$$\sin\theta = \frac{y}{r} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{y}{x} = \frac{5}{-5} = -1$$

The remaining functions can be evaluated using the reciprocal relationships.

$$\csc\theta = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \sec\theta = -\frac{2}{\sqrt{2}} = -\sqrt{2} \qquad \cot\theta = -1$$

Note the connection between these results and the standard values for $\theta = 45°$.

NOW TRY EXERCISES 9 THROUGH 24 ▶

**EXAMPLE 3** ▶ Given that $P(x, y)$ is a point on the terminal side of angle $\theta$ in standard position, find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ if (a) the terminal side is in QII and coincident with the line $y = -\frac{12}{5}x$ and (b) the terminal side is in QIV and coincident with the line $y = -\frac{12}{5}x$.

**Solution:** ▶ **a.** Select any convenient point in QII that satisfies this equation. We select $x = -5$ since $x$ is negative in QII, which gives $y = 12$ and the point $(-5, 12)$. Solving for $r$ gives $r = \sqrt{(-5)^2 + (12)^2} = 13$. The ratios are:

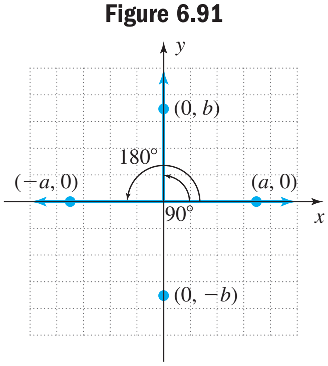
$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{13} \qquad \tan\theta = \frac{y}{x} = \frac{12}{-5}$$

**b.** In QIV we select $x = 10$ since $x$ is positive in QIV, giving $y = -24$ and the point $(10, -24)$. Next we find $r$: $r = \sqrt{(10)^2 + (-24)^2} = 26$. The ratios are:

$$\sin\theta = \frac{y}{r} = \frac{-24}{26} = -\frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{10}{26} = \frac{5}{13} \qquad \tan\theta = \frac{y}{x} = \frac{-24}{10} = -\frac{12}{5}$$

NOW TRY EXERCISES 25 THROUGH 28 ▶

**Figure 6.91**




In Example 3, note the ratios are the same in QII and QIV *except for their sign*. We will soon use this observation to great advantage.

As with our unit circle study in Section 6.2, certain trig functions are undefined for quadrantal angles. For 90° and 270°, any point on the terminal side of the angle has an *x-value* of zero, meaning tan 90°, sec 90°, tan 270°, and sec 270° are all undefined. Similarly, at 180° and 360°, the *y*-value of any point on the terminal side is zero, so cot 180°, csc 180°, cot 360°, and csc 360° are also undefined (see Figure 6.91).

**EXAMPLE 4** Evaluate the six trig functions for $\theta = 270°$.

**Solution:** Here, $\theta$ is the quadrantal angle whose terminal side separates QIII and QIV. Since the evaluation is independent of the point chosen on this side, we choose $(0, -1)$ for convenience, giving $r = 1$. For $x = 0$, $y = -1$, and $r = 1$ we obtain:

$$\sin\theta = \frac{-1}{1} = -1 \qquad \cos\theta = \frac{0}{-1} = 0 \qquad \tan\theta = \frac{-1}{0}\ (\textit{undefined})$$

The remaining ratios can be evaluated using the reciprocal relationships.

$$\csc\theta = -1 \qquad \sec\theta = (\textit{undefined}) \qquad \cot\theta = \frac{0}{-1} = 0$$

NOW TRY EXERCISES 29 AND 30

Results for the quadrantal angles are summarized in Table 6.13.

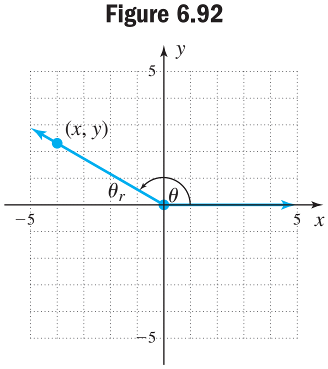
**Table 6.13**

| $\theta$ | $\sin\theta = \frac{y}{r}$ | $\cos\theta = \frac{x}{r}$ | $\tan\theta = \frac{y}{x}$ | $\csc\theta = \frac{r}{y}$ | $\sec\theta = \frac{r}{x}$ | $\cot\theta = \frac{x}{y}$ |
|---|---|---|---|---|---|---|
| 0°/360° (1, 0) | 0 | 1 | 0 | undefined | 1 | undefined |
| 90° (0, 1) | 1 | 0 | undefined | 1 | defined | 0 |
| 180° (−1, 0) | 0 | −1 | 0 | undefined | −1 | undefined |
| 270° (0, −1) | −1 | 0 | undefined | −1 | undefined | 0 |

## B. Reference Angles and the Trig Functions of Any Angle

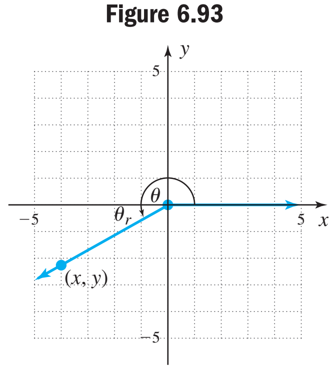
Recall from Section 6.2 that for any angle $\theta$ in standard position, the acute angle $\theta_r$ formed by the terminal side and the nearest $x$-axis is called the reference angle. Several examples of this concept are illustrated in Figures 6.92 through 6.95 for $\theta > 0$ and a point $(x, y)$ on the terminal side. Note the strong resemblance to the reference angles and reference arcs utilized previously.

**Figure 6.92**



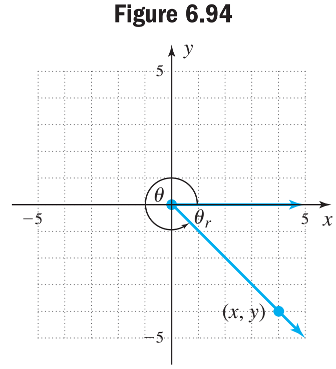

$90° < \theta < 180°$
$\theta_r = 180° - \theta$

**Figure 6.93**



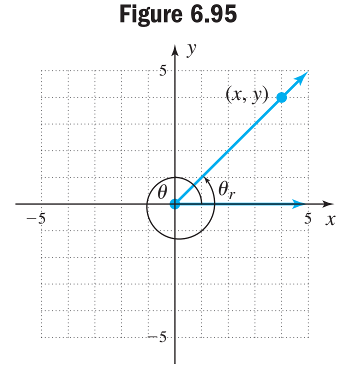

$180° < \theta < 270°$
$\theta_r = \theta - 180°$

**Figure 6.94**




$270° < \theta < 360°$
$\theta_r = 360° - \theta$

**Figure 6.95**




$360° < \theta < 450°$
$\theta_r = \theta - 360°$

**EXAMPLE 5** Determine the reference angle for (a) 315°, (b) 150°, (c) −121°, and (d) 425°.

**Solution:**

**a.** 315° is a QIV angle:

$\theta_r = 360° - 315° = 45°$

**b.** 150° is a QII angle:

$\theta_r = 180° - 150° = 30°$

**c.** −121° is a QIII angle:

$\theta_r = 180° - 121° = 59°$

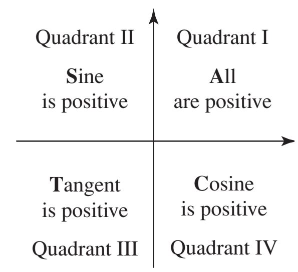
**d.** 425° is a QI angle:

$\theta_r = 425° - 360° = 65°$

NOW TRY EXERCISES 31 THROUGH 42

The reference angles from Examples 5(a) and 5(b) were standard angles, which means we automatically know the absolute value of the trig ratios using $\theta_r$. The best way to remember the signs of the trig functions is to keep in mind that sine is associated with $y$, cosine with $x$, and tangent with both. In addition, there are several mnemonic devices (memory tools) to assist you. One is to use the first letter of the function that is positive in each quadrant and create a catchy acronym. For instance **ASTC** → **A**ll **S**tudents **T**ake **C**lasses (see Figure 6.96). Note that a trig function and its reciprocal function will always have the same sign.

**Figure 6.96**

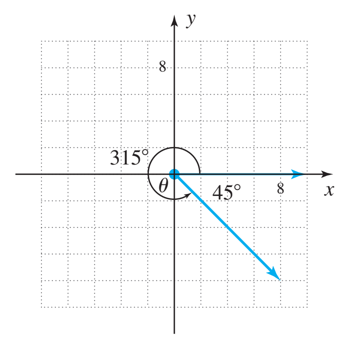



**EXAMPLE 6** Use a reference angle to evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 315°$.

**Solution:** The terminal side is in QIV where $x$ is positive and $y$ is negative. With $\theta_r = 45°$, we have

$$\sin 315° = -\frac{\sqrt{2}}{2} \qquad \cos 315° = \frac{\sqrt{2}}{2}$$

$$\tan 315° = -1$$




NOW TRY EXERCISES 43 THROUGH 54

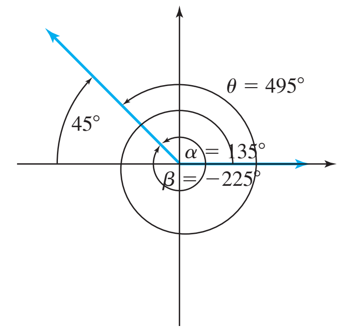
**EXAMPLE 7** Given $\sin\theta = \frac{5}{13}$ and $\cos\theta < 0$, state the values of the five remaining ratios.

**Solution:** Always begin with a quadrant analysis: $\sin\theta$ is positive in QI and QII, while $\cos\theta$ is negative in QII and QIII. Both conditions are satisfied in QII only. For $r = 13$ and $y = 5$, the Pythagorean theorem shows $x = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. With $\theta$ in QII this gives $\cos\theta = \frac{-12}{13}$ and $\tan\theta = \frac{5}{-12}$. The reciprocal values are $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{-12}$, and $\cot\theta = \frac{-12}{5}$.

NOW TRY EXERCISES 55 THROUGH 62

In our everyday experience, there are many actions and activities where angles greater than or equal to 360° are applied. Some common instances are a professional basketball player that "does a three-sixty" (360°) while going to the hoop, a diver that completes a "two-and-a-half" (900°) off the high board, and a skater that executes a perfect triple axel ($3\frac{1}{2}$ turns or 1260°). As these examples suggest, angles greater than 360° must still terminate in one of the four quadrants, allowing a reference angle to be found and the functions to be evaluated for any angle *regardless of size.* Recall that two angles in standard position that share the same terminal side are called coterminal angles. Figure 6.97 illustrates that $\alpha = 135°$, $\beta = -225°$, and $\theta = 495°$ are all coterminal, with *all three having a reference angle of* 45°.

**Figure 6.97**




**EXAMPLE 8** Evaluate sin 135°, cos −225°, and tan 495°.

**Solution:** The angles are coterminal and terminate in QII, where $x < 0$ and $y > 0$. With $\theta_r = 45°$ we have $\sin 135° = \frac{\sqrt{2}}{2}$, $\cos -225° = -\frac{\sqrt{2}}{2}$, and $\tan 495° = -1$.

**NOW TRY EXERCISES 63 THROUGH 74**

Since 360° is one full rotation, all angles $\theta + 360°k$ will be coterminal for any integer $k$. For angles with a very large magnitude, we can find the quadrant of the terminal side by subtracting as many integer multiples of 360° as needed from the angle.

**EXAMPLE 9** Find the coterminal angle $\theta$ where $0° \le |\theta| \le 360°$, then name the quadrant of the terminal side and the related reference angle.

**a.** $\alpha = 1908°$ **b.** $\beta = -1125°$

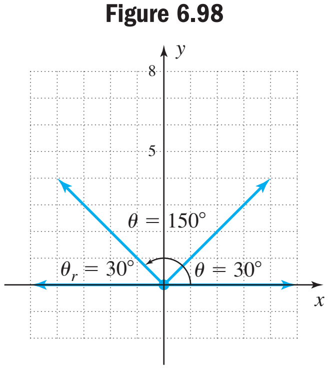
**Solution:**

**a.** For $\alpha = 1908°$, $\frac{1980}{360} = 5.3$ and $1908 - 360(5) = 108°$. This angle is in QII with $\theta_r = 72°$.

**b.** For $\beta = -1125°$, $\frac{-1125}{360} = -3.125$ and $-1125 - 360(-3) = -45°$. This angle is in QIV with $\theta_r = 45°$.

**NOW TRY EXERCISES 75 THROUGH 90**

**Figure 6.98**




## C. Applications of the Trig Functions of Any Angle

One of the most basic uses of coterminal angles is determining all values of $\theta$ that satisfy a stated relationship. For example, by now you are aware that if $\sin\theta = \frac{1}{2}$ (positive one-half), then $\theta = 30°$ or $\theta = 150°$ (see Figure 6.98). But this is also true for all angles coterminal with these two, and we would write the solutions as $\theta = 30° + 360°k$ and $\theta = 150° + 360°k$ for all integers $k$.

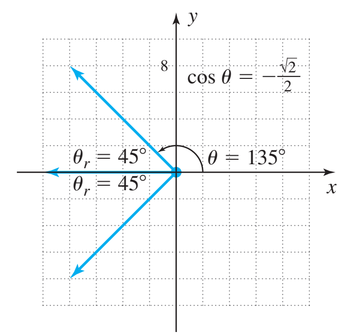
**EXAMPLE 10** Find all angles satisfying the relationship. Answer in degrees.

**a.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **b.** $\sin\theta = 0.3987$ **c.** $\tan\theta = -1.4654$

**Solution:**

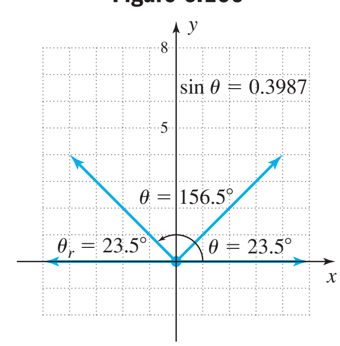
**a.** Cosine is negative in QII and QIII. Recognizing $\cos 45° = \frac{\sqrt{2}}{2}$, we reason $\theta_r = 45°$ and two solutions are $\theta = 135°$ from QII and $\theta = 225°$ from QIII. For all values of $\theta$ satisfying the relationship, we have $\theta = 135° + 360°k$ and $\theta = 225° + 360°k$. See Figure 6.99.

**Figure 6.99**



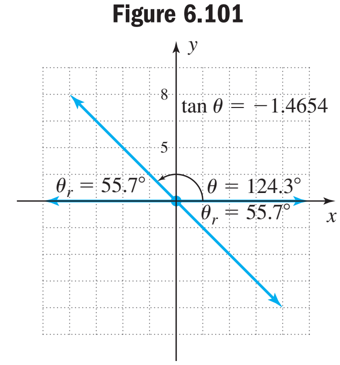

**b.** Sine is positive in QI and QII. Since 0.3987 is not a standard value, we find $\theta_r$ using a calculator as in Section 6.6. The keystrokes 2nd SIN 0.3987 ENTER show $\sin^{-1} 0.3987 \approx 23.5 = \theta_r$, and two solutions are $\theta = 23.5°$ from QI and $\theta = 180° - 23.5° = 156.5°$ from QII. The solution is $\theta = 23.5° + 360°k$ and $\theta = 156.5° + 360°k$. See Figure 6.100.

**Figure 6.100**




**c.** Tangent is negative in QII and QIV. For −1.4654 we again find $\theta_r$ using a calculator: 2nd TAN −1.4654 ENTER shows $\tan^{-1} - 1.4654 \approx -55.7$, so $\theta_r = 55.7°$. Two solutions are $\theta = 180° - 55.7° = 124.3°$ from QII, and in QIV $\theta = 360° - 55.7° = 304.3°$. The result is $\theta = 124.3° + 360°k$ and $\theta = 304.3° + 360°k$. See Figure 6.101.

**Figure 6.101**




NOW TRY EXERCISES 93 THROUGH 100

We close this section with two additional applications of the concepts related to trigonometric functions of any angle.

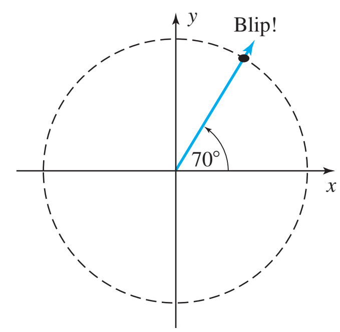
**EXAMPLE 11** A figure skater is setting up to execute a jump. As he lowers into the crouch position, getting ready to "pop and turn," he is facing the east corner of the pavilion at an angle of 0°. If he executes a perfect jump and rotates through an angle of 1260°: (a) How many rotations did he turn through? (b) What direction is he facing when he lands?

**Solution:** **a.** Using $\frac{1260}{360} = 3.5$, he completed three and one-half rotations.

**b.** Since $1260° - 360°(3) = 180°$, the skater was facing to the west.

**NOW TRY EXERCISES 101 AND 102**

**EXAMPLE 12** A radar operator calls the captain over to her screen saying, "Sir, we have an unidentified bogey at bearing 20° (20° east of due north or 70° on the coordinate grid). I think it's a UFO." The captain replies, "What makes you think so?" "Because it's at 25,000 ft and not moving!" replied the sailor. Name all angles for which the UFO causes a "blip" to occur on the radar screen.




**Solution:** Since radar typically sweeps out 360° angles, a blip will occur on the screen for all angles $\theta = 70° + 360°k$, where $k$ is a positive integer.

**NOW TRY EXERCISES 103 THROUGH 106**

## TECHNOLOGY HIGHLIGHT

## *x*, *y*, *r*, and functions of any angle

*The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.*

Graphing calculators offer a number of features that can assist a study of the trig functions of any angle. On the TI-84 Plus, the keystrokes 2nd APPS (**ANGLE**) will bring up the menu shown in Figure 6.102. Options 1 through 4 are basically used for angle conversions (DMS degrees to decimal degrees, degrees to radians, and so on). Of interest to us here are options 5 and 6, which can be used to determine the radius $r$ (option 5) or the angle $\theta$ (option 6) related to a given point $(x, y)$. For $(-12, 35)$, CLEAR the home screen and press 2nd APPS **(ANGLE) 5:R▶Pr(,** which will place the option on the home screen. This feature supplies the left parenthesis of the ordered pair, and you simply complete it: **5:R▶Pr(−12, 35)**. As shown in Figure 6.103, the calculator returns 37 and indeed $(-12)^2 + (35)^2 = (37)^2$. To find the related angle, it is assumed that $\theta$ is in standard position and $(x, y)$ is on the terminal side. Pressing 2nd APPS **(ANGLE) 6:R▶Pθ(** and completing the ordered pair as before, shows the corresponding angle is approximately 108.9° (Figure 6.103). Note

**Figure 6.102**

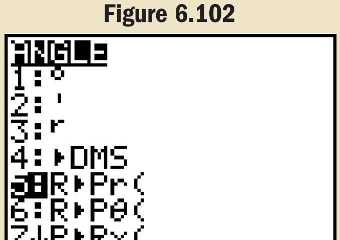
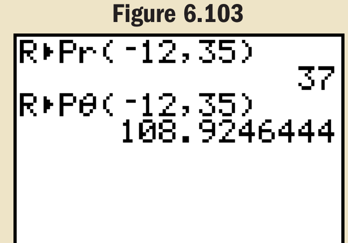



**Figure 6.103**



this is a QII angle as expected, since $x < 0$ and $y > 0$. Use these features to complete the following exercises.

Exercise 1: Find the radius corresponding to the point $(-5, 5\sqrt{3})$.

Exercise 2: Find the radius corresponding to the point $(-28, -45)$.

Exercise 3: Find the angle corresponding to $(-5, 5\sqrt{3})$, then use a calculator to evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for this angle. Compare each result to the values given by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

Exercise 4: Find the angle corresponding to $(-28, -45)$, then use a calculator to evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for this angle. Compare each result to the values given by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

# 6.7 EXERCISES

## CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

**1.** An angle is in standard position if its vertex is at the ________ and the initial side is along the ________.

**2.** A(n) ________ angle is one where the ________ side is coincident with one of the coordinate axes.

**3.** Angles formed by a counterclockwise rotation are ________ angles. Angles formed by a ________ rotation are negative angles.

**4.** For any angle $\theta$, its reference angle $\theta_r$ is the positive ________ angle formed by the ________ side and the nearest $x$-axis.

**5.** Discuss the similarities and differences between the trigonometry of right triangles and the trigonometry of *any* angle.

**6.** Let $T(x)$ represent any one of the six basic trig functions. Explain why the equation $T(x) = k$ will always have exactly two solutions in $[0, 2\pi)$ if $x$ is not a quadrantal angle.

## DEVELOPING YOUR SKILLS

**7.** Draw a 30-60-90 triangle with the 60° angle at the origin and the short side along the $x$-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 60°, cos 60°, and tan 60°. Comment on what you notice.

**8.** Draw a 45-45-90 triangle with a 45° angle at the origin and one side along the $x$-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 45°, cos 45°, and tan 45°. Comment on what you notice.

Find the value of the six trigonometric functions given $P(x, y)$ is on the terminal side of angle $\theta$, with $\theta$ in standard position.

**9.** $(8, 15)$ **10.** $(7, 24)$ **11.** $(-20, 21)$ **12.** $(-3, -1)$

**13.** $(7.5, -7.5)$ **14.** $(9, -9)$ **15.** $(4\sqrt{3}, 4)$ **16.** $(-6, 6\sqrt{3})$

**17.** $(2, 8)$ **18.** $(6, -15)$ **19.** $(-3.75, -2.5)$ **20.** $(6.75, 9)$

**21.** $\left(-\frac{5}{9}, \frac{2}{3}\right)$ **22.** $\left(\frac{3}{4}, -\frac{7}{16}\right)$ **23.** $\left(\frac{1}{4}, -\frac{\sqrt{5}}{2}\right)$ **24.** $\left(-\frac{\sqrt{3}}{5}, \frac{22}{25}\right)$

Use a slope triangle to help graph each linear equation. Then state the quadrants it traverses and pick one point on the line from each quadrant and evaluate the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ using these points.

**25.** $y = \frac{3}{4}x$ **26.** $y = \frac{5}{12}x$ **27.** $y = -\frac{\sqrt{3}}{3}x$ **28.** $y = -\frac{\sqrt{3}}{2}x$

**29.** Evaluate the six trig functions in terms of $x$, $y$, and $r$ for $\theta = 90°$.

**30.** Evaluate the six trig functions in terms of $x$, $y$, and $r$ for $\theta = 180°$.

Name the reference angle $\theta_r$ for the angle $\theta$ given.

**31.** $\theta = 120°$ **32.** $\theta = 210°$ **33.** $\theta = 135°$ **34.** $\theta = 315°$

**35.** $\theta = -45°$ **36.** $\theta = -240°$ **37.** $\theta = 112°$ **38.** $\theta = 179°$

**39.** $\theta = 500°$ **40.** $\theta = 750°$ **41.** $\theta = -168.4°$ **42.** $\theta = -328.2°$

State the quadrant of the terminal side of $\theta$, using the information given.

**43.** $\sin\theta > 0$, $\cos\theta < 0$ **44.** $\cos\theta < 0$, $\tan\theta < 0$ **45.** $\tan\theta < 0$, $\sin\theta > 0$ **46.** $\sec\theta > 0$, $\tan\theta > 0$

Find the exact value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

**47.** $\theta = 330°$ **48.** $\theta = 390°$ **49.** $\theta = -45°$ **50.** $\theta = -120°$

**51.** $\theta = 240°$ **52.** $\theta = 315°$ **53.** $\theta = -150°$ **54.** $\theta = -210°$

For the information given, find the related values of $x$, $y$, and $r$. Clearly indicate the quadrant of the terminal side of $\theta$, then state the values of the six trig functions of $\theta$.

**55.** $\cos\theta = \frac{4}{5}$ and $\sin\theta < 0$

**56.** $\tan\theta = -\frac{12}{5}$ and $\cos\theta > 0$

**57.** $\csc\theta = -\frac{37}{35}$ and $\tan\theta > 0$

**58.** $\sin\theta = -\frac{20}{29}$ and $\cot\theta < 0$

**59.** $\csc\theta = 3$ and $\cos\theta > 0$

**60.** $\csc\theta = -2$ and $\cos\theta > 0$

**61.** $\sin\theta = -\frac{7}{8}$ and $\sec\theta < 0$

**62.** $\cos\theta = \frac{5}{12}$ and $\sin\theta < 0$

Find two positive and two negative angles that are coterminal with the angle given. Answers will vary.

**63.** 52° **64.** 12° **65.** 87.5° **66.** 22.8°

**67.** 225° **68.** 175° **69.** −107° **70.** −215°

Evaluate in exact form as indicated.

**71.** $\sin 120°$, $\cos -240°$, $\tan 480°$

**72.** $\sin 225°$, $\cos 585°$, $\tan -495°$

**73.** $\sin -30°$, $\cos -390°$, $\tan 690°$

**74.** $\sin 210°$, $\cos 570°$, $\tan -150°$

Find the exact value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

**75.** $\theta = 600°$ **76.** $\theta = 480°$ **77.** $\theta = -840°$ **78.** $\theta = -930°$

**79.** $\theta = 570°$ **80.** $\theta = 495°$ **81.** $\theta = -1230°$ **82.** $\theta = 3270°$

For each exercise, state the quadrant of the terminal side and the sign of the function in that quadrant. Then evaluate the expression using a calculator. Round function values to four decimal places.

**83.** $\sin 719°$ **84.** $\cos 528°$ **85.** $\tan -419°$ **86.** $\sec -621°$

**87.** $\csc 681°$ **88.** $\tan 995°$ **89.** $\cos 805°$ **90.** $\sin 772°$

## WORKING WITH FORMULAS

**91. The area of a parallelogram: $A = ab\sin\theta$**

The area of a parallelogram is given by the formula shown, where $a$ and $b$ are the lengths of the sides and $\theta$ is the angle between them. Use the formula to complete the following: (a) find the area of a parallelogram with sides $a = 9$ and $b = 21$, given $\theta = 50°$. (b) What is the smallest integer value of $\theta$ where the area is greater than 150 units$^2$? (c) State what happens when $\theta = 90°$. (d) How can you find the area of a triangle using this formula?

**92. The angle between two lines in the plane: $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$**

Given line 1 and line 2 with slopes $m_1$ and $m_2$, respectively, the angle between the two lines is given by the formula shown. Find the angle $\theta$ if the equation of line 1 is $y_1 = \frac{3}{4}x + 2$ and line 2 has equation $y = -\frac{2}{3}x + 5$.

## APPLICATIONS

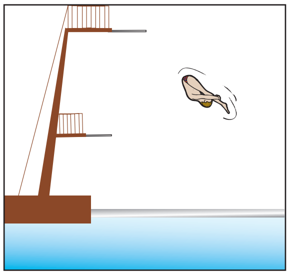
Find all angles satisfying the stated relationship. For standard angles, express your answer in exact form. For nonstandard values, use a calculator and round function values to tenths.

**93.** $\cos\theta = \dfrac{1}{2}$ **94.** $\sin\theta = \dfrac{\sqrt{2}}{2}$ **95.** $\sin\theta = -\dfrac{\sqrt{3}}{2}$ **96.** $\tan\theta = -\dfrac{\sqrt{3}}{1}$

**97.** $\sin\theta = 0.8754$ **98.** $\cos\theta = 0.2378$ **99.** $\tan\theta = -2.3512$ **100.** $\cos\theta = -0.0562$

**101. High dives:** As part of a diving competition, David executes a perfect reverse two-and-a-half flip. Does he enter the water feet first or head first? Through what angle did he turn from takeoff until the moment he entered the water?

**Exercise 101**



**102. Gymnastics:** While working out on a trampoline, Charlene does three complete, forward flips and then belly-flops on the trampoline before returning to the upright position. What angle did she turn through from the flip to the belly-flop?

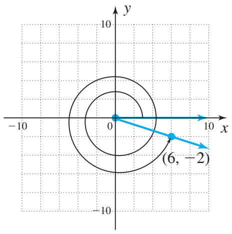
**103. Nonacute angles:** At a recent carnival, one of the games on the midway was played using a large spinner that turns clockwise. On Jorge's spin the number 25 began at the 12 o'clock (top/center) position, returned to this position five times during the spin and stopped at the 3 o'clock position. What angle $\theta$ did the spinner spin through? Name all angles that are coterminal with $\theta$.

**Exercise 103**

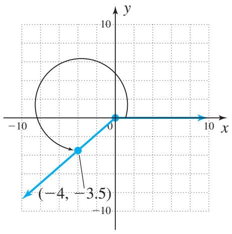
**104. Nonacute angles:** One of the four blades on a ceiling fan has a decal on it and begins at a designated "12 o'clock" position. Turning the switch on and then immediately off, causes the blade to make over three complete, counterclockwise rotations, with the blade stopping at the 8 o'clock position. What angle $\theta$ did the blade turn through? Name all angles that are coterminal with $\theta$.

**105. Spiral of Archimedes:** The graph shown is called the spiral of Archimedes. Through what angle $\theta$ has the spiral turned, given the spiral terminates at $(6, -2)$ as indicated?

**Exercise 105**



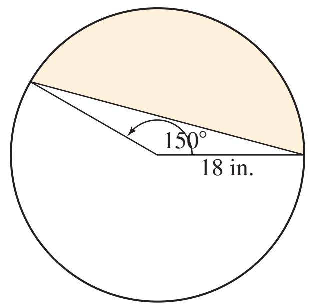

**Exercise 106**



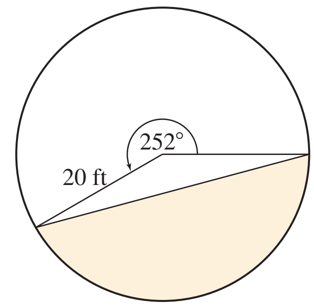
**106.** **Involute of a circle:** The graph shown is called the involute of a circle. Through what angle $\theta$ has the involute turned, given the graph terminates at $(-4, -3.5)$ as indicated?

**Area bounded by chord and circumference:** Find the area of the shaded region, rounded to the nearest 100th. Note the area of a triangle is one-half the area of a parallelogram (see Exercise 91).

**107.**




**108.**



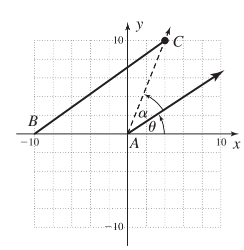

## WRITING, RESEARCH, AND DECISION MAKING

**109.** Each of our eyes can individually see through a cone approximately 150° wide, called the **cone of vision.** When the cone from the right and left eyes overlap, the eyes can gather light (*see*) from almost 180° (including the peripheral vision), although the clearest focus is within a central 60° cone. Use the Internet or the resources of a local library to investigate and research *cones of vision*. Try to determine what trigonometric principles are involved in studies of the eye, and its ability to see and interpret light. Prepare a short summary on what you find.

**110.** In an elementary study of trigonometry, the hands of a clock are often studied because of the angle relationship that exists between the hands. For example, at 3 o'clock, the angle between the two hands is a right angle and measures 90°. (a) What is the angle between the two hands at 1 o'clock? 2 o'clock? Explain why. (b) What is the angle between the two hands at 6:30? 7:00? 7:30? Explain why. (c) Name four times at which the hands will form a 45° angle. (d) What other questions of interest can you think of regarding the angle between the hands of a clock?

## EXTENDING THE CONCEPT

**111.** In the diagram shown, the indicated ray is of arbitrary length. (a) Through what additional angle $\alpha$ would the ray have to be rotated to create triangle $ABC$? (b) What will be the length of side $AC$ once the triangle is complete?




**112.** Referring to Exercise 104, suppose the fan blade had a radius of 20 in. and is turning at a rate of 12 revolutions per second. (a) Find the angle the blade turns through in 3 sec. (b) Find the circumference of the circle traced out by the tip of the blade. (c) Find the total distance traveled by the blade tip in 10 sec. (d) Find the speed, in miles per hour, that the tip of the blade is traveling.

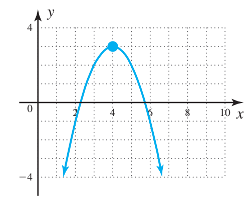
## MAINTAINING YOUR SKILLS

**113.** (5.4) Solve for $x$. Answer in both exact and approximate form: $-250 = -150e^{-0.05t} - 202$.

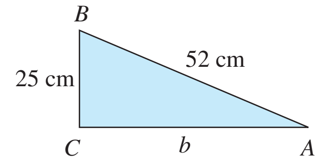
**114.** (1.5) Solve by completing the square. Answer in exact form: $2x^2 - 12x + 15 = 0$.

**115.** (6.2) Verify that $\left(\frac{12}{37}, \frac{35}{37}\right)$ is a point on the unit circle and find the value of the six trig functions using this point.

**116.** (3.3) Find the equation of the function graphed in the figure, assuming it came from the function family $y = x^2$ and has not been vertically stretched or compressed.




**117.** (6.6) Solve the triangle shown. Express all angles to tenths and all sides to hundredths of a unit. Answer in table form.




**118.** (2.1) Find the equation of the line perpendicular to $4x - 5y = 15$ that contains the point $(4, -3)$.

# SUMMARY AND CONCEPT REVIEW

## SECTION 6.1 Angle Measure, Special Triangles, and Special Angles

### KEY CONCEPTS

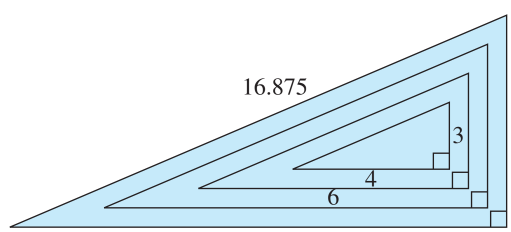
- An angle is defined as the joining of two rays at a common endpoint called the vertex.
- Angle measure can be viewed as the amount of inclination between two intersecting lines, or the amount of rotation from a fixed (initial) side to a terminal side.
- Angles are often named using a capital letter at the vertex or using Greek letters in the interior. The most common are $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma), and $\theta$ (theta).
- An angle in standard position has its vertex at the origin and its initial side coterminal with the $x$-axis.
- Two angles in standard position are coterminal if they have the same terminal side.
- A counterclockwise rotation gives a positive angle, a clockwise rotation gives a negative angle.
- One $(1°)$ degree is defined to be $\frac{1}{360}$ of a full revolution. One (1) radian is the measure of a central angle subtended by an arc equal in length to the radius.
- Degrees can be divided into a smaller unit called minutes: $1° = 60'$; minutes can be divided into a smaller unit called seconds: $1' = 60''$. This implies $1° = 3600''$.
- Straight angles measure 180°; right angles measure 90°.
- Two angles are complementary if they sum to 90° and supplementary if they sum to 180°.
- Properties of triangles: (I) the sum of the angles is 180°; (II) the combined length of any two sides must exceed that of the third side and; (III) larger angles are opposite larger sides.
- Given two triangles, if all three angles are equal, the triangles are said to be similar. If two triangles are similar, then corresponding sides are in proportion.
- If $\Delta ABC$ is a right triangle with hypotenuse $C$, then $A^2 + B^2 = C^2$ (Pythagorean theorem). For $\Delta ABC$ with longest side $C$, if $A^2 + B^2 = C^2$, then $\Delta ABC$ is a right triangle (converse of Pythagorean theorem).
- In a 45-45-90 triangle, the sides are in the proportion $1x{:}1x{:}\sqrt{2}x$.
- In a 30-60-90 triangle, the sides are in the proportion $1x{:}\sqrt{3}x{:}2x$.
- Since $C = 2\pi r$, there are $2\pi$ rad in a complete revolution (the radius can be laid out along the circumference $2\pi$ times).

- The formula for arc length in degrees: $s = \frac{\theta}{360°}2\pi r$; in radians: $s = r\theta$.
- The formula for the area of a circular sector in degrees: $A = \frac{\theta}{360°}\pi r^2$; in radians: $A = \frac{1}{2}r^2\theta$.
- To convert degree measure to radians, multiply by $\frac{\pi}{180}$; for radians to degrees, multiply by $\frac{180}{\pi}$.
- Standard conversions should be committed to memory: $30° = \frac{\pi}{6}$; $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$ $90° = \frac{\pi}{2}$.
- A location north or south of the equator is given in degrees latitude; a location east or west of the Greenwich Meridian is given in degrees longitude.
- Angular velocity is a rate of rotation per unit time: $\omega = \frac{\theta}{t}$. Linear velocity is a change in distance per unit time: $V = \frac{\theta r}{t}$ or $V = r\omega$.

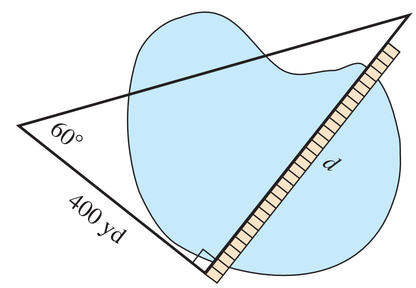
## EXERCISES

1. Convert 147°36′48″ to decimal degrees.
2. Convert 32.87° to degrees, minutes, and seconds
3. All of the right triangles given are similar. Find the dimensions of the largest triangle.

**Exercise 3**



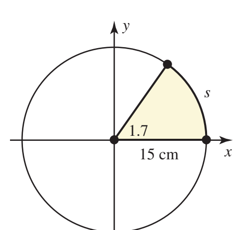

**Exercise 4**



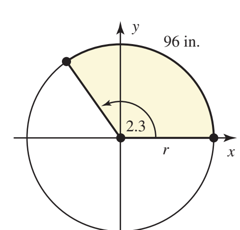

4. Use special angles/special triangles to find the length of the bridge needed to cross the lake shown in the figure.
5. Convert to degrees: $\frac{2\pi}{3}$
6. Convert to radians: 210°
7. Find the arc length if $r = 5$ and $\theta = 57°$.
8. Evaluate without using a calculator: $\sin\left(\frac{7\pi}{6}\right)$.

Find the angle, radius, arc length, and/or area as needed, until all values are known.

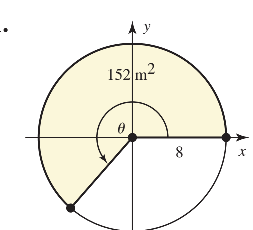
**9.**




**10.**




**11.**




12. With great effort, 5-year-old Mackenzie has just rolled her bowling ball down the lane, and it is traveling painfully slow. So slow, in fact, that you can count the number of revolutions the ball makes using the finger holes as a reference. (a) If the ball is rolling at 1.5 revolutions per second, what is the angular velocity? (b) If the ball's radius is 5 in., what is its linear velocity in feet per-second? (c) If the distance to the first pin is 60 feet and the ball is true, how long until it hits?

## SECTION 6.2 Unit Circles and the Trigonometry of Real Numbers

### KEY CONCEPTS

- A central unit circle refers to a circle of radius 1 with its center at the origin and can be graphed by drawing a circle through its quadrantal points: (1, 0), (0, 1), $(-1, 0)$, and $(0, -1)$.
- A central circle is symmetric to both axes and the origin. This means that if $(a, b)$ is a point on the circle, then $(-a, b)$, $(a, -b)$, $(-a, -b)$, and $(a, -b)$ are also on the circle and satisfy the equation of the circle.
- Points on a unit circle can be located using a right triangle of radius 1, with the vertex of an acute angle at the center. The coordinates of the point are $(x, y)$, where $x$ and $y$ are the lengths of the legs.
- On a unit circle with $\theta$ in radians, the length of a subtended arc is numerically the same as the subtended angle, making the arc a "circular number line" and allowing us to treat the trig functions as functions of a real number.
- A reference angle is defined to be the acute angle formed by the terminal side of a given angle and the nearest $x$-axis. As functions of a real number we refer to a reference arc rather than a reference angle.
- For any real number $t$ and a point on the unit circle associated with $t$, we have:

  $\cos t = x \qquad \sin t = y \qquad \tan t = \frac{y}{x} \qquad \sec t = \frac{1}{x} \qquad \csc t = \frac{1}{y} \qquad \cot t = \frac{x}{y}$
- The domain of each trig function must exclude division by zero (see box on page 590).
- Given the specific value of any function, the related real number $t$ or angle $\theta$ can be found using a reference arc/angle, or the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ features of a calculator for nonstandard values.

### EXERCISES

**13.** Given $\left(\frac{\sqrt{13}}{7}, y\right)$ is on a unit circle, find $y$ if the point is in QIV, then use the symmetry of the circle to locate three other points.

**14.** Given $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$ is on the unit circle, find the value of all six trig functions of $t$ without the use of a calculator.

**15.** Without using a calculator, find two values in $[0, 2\pi)$ that make the equation true: $\csc t = \frac{2}{\sqrt{3}}$.

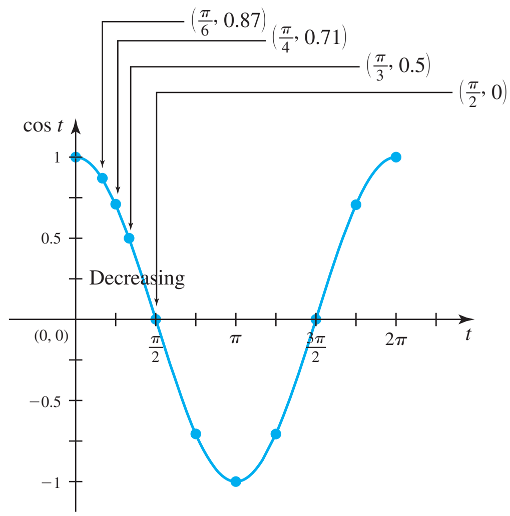
**16.** Use a calculator to find the value of $t$ that corresponds to the situation described: $\cos t = -0.7641$ with $t$ in QII.

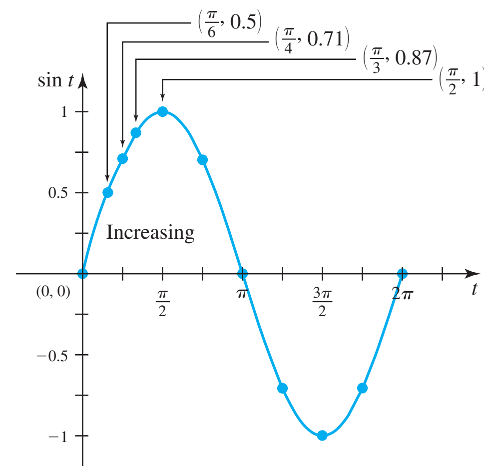
**17.** A crane used for lifting heavy equipment has a winch-drum with a 1-yd radius. (a) If 59 ft of cable has been wound in while lifting some equipment to the roof-top of a building, what angle has the drum turned through? (b) What angle must the drum turn through to wind in 75 ft of cable?

## SECTION 6.3 Graphs of the Sine and Cosine Functions

### KEY CONCEPTS

- On a unit circle, $\cos t = x$ and $\sin t = y$. Graphing these functions using the $x$- and $y$-coordinates of points on the unit circle, results in a periodic, wavelike graph with domain $(-\infty, \infty)$.




- The characteristics of each graph play a vital role in their contextual application, and these are repeated here ($k$ is any integer):

**CHARACTERISTICS OF $y = \cos t$**

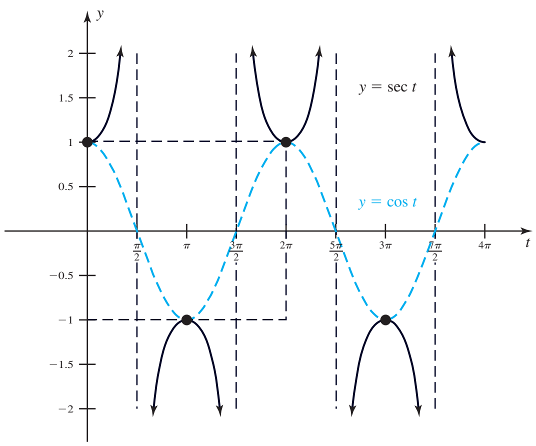
| Unit Circle Definition | Domain | Symmetry | Maximum values | Increasing: $t \in (0, 2\pi)$ |
|---|---|---|---|---|
| $\cos t = x$ | $t \in \mathbb{R}$ | Even: $\cos(-t) = \cos t$ | $\cos t = 1$ at $t = 2\pi k, k \in \mathbb{Z}$ | $t \in (\pi, 2\pi)$ |
| **Period** | **Range** | **Zeroes:** | **Minimum values** | **Decreasing: $t \in (0, 2\pi)$** |
| $2\pi$ | $y \in [-1, 1]$ | $t = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ | $\cos t = -1$ at $t = \pi + 2\pi k, k \in \mathbb{Z}$ | $t \in (0, \pi)$ |

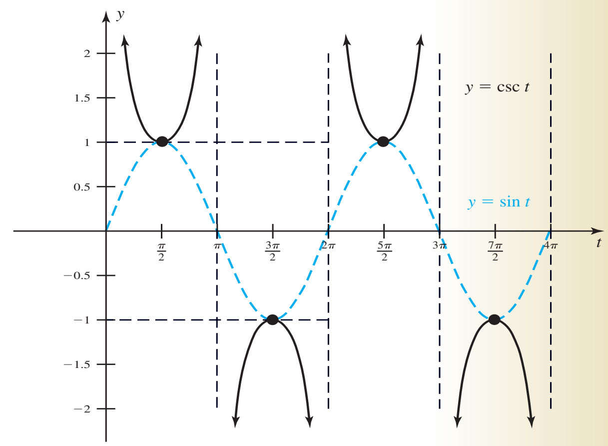
**CHARACTERISTICS OF $y = \sin t$**

| Unit Circle Definition | Domain | Symmetry | Maximum values | Increasing: $t \in (0, 2\pi)$ |
|---|---|---|---|---|
| $\sin t = y$ | $t \in \mathbb{R}$ | Odd: $\sin(-t) = -\sin t$ | $\sin t = 1$ at $t = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$ | $t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ |
| **Period** | **Range** | **Zeroes:** | **Minimum values** | **Decreasing: $t \in (0, 2\pi)$** |
| $2\pi$ | $y \in [-1, 1]$ | $t = k\pi, k \in \mathbb{Z}$ | $\sin t = -1$ at $t = \frac{3\pi}{2} + 2\pi k, k \in \mathbb{Z}$ | $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ |

- The amplitude of a sine or cosine graph is the maximum displacement from the average value. For $y = A\sin(Bt)$ and $y = A\cos(Bt)$, where no vertical shift has been applied, the average value is $y = 0$ ($x$-axis).
- When sine and cosine functions are applied in context, they often have amplitudes other than 1 and periods other than $2\pi$. For $y = A\sin(Bt)$ and $y = A\cos(Bt)$, $|A|$ gives the amplitude; $P = \frac{2\pi}{B}$ gives the period.
- If $|A| > 1$, the graph is vertically stretched, if $0 < |A| < 1$ the graph is vertically compressed, and if $A < 0$ the graph is reflected across the $x$-axis, just as with algebraic functions.

- If $B > 1$, the graph is horizontally compressed (the period is smaller/shorter), if $B < 1$ the graph is horizontally stretched (the period is larger/longer).
- To graph $y = A\sin(Bt)$ or $A\cos(Bt)$, draw a reference rectangle $2A$ units high and $P = \frac{2\pi}{B}$ units wide, then use the *rule of fourths* to locate zeroes and max/min values. Connect these points with a smooth curve.
- The graph of $y = \sec t = \frac{1}{\cos t}$ will be asymptotic every where $\cos t = 0$, increasing where $\cos t$ is decreasing, and decreasing where $\cos t$ is increasing. It will also "share" the max/min values of $\cos t$.




- The graph of $y = \csc t = \frac{1}{\sin t}$ will be asymptotic everywhere $\sin t = 0$, increasing where $\sin t$ is decreasing, and decreasing where $\sin t$ is increasing. It will also "share" the max/min values of $\sin t$.

## EXERCISES

Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of the following functions through at least one full period. Clearly state the amplitude (as applicable) and period as you begin.

**18.** $y = 3\sin t$

**19.** $y = 3\sec t$

**20.** $y = -\cos(2t)$

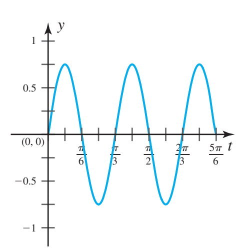
**21.** $y = 1.7\sin(4t)$

**22.** $f(t) = 2\cos(4\pi t)$

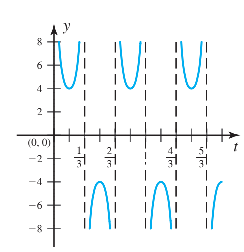
**23.** $g(t) = 3\sin(398\pi t)$

The given graphs are of the form $y = A\sin(Bt)$ and $y = A\csc(Bt)$. Determine the equation of each graph.

**24.**



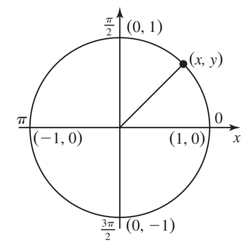
**25.**



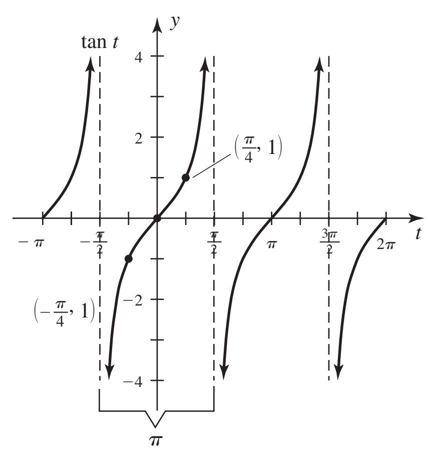
**26.** Referring to the chart of colors visible in the electromagnetic spectrum (page 616), what color is represented by the equation $y = \sin\left(\frac{\pi}{270}t\right)$? By $y = \sin\left(\frac{\pi}{320}t\right)$?

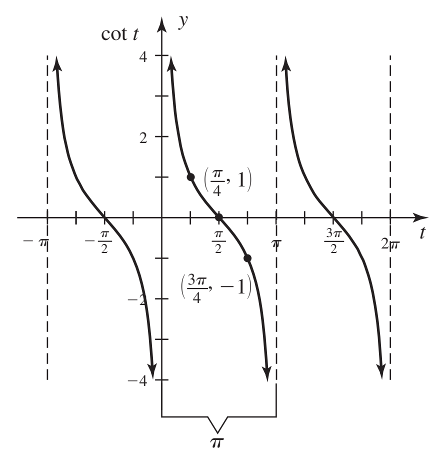
# SECTION 6.4 Graphs of the Tangent and Cotangent Functions

## KEY CONCEPTS




- Since tan $t$ is defined in terms of the ratio $\frac{y}{x}$, the graph will be asymptotic everywhere $x = 0$ on the unit circle, meaning all odd multiples of $\frac{\pi}{2}$.
- Since cot $t$ is defined in terms of the ratio $\frac{x}{y}$, the graph will be asymptotic everywhere $y = 0$ on the unit circle, meaning all integer multiples of $\pi$.
- The graph of $y = \tan t$ is increasing everywhere it is defined, the graph of $y = \cot t$ is decreasing everywhere it is defined.




- The characteristics of each graph plays a vital role in their contextual application, and these are repeated here ($k$ is any integer):

**CHARACTERISTICS OF $y = \tan t$ AND $y = \cot t$**

| | $y = \tan t$ | | | $y = \cot t$ | |
|---|---|---|---|---|---|
| Unit Circle Definition | Domain | Range | Unit Circle Definition | Domain | Range |
| $\tan t = \frac{y}{x}$ | $t \neq \frac{(2k+1)\pi}{2}$; $k \in \mathbb{Z}$ | $y \in (-\infty, \infty)$ | $\cot t = \frac{x}{y}$ | $t \neq k\pi$; $k \in \mathbb{Z}$ | $y \in (-\infty, \infty)$ |
| Period | Behavior | Symmetry | Period | Behavior | Symmetry |
| $\pi$ | increasing | Odd $\tan(-t) = -\tan t$ | $\pi$ | decreasing | Odd $\cot(-t) = -\cot t$ |

- For the more general tangent and cotangent $y = A\tan(Bt)$ and $y = A\cot(Bt)$, if $|A| > 1$, the graph is vertically stretched, if $0 < |A| < 1$ the graph is vertically compressed, and if $A < 0$ the graph is reflected across the $x$-axis, just as with algebraic functions.
- If $B > 1$, the graph is horizontally compressed (the period is smaller/shorter), if $B < 1$ the graph is horizontally stretched (the period is larger/longer).

- To graph $y = A\tan(Bt)$, note $A\tan(Bt)$ is still zero at $t = 0$. Compute the period $P = \frac{\pi}{B}$ and draw asymptotes a distance of $\frac{P}{2}$ on either side of the $y$-axis (zeroes occur halfway between asymptotes). Plot other convenient points and use the symmetry and period of $\tan t$ to complete the graph.
- To graph $y = A\cot(Bt)$, note it is asymptotic at $t = 0$. Compute the period $P = \frac{\pi}{B}$ and draw asymptotes a distance $P$ on either side of the $y$-axis (zeroes occur halfway between asymptotes). Other convenient points along with the symmetry and period can be used to complete the graph.

## EXERCISES

**27.** State the value of each expression without the aid of a calculator:

$\tan\left(\frac{7\pi}{4}\right)$ $\qquad$ $\cot\left(\frac{\pi}{3}\right)$

**28.** State the value of each expression without the aid of a calculator, given that $t$ terminates in QII.

$\tan^{-1}(-\sqrt{3})$ $\qquad$ $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

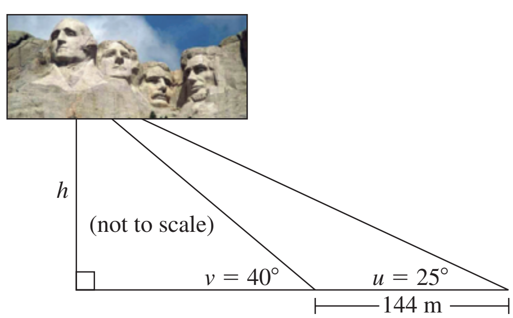
**29.** Graph $y = 6\tan\left(\frac{1}{2}t\right)$ in the interval $[-2\pi, 2\pi]$.

**30.** Graph $y = \frac{1}{2}\cot(2\pi t)$ in the interval $[-1, 1]$.

**31.** Use the period of $y = \cot t$ to name three additional solutions to $\cot t = 0.0208$, given $t = 1.55$ is a solution. Many solutions possible.

**32.** Given $t = 0.4444$ is a solution to $\cot^{-1}(t) = 2.1$, use an analysis of signs and quadrants to name an additional solution in $[0, 2\pi]$.

**33.** Find the height of Mount Rushmore, using the formula $h = \frac{d}{\cot u - \cot v}$ and the values shown.




**34.** Model the data in the table using a tangent function. Clearly state the period, the value of $A$ and the location of the asymptotes.

| Input | Output | Input | Output |
|---|---|---|---|
| −6 | −∞ | 1 | 1.4 |
| −5 | −19.4 | 2 | 3 |
| −4 | −9 | 3 | 5.2 |
| −3 | −5.2 | 4 | 9 |
| −2 | −3 | 5 | 19.4 |
| 1 | −1.4 | 6 | ∞ |
| 0 | 0 | | |

# SECTION 6.5 Transformations and Applications of Trigonometric Graphs

## KEY CONCEPTS

- Many everyday phenomena follow a sinusoidal pattern, or a pattern that can be modeled by a sine or cosine function (e.g., daily temperatures, hours of daylight, and some animal populations).
- To obtain accurate equation models of these phenomena, we study two additional transformations of sine and cosine functions—vertical shifts and horizontal shifts.
- The equation $y = A\sin(Bt \pm C) + D$ is called the *standard form* of a general sinusoid. The equation $y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$ is called the *shifted form* of a general sinusoid.
- In either form, $D$ represents the average value of the function and a vertical shift $D$ units upward if $D > 0$, $D$ units downward if $D < 0$. For a maximum value $M$ and minimum value $m$, $\frac{M + m}{2} = D$.

- The shifted form $y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$ enables us to quickly identify the horizontal shift of the function: $\frac{C}{B}$ units in a direction opposite the given sign.
- To graph a shifted sinusoid, locate the primary interval by solving $0 \le Bt + C < 2\pi$, then use a reference rectangle along with the rule of fourths to sketch the graph in this interval. The graph can then be extended as needed in either direction, then shifted vertically $D$ units.
- One basic application of sinusoidal graphs involves phenomena in harmonic motion, or motion that can be modeled by functions of the form $y = A\sin(Bt)$ or $y = A\cos(Bt)$ (with no horizontal or vertical shift).

### EXERCISES

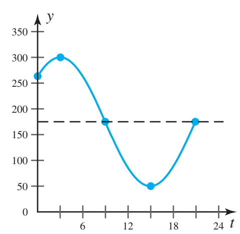
For each equation given, (a) identify/clearly state the amplitude, period, horizontal shift, and vertical shift; then (b) graph the equation using the primary interval, a reference rectangle, and rule of fourths.

**35.** $y = 240\sin\left[\frac{\pi}{6}(t - 3)\right] + 520$

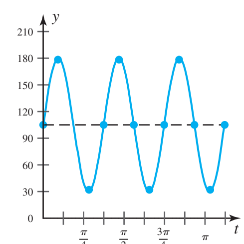
**36.** $y = 3.2\cos\left(\frac{\pi}{4}t + \frac{3\pi}{2}\right) + 6.4$

For each graph given, identify the amplitude, period, horizontal shift, and vertical shift and give the equation of the graph.

**37.**




**38.**



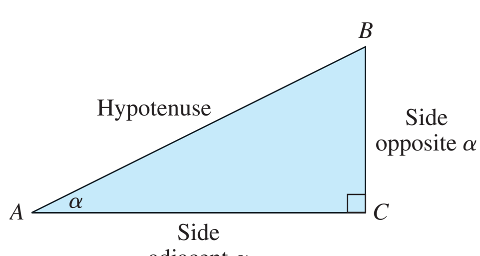

**39.** Monthly precipitation in Cheyenne, Wyoming can be modeled by a sine function, by using the average precipitation for July (2.26 in.) as a maximum (actually slightly higher in May), and the average precipitation for February (0.44 in.) as a minimum. Assume $t = 0$ corresponds to March. (a) Use the information to construct a sinusoidal model, and (b) Use the model to estimate the inches of precipitation Cheyenne receives in August ($t = 5$) and December ($t = 9$).

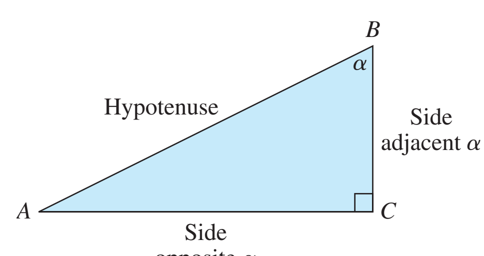
Source: 2004 Statistical Abstract of the United States, Table 380

## SECTION 6.6 The Trigonometry of Right Triangles

### KEY CONCEPTS

- The sides of a triangle can be named according to their location with respect to a given angle.




- The ratios of two sides with respect to a given angle are named as follows:

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$

- The reciprocal of the ratios above play a vital role and are likewise given special names:

$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\alpha = \frac{\text{adjacent}}{\text{opposite}}$$

$$\csc\alpha = \frac{1}{\sin(\alpha)} \qquad \sec\alpha = \frac{1}{\cos(\alpha)} \qquad \cot\alpha = \frac{1}{\tan(\alpha)}$$

- The cofunctions of these ratios are so named because each function of α is equal to the cofunction of its complement. For instance, the compliment of sine is *co*sine and $\sin\alpha = \cos(90° - \alpha)$.
- To solve a right triangle means to apply any combination of the sine, cosine, and tangent ratios, along with the Pythagorean theorem, until all three sides and all three angles are known.
- An angle of elevation is the angle formed by a counterclockwise rotation above a horizontal line of sight (parallel level ground). An angle of depression is the angle formed by a clockwise rotation below a horizontal line of sight.

## EXERCISES

**40.** Use a calculator to solve for $A$:

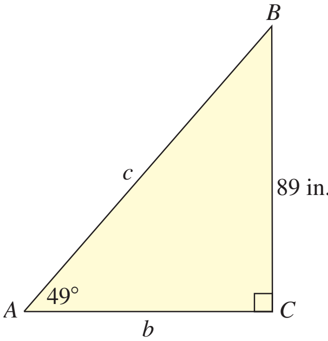
**a.** $\cos 37° = A$ **b.** $\cos A = 0.4340$

**41.** Rewrite each expression in terms of a cofunction.

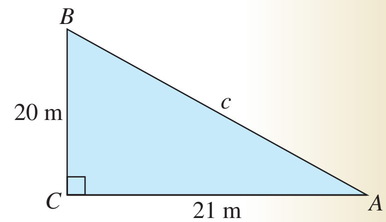
**a.** $\tan 57.4°$ **b.** $\sin(19°30'15'')$

Solve each triangle. Round angles to the nearest tenth and sides to the nearest hundredth.

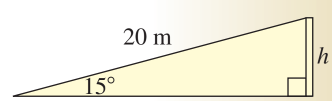
**42.**



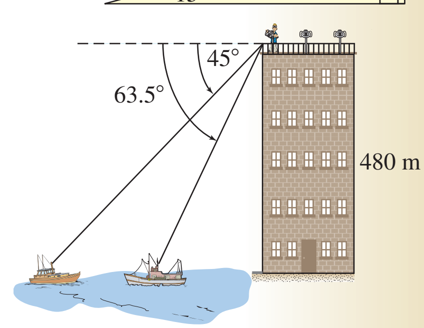

**43.**




**44.** Josephine is to weld a support to a 20-m ramp so that the incline is exactly 15°. What is the height $h$ of the support that must be used?




**45.** From the observation deck of a seaside building 480 m high, Armando sees two fishing boats in the distance, along a single line of sight. The angle of depression to the nearer boat is 63.5°, while for the boat farther away the angle is 45°. (a) How far out to sea is the nearer boat? (b) How far apart are the two boats?




**46.** A slice of bread is roughly 14 cm by 10 cm. A sandwich is made and then cut diagonally. What two acute angles are formed?

# SECTION 6.7 Trigonometry and the Coordinate Plane

## KEY CONCEPTS

- In standard position, the terminal sides of 90°, 180°, 270°, and 360° angles coincide with one of the axes and are called quadrantal angles.

- By placing a right triangle in the coordinate plane with one acute angle at the origin and one side along the $x$-axis, we note the trig functions can be defined in terms of a point $P(x, y)$.
- Given $P(x, y)$ is any point on the terminal side of an angle $\theta$ in standard position, with $r = \sqrt{x^2 + y^2}$ the distance from the origin to this point. The six trigonometric functions of $\theta$ are:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \qquad \csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$
$$\qquad\qquad\qquad\qquad x \neq 0 \qquad y \neq 0 \qquad x \neq 0 \qquad y \neq 0$$

- Reference angles can be used to evaluate the trig functions of any angle, since the values are fixed by the ratio of sides and the signs are dictated by the quadrant of the terminal side.
- As in Section 6.2, if a specific value of a trig function is known, the related angle $\theta$ can be found using a reference arc/angle, or the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ features of a calculator for nonstandard values.
- If $\theta$ is a solution to $\sin\theta = k$, then $\theta + 360k$ is also a solution for any integer $k$.

## EXERCISES

**47.** Find two positive angles and two negative angles that are coterminal with $\theta = 207°$.

**48.** Name the reference angle for the angles given:

$\theta = -152°$ $\quad\theta = 521°$ $\quad\theta = 210°$

**49.** Find the value of the six trigonometric functions, given $P(x, y)$ is on the terminal side of angle $\theta$ in standard position.

**a.** $P(-12, 35)$ **b.** $(12, -18)$

**50.** Find the value of $x$, $y$, and $r$ using the information given, and state the quadrant of the terminal side of $\theta$. Then state the values of the six trig functions of $\theta$.

**a.** $\cos\theta = \frac{4}{5}$; $\sin\theta < 0$

**b.** $\tan\theta = -\frac{12}{5}$; $\cos\theta > 0$

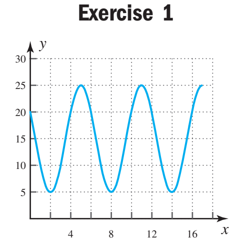
**51.** Find all angles satisfying the stated relationship. For standard angles, express your answer in exact form. For nonstandard angles, use a calculator and round to the nearest tenth.

**a.** $\tan\theta = -1$ **b.** $\cos\theta = \frac{\sqrt{3}}{2}$ **c.** $\tan\theta = 4.0108$ **d.** $\sin\theta = -0.4540$

# MIXED REVIEW

**1.** For the graph of periodic function $f$ given, state the (a) amplitude, (b) average value, (c) period, and (d) value of $f(4)$.

**Exercise 1**




**2.** Name two values in $[0, 2\pi]$ where $\tan t = 1$.

**3.** Name two values in $[0, 2\pi]$ where $\cos t = -\frac{1}{2}$.

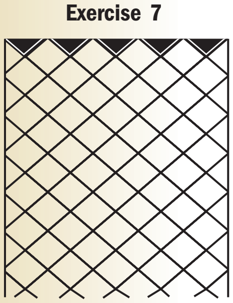
**4.** Given $\sin\theta = \frac{8}{\sqrt{185}}$ with $\theta$ in QII, state the value of the other five trig functions.

**5.** Convert to DMS form: $220.813\overline{8}°$.

**6.** Find two negative angles and two positive angles that are coterminal with (a) 57° and (b) 135°.

**7.** To finish the top row of the tile pattern on our bathroom wall, 12″ by 12″ tiles must be cut diagonally. Use a standard triangle to find the length of each cut and the width of the wall covered by tiles.

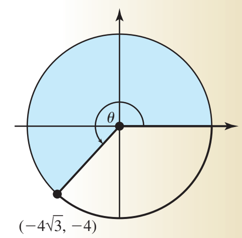
**Exercise 7**



**8.** The service door into the foyer of a large office building is 36″ wide by 78″ tall. The building manager has ordered a large wall painting 85″ by 85″ to add some atmosphere to the foyer area. (a) Can the painting be brought in the service door? (b) If so, at what two integer-valued angles (with respect to level ground) could the painting be tilted?

**9.** Find the arc length and area of the shaded sector.

**Exercise 9**




**10.** Monthly precipitation in Minneapolis, Minnesota can be modeled by a sine function, by using the average precipitation for August (4.05 in.) as a maximum (actually slightly higher in June), and the average precipitation for February (0.79 in.) as a minimum. Assume $t = 0$ corresponds to April. (a) Use the information to construct a sinusoidal model, and (b) Use the model to approximate the inches of precipitation Minneapolis receives in July ($t = 3$) and December ($t = 8$).

Source: 2004 Statistical Abstract of the United States, Table 380

**11.** Convert from DMS to decimal degrees: 86° 54′ 54″.

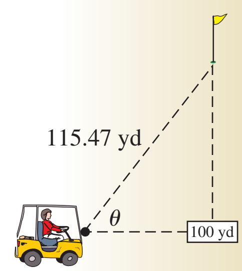
**12.** Name the reference angle $\theta_r$ for the angle $\theta$ given.

**a.** $735°$ **b.** $-135°$ **c.** $\frac{5\pi}{6}$ **d.** $-\frac{5\pi}{3}$

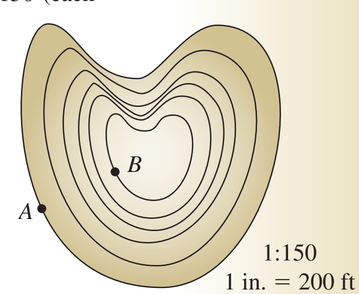
**13.** Find the value of all six trig functions of $\theta$, given the point $(15, -8)$ is on the terminal side.

**14.** Verify that $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is a point on the unit circle and find the value of all six trig functions at this point.

**15.** On your approach shot to the ninth green, the Global Positioning System (GPS) your cart is equipped with tells you the pin is 115.47 yd away. The distance plate states the straight line distance to the hole is 100 yd (see the diagram). Relative to a parallel line between the plate and the hole, at what complementary angle $\theta$ should you hit the shot?




**16.** The electricity supply lines to the top of Lone Eagle Plateau must be replaced, and the new lines will be run in conduit buried slightly beneath the surface. The scale of elevation is 1:150 (each concentric line indicates an increase in 150 ft of elevation), and the scale of horizontal distance is 1 in. = 200 ft. (a) Find the increase in elevation from point $A$ to point $B$, (b) use a proportion to find the horizontal distance from $A$ to $B$ if the measured distance on the map is $2\frac{1}{4}$ inches, (c) draw the corresponding right triangle and use it to estimate the length of conduit needed from $A$ to $B$ and the angle of incline the installers will experience while installing the conduit.




**17.** A salad spinner consists of a colander basket inside a large bowl, and is used to wash and dry lettuce and other salad ingredients. In vigorous use, the spinner is turned at about 3 revolutions per second. (a) Find the angular velocity and (b) find the linear velocity of a point of the circumference if the basket has a 20 cm radius.

**18.** Solve each equation in $[0, 2\pi)$ without the use of a calculator. If the expression is undefined, so state.

**a.** $x = \sin\left(-\frac{\pi}{4}\right)$ **b.** $\sec x = \sqrt{2}$ **c.** $\cot\left(\frac{\pi}{2}\right) = x$

**d.** $\cos \pi = x$ **e.** $\csc x = \frac{2\sqrt{3}}{3}$ **f.** $\tan\left(\frac{\pi}{2}\right) = x$

**19.** State the amplitude, period, horizontal shift, vertical shift, and endpoints of the primary interval, then sketch the graph using a reference rectangle and the rule of fourths.

**a.** $y = 5\cos(2t) - 8$ **b.** $y = \frac{7}{2}\sin\left[\frac{\pi}{2}(x - 1)\right]$

**20.** State the period and phase shift, then sketch the graph of each function.

**a.** $y = 2\tan\left(\frac{1}{4}t\right)$ **b.** $y = 3\sec\left(x - \frac{\pi}{2}\right)$

# PRACTICE TEST

**1.** State the complement and supplement of a 35° angle.

**2.** Find two negative angles and two positive angles that are coterminal with $\theta = 30°$. Many solutions are possible.

**3.** Name the reference angle of each angle $\theta$ given.

**a.** 225° **b.** −510° **c.** $\frac{7\pi}{6}$ **d.** $\frac{25\pi}{3}$

**4.** Convert from DMS to decimal degrees or decimal degrees to DMS as indicated.

**a.** 100°45′18″ to decimal degrees

**b.** 48.2125° to DMS

**5.** Four Corners USA is the point at which Utah, Colorado, Arizona, and New Mexico meet. Using the southern border of Colorado, the western border of Kansas, and the point $P$ where Colorado, Nebraska, and Kansas meet, very nearly approximates a 30-60-90 triangle. If the western border of Kansas is 215 mi long, (a) what is the distance from Four Corners USA to point $P$? (b) How long is Colorado's southern border?

**Exercise 5**


**6.** Complete the table from memory using exact values. If a function is undefined, so state.

| $t$ | sin $t$ | cos $t$ | tan $t$ | csc $t$ | sec $t$ | cot $t$ |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| $\frac{2\pi}{3}$ | | | | | | |
| $\frac{7\pi}{6}$ | | | | | | |
| $\frac{5\pi}{4}$ | | | | | | |
| $\frac{5\pi}{3}$ | | | | | | |
| $\frac{7\pi}{4}$ | | | | | | |
| $\frac{13\pi}{6}$ | | | | | | |

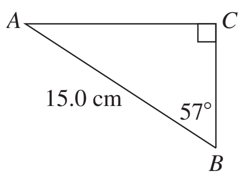
7. Given $\cos\theta = \frac{2}{5}$ and $\tan\theta < 0$, find the value of the other five trig functions of $\theta$.

8. Verify that $\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ is a point on the unit circle, then find the value of all six trig functions associated with this point.

9. In order to take pictures of a dance troupe as it performs, a camera crew rides in a cart on tracks that trace a circular arc. The radius of the arc is 75 ft, and from end to end the cart sweeps out an angle of 172.5° in 20 seconds. Use this information to find (a) the length of the track in feet and inches, (b) the angular velocity of the cart, and (c) the linear velocity of the cart in both ft/sec and mph.

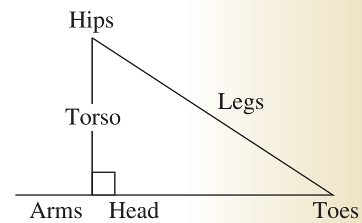
10. Solve the triangle shown. Answer in table form.

**Exercise 10**



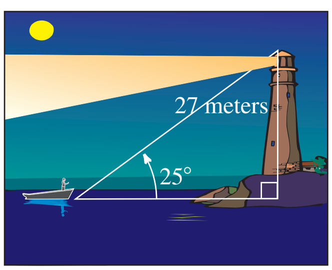

11. The "plow" is a yoga position in which a person lying on their back brings their feet up, over, and behind their head and touches them to the floor. If distance from hip to shoulder (at the right angle) is 57 cm and from hip to toes is 88 cm, find the distance from shoulders to toes and the angle formed at the hips.

**Exercise 11**




12. While doing some night fishing, you round a peninsula and a tall light house comes into view. Taking a sighting, you find the angle of elevation to the top of the lighthouse is 25°. If the lighthouse is known to be 27 m tall, how far away from shore are you?

**Exercise 12**




13. Find the value of $t \in [0, 2\pi)$ satisfying the conditions given.

**a.** $\sin t = -\frac{1}{2}$, $t$ in QIII **b.** $\sec t = \frac{2\sqrt{3}}{3}$, $t$ in QIV **c.** $\tan t = -1$, $t$ in QII

14. In arid communities, daily water usage can often be approximated using a sinusoidal model. Suppose water consumption in the city of Caliente del Sol reaches a maximum of 525,000 gallons in the heat of the day, with a minimum usage of 157,000 gallons in the cool of the night. Assume $t = 0$ corresponds to 6:00 A.M. (a) Use the information to construct a sinusoidal model, and (b) Use the model to approximate water usage at 4:00 P.M. and 4:00 A.M.

15. State the domain, range, period, and amplitude (if it exists), then graph the function over 1 period.

**a.** $y = 2\sin\left(\frac{\pi}{5}t\right)$ **b.** $y = \sec t$ **c.** $y = 2\tan(3t)$

16. State the amplitude, period, horizontal shift, vertical shift, and endpoints of the primary interval. Then sketch the graph using a reference rectangle and the rule of fourths:
$y = 12\sin\left(3t - \frac{\pi}{4}\right) + 19$.

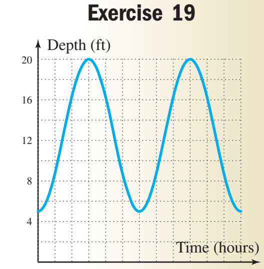
17. An athlete throwing the shot-put begins his first attempt facing due east, completes three and one-half turns and launches the shot facing due west. What angle did his body turn through?

18. State the domain, range, and period, then sketch the graph in $[0, 2\pi)$.

**a.** $y = \tan(2t)$ **b.** $y = \cot\left(\frac{1}{2}t\right)$

19. Due to tidal motions, the depth of water in Brentwood Bay varies sinusoidally as shown in the diagram, where time is in hours and depth is in feet. Find an equation that models the depth of water at time $t$.

**Exercise 19**




20. Find the value of $t$ satisfying the given conditions.

**a.** $\sin t = -0.7568$; $t$ in QIII

**b.** $\sec t = -1.5$; $t$ in QII

# Calculator Exploration and Discovery

## Variable Amplitudes and Modeling the Tides

As mentioned in the *Point of Interest* from Section 6.5, tidal motion is often too complex to be modeled by a single sine function. In this *Exploration and Discovery*, we'll look at a method that combines two sine functions to help model a tidal motion with variable amplitude. In the process, we'll use much of what we know about the amplitude, horizontal shifts and vertical shifts of a sine function, helping to reinforce these important concepts and broaden our understanding about how they can be applied. The graph in Figure 6.104 shows three days of tidal motion for Davis Inlet, Canada.

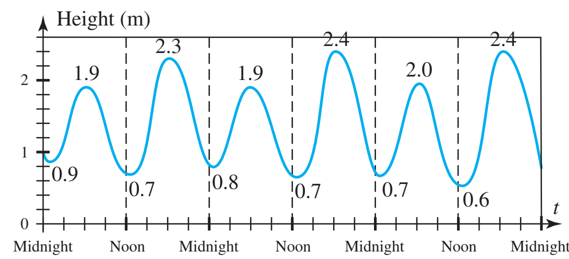



**Figure 6.104**

As you can see, the amplitude of the graph varies, and there is no *single* sine function that can serve as a model. However, notice that the amplitude *varies predictably*, and that the high tides and low tides can independently be modeled by a sine function. To simplify our exploration, will use the assumption that tides have an exact 24 hr period (close, but no), that variations between high and low tides takes place every 12 hr (again close but not exactly true), and the variation between the "low-high" (1.9 m) and the "high-high" (2.4 m) is uniform. A similar assumption is made for the low tides. The result is the graph in Figure 6.105.

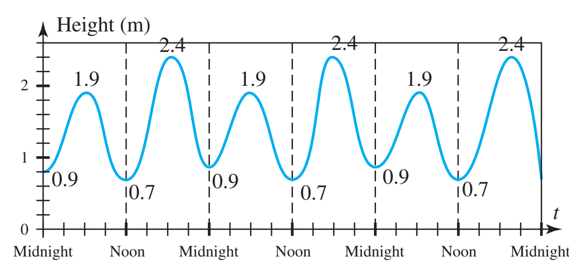
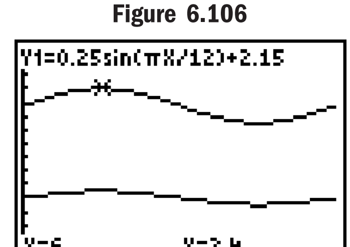



**Figure 6.105**

First consider the high tides, which vary from a maximum of 2.4 to a minimum of 1.9. Using the ideas from Section 6.5 to construct an equation model gives $A = \frac{2.4 - 1.9}{2} = 0.25$ and $D = \frac{2.4 + 1.9}{2} = 2.15$. With a period of $P = 24$ hr we obtain the equation $Y_1 = 0.25 \sin\left(\frac{\pi}{12}x\right) + 2.15$. Using 0.9 and 0.7 as the maximum and minimum low tides, similar calculations yield the equation $Y_2 = 0.1 \sin\left(\frac{\pi}{12}x\right) + 0.8$ (verify this). Graphing these two functions over a 24-hour period yields the graph in Figure 6.106, where we note the high and low values are correct, but the two functions are in phase with each other. As can be determined from Figure 6.105, we want the high tide model to start at the average value and decrease, and the low tide equation model to start at high-low and decrease. Replacing $x$ with $x - 12$ in $Y_1$ and $x$ with $x + 6$ in $Y_2$ accomplishes this result (see Figure 6.107). Now comes the fun part! Since $Y_1$ represents the low/high maximum values for high tide, and $Y_2$ represents the low/high minimum values for low tide, *the amplitude and average value for the tidal*

**Figure 6.106**



*motion at Davis Inlet are* $A = \frac{Y_1 - Y_2}{2}$ and $D = \frac{Y_1 + Y_2}{2}$*!*

By entering $Y_3 = \frac{Y_1 - Y_2}{2}$ and $Y_4 = \frac{Y_1 + Y_2}{2}$, the equation for the tidal motion (with its variable amplitude) will have the form $Y_5 = Y_3 \sin(Bx \pm C) + Y_4$, where the value of $B$ and $C$ must be determined. The key here is to note there is only a 12-hr difference between the changes in amplitude, so $P = 12$ (instead of 24) and $B = \frac{\pi}{6}$ for this function. Also, from the graph (Figure 6.105) we note the tidal motion begins at a minimum and increases, indicating a shift of 3 units to the right is required. Replacing $x$ with $x - 3$ gives the equation modeling these tides, and the final equation is $Y_5 = Y_3 \sin\left[\frac{\pi}{6}(x - 3)\right] + Y_4$. Figure 6.108 gives a screen shot of $Y_1$, $Y_2$, and $Y_5$ in the interval [0, 24]. The tidal graph from Figure 6.105 is shown in Figure 6.109 with $Y_3$ and $Y_4$ superimposed on it.

**Figure 6.107**

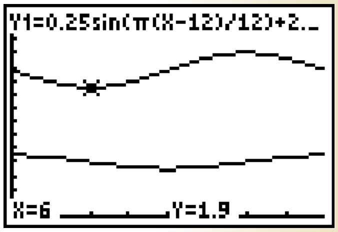
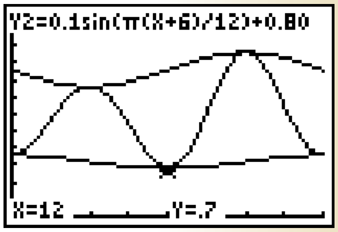



**Figure 6.108**



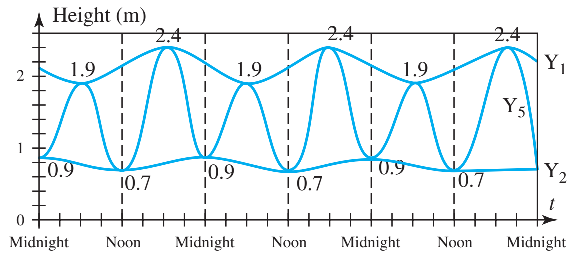




**Figure 6.109**

**Exercise 1:** The website www.tides.com/tcpred.htm offers both *t*ide and *c*urrent *pred*ictions for various locations around the world, in both numeric and graphical form. In addition, data for the "two" high tides and "two" low tides are clearly highlighted. Select a coastal area where tidal motion is similar to that of Davis Inlet, and repeat this exercise. Compare your model to the actual data given on the website. How good was the fit?

# STRENGTHENING CORE SKILLS

## Standard Angles, Reference Angles, and the Trig Functions

A review of the main ideas discussed in this chapter indicates there are four of what might be called "core skills." These are skills that (a) play a fundamental part in the acquisition of concepts, (b) hold the overall structure together as we move from concept to concept, and (c) are ones we return to again and again throughout our study. The first of these is *(1) knowing the standard angles and standard values.* These values are "standard" because no estimation, interpolation or special methods are required to name their value, and each can be expressed as a single factor. This gives them a great advantage in that further conceptual development can take place without the main points being obscured by large expressions or decimal approximations. Knowing the value of the trig functions for each standard angle will serve you very well throughout this study. *Know* the chart on page 659 and the ideas that led to it.

The standard angles/values brought us to the trigonometry of any angle, forming a strong bridge to the second core skill: *(2) using reference angles to determine the value of the trig functions in each quadrant.* For review, a 30-60-90 triangle will always have sides that are in the proportion $1x{:}\sqrt{3}x{:}2x$, regardless of its size. This means for any angle $\theta$, where $\theta_r = 30°$, $\sin\theta = \frac{1}{2}$

or $\sin\theta = -\frac{1}{2}$ since the *ratio is fixed* but the sign *depends on the quadrant of* $\theta$ : $\sin 30° = \frac{1}{2}$ [QI], $\sin 150° = \frac{1}{2}$ [QII], $\sin 210° = -\frac{1}{2}$ [QIII], $\sin 330° = -\frac{1}{2}$ [QII], and so on (see Figure 6.110).

In turn, the reference angles led us to a third core skill, helping us realize that if $\theta$ was not a quadrantal angle, *(3) equations like* $\cos(\theta) = -\frac{\sqrt{3}}{2}$ *must have two solutions in* $[0, 360°)$. From the standard angles and standard values we learn to recognize that for $\cos\theta = -\frac{\sqrt{3}}{2}$, $\theta_r = 30°$, which will occur as a reference angle in the two quadrants where cosine is negative, QII and QIII. The solutions in $[0, 360°)$ are $\theta = 150°$ and $\theta = 210°$ (see Figure 6.111).

Of necessity, this brings us to the fourth core skill, *(4) effective use of a calculator.* The standard angles are a wonderful vehicle for introducing the basic ideas of trigonometry, and actually occur quite frequently in real-world applications. But by far, most of the values we encounter will be nonstandard values where $\theta_r$ must be found using a calculator. However, once $\theta_r$ is found, the reason and reckoning inherent in the ideas above can be directly applied.

The *Summary and Concept Review Exercises*, as well as the *Practice Test* offer ample opportunities to refine these skills, so that they will serve you well in future chapters as we continue our attempts to explain and understand the world around us in mathematical terms.

**Figure 6.110**

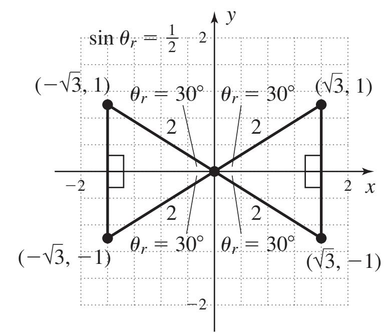
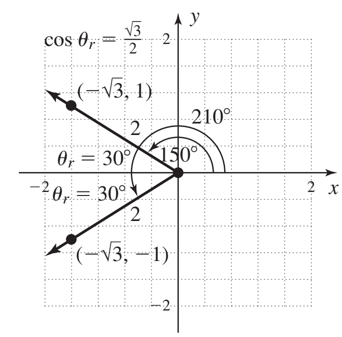



**Figure 6.111**




**Exercise 1:** Fill in the table from memory.

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin t = y$ | | | | | | | | | | | |
| $\cos t = x$ | | | | | | | | | | | |
| $\tan t = \frac{y}{x}$ | | | | | | | | | | | |

**Exercise 2:** Solve each equation in $[0, 2\pi)$ without the use of a calculator.

**a.** $2\sin t + \sqrt{3} = 0$

**b.** $-3\sqrt{2}\cos t + 4 = 1$

**c.** $-\sqrt{3}\tan t + 2 = 1$

**d.** $\sin^{-1}\left(\frac{1}{2}\right) = t$

**Exercise 3:** Solve each equation in $[0, 2\pi)$ using a calculator and rounding answers to four decimal places.

**a.** $\sqrt{6}\sin t - 2 = 1$

**b.** $-3\sqrt{2}\cos t + \sqrt{2} = 0$

**c.** $3\tan t + \frac{1}{2} = -\frac{1}{4}$

**d.** $2\sec t = -5$

# Cumulative Review Chapters 1–6

**1.** Solve the inequality given: $2|x + 1| - 3 < 5$

**2.** Find the domain of the function: $y = \sqrt{x^2 - 2x - 15}$

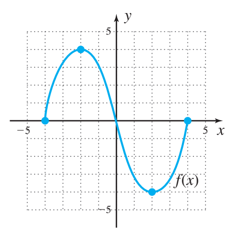
**3.** Given that $\tan\theta = \frac{80}{39}$, draw a right triangle that corresponds to this ratio, then use the Pythagorean theorem to find the length of the missing side. Finally, find the two acute angles.

**4.** Without a calculator, what values in $[0, 2\pi)$ make the equation true: $\sin t = -\frac{\sqrt{3}}{2}$?

**5.** Given $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$ is a point on the unit circle corresponding to $t$, find all six trig functions of $t$.

State the domain and range of each function shown:

**6.** $y = f(x)$




**7. a.** $f(x) = \sqrt{2x - 3}$

**b.** $g(x) = \frac{2x}{x^2 - 49}$

**8.** $y = T(x)$

| *x* | *T*(*x*) |
|---|---|
| 0 | −7 |
| 1 | −5 |
| 2 | −3 |
| 3 | −1 |
| 4 | 1 |
| 5 | 3 |
| 6 | 5 |

**9.** Analyze the graph of the function in Exercise 6, including: (a) maximum and minimum values; (b) intervals where $f(x) \geq 0$ and $f(x) < 0$; (c) intervals where $f$ is increasing or decreasing; and (d) any symmetry noted. Assume the features you are to describe have integer values.

**10.** The attractive force that exists between two magnets varies inversely as the square of the distance between them. If the attractive force is 1.5 newtons (N) at a distance of 10 cm, how close are the magnets when the attractive force reaches 5 N?

**11.** The world's tallest indoor waterfall is in Detroit, Michigan, in the lobby of the International Center Building. Standing 66 ft from the base of the falls, the angle of elevation is 60°. How tall is the waterfall?

**12.** It's a warm, lazy Saturday and Hank is watching a county maintenance crew mow the park across the street. He notices the mower takes 29 sec to pass through 77° of rotation from one end of the park to the other. If the corner of the park is 60 ft directly across the street from his house, (a) how wide is the park? (b) How fast (in mph) does the mower travel as it cuts the grass?

**13.** Graph using transformations of a parent function: $f(x) = \frac{1}{x + 1} - 2$.

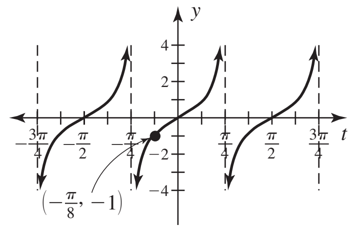
**14.** Graph using transformations of a parent function: $g(x) = e^{x - 1} - 2$.

**15.** Find $f(\theta)$ for all six trig functions, given the point $P(-9, 40)$ is a point on the terminal side of the angle. Then find the angle $\theta$ in degrees, rounded to tenths.

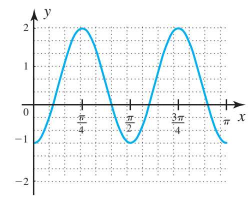
**16.** Given $t = 5.37$, (a) in what quadrant does the arc terminate? (b) What is the reference arc? (c) Find the value of $\sin t$ rounded to four decimal places.

**17.** A jet-stream water sprinkler shoots water a distance of 15 m and turns back-and-forth through an angle of $t = 1.2$ rad. (a) What is the length of the arc that the sprinkler reaches? (b) What is the area in $m^2$ of the yard that is watered?

**18.** Determine the equation of graph shown given it is of the form $y = A\tan(Bt)$.

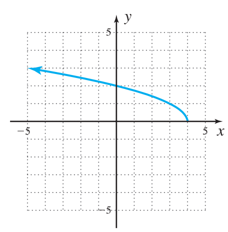



**19.** Determine the equation of the graph shown given it is of the form $y = A\sin(Bt \pm C) + D$.

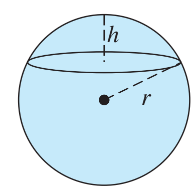



**20.** In London, the average temperatures on a summer day range from a high of 72°F to a low of 56°F [Source: 2004 Statistical Abstract of the United States, Table 1331]. Use this information to write a sinusoidal equation model, assuming the low temperature occurs at 6:00 A.M. Clearly state the amplitude, average value, period, and horizontal shift.

**21.** The graph of a function $f(x)$ is given. Sketch the graph of $f^{-1}(x)$.




**22.** The volume of a spherical cap is given by $V = \frac{\pi h^2}{3}(3r - h)$. Solve for $r$ in terms of $V$ and $h$.




**23.** Find the slope and $y$-intercept: $3x - 4y = 8$.

**24.** Solve by factoring: $4x^3 - 8x^2 - 9x + 18 = 0$.

**25.** At what interest rate will $1000 grow to $2275 in 12 yr if compounded continuously?