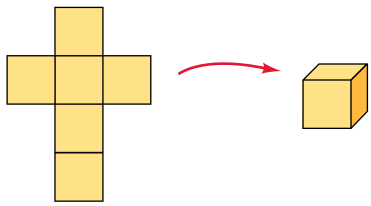
MATH ACTIVITY 9.3

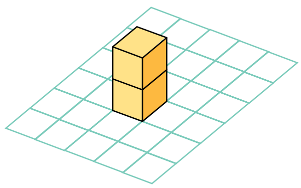
Nets for Three-Dimensional Figures

Materials: Sheets of 2-centimeter grid paper (copy from the website) and scissors. The 2-centimeter cubes for building figures are optional.

*1. A cube can be formed by creasing and folding along the lines of the pattern shown here. Use your grid paper to form and cut out several different types of patterns that will fold into a cube with no overlaps. Show sketches of your patterns. Patterns for three-dimensional figures are called **nets.**



2. Form and cut out a net that will fold into the two-cube stack shown here. (*Hint:* One way is to imagine this stack sitting on a square of the grid and visualize the squares that would need to be folded up to cover the stack.) Sketch your net.

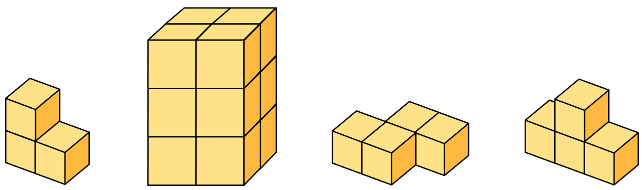


NCTM Standards

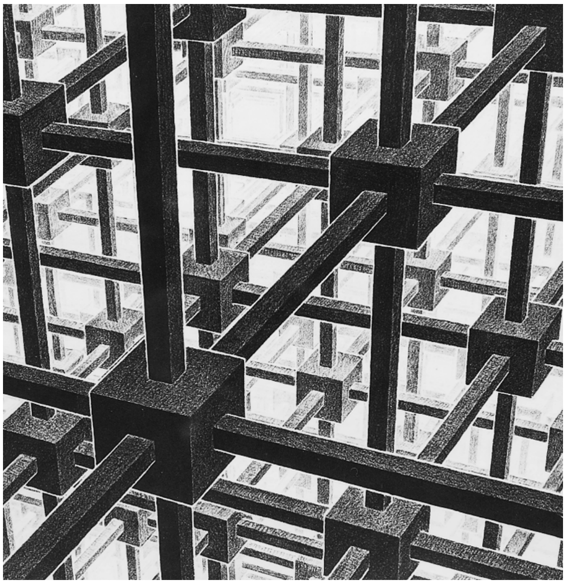
By representing three-dimensional shapes in two dimensions and constructing three-dimensional shapes from two-dimensional representations, students learn about the characteristics of shapes. p. 168

3. Visualize a stack of n cubes, and describe a net of squares that will fold and cover this stack. Write an algebraic expression for the number of squares in this net.

4. Select two of the following figures, and form their nets on grid paper. Show sketches of your nets. The number of cubes in the figure is the **volume** of the figure, and the number of squares in the net is the **surface area** of the figure. Determine the volume and surface area of each figure you select.



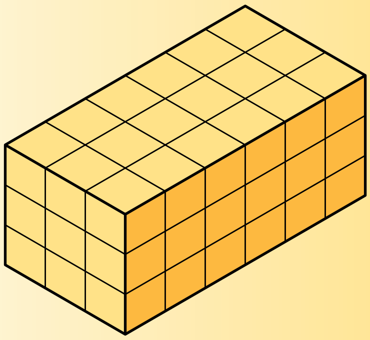
Section 9.3 SPACE FIGURES



M. C. Escher's "Cubic Space Division"

PROBLEM OPENER

This is a sketch of a three-dimensional figure that contains 54 small cubes. If the outside of the figure is painted and then the figure is disassembled into 54 individual cubes, how many cubes will have paint on one face, two faces, three faces, and no faces?



In the above lithograph by the Dutch artist Maurits C. Escher (1898–1970), the girders intersect at right angles to form the edges of large cubes. The Canadian mathematician H. S. M. Coxeter calls it the *cubic honeycomb.* By representing space as being filled with cubes of the same size, Escher gives a wonderful sense of infinite space.

The notion of **space** in geometry is an undefined term, just as the ideas of point, line, and plane are undefined. We intuitively think of space as three-dimensional and of a plane as only two-dimensional. In his theory of relativity, Einstein tied together the three dimensions of space and the fourth dimension of time. He showed that space and time affect each other and give us a four-dimensional universe.

NCTM Standards

In NCTM's K–4 Standard, *Geometry and Spatial Sense,* the importance of spatial understanding is discussed:

> Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas, as well as other advanced mathematical topics.*

**Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics 1989), p. 48.

Sonya Kovalevsky, 1850–1891

HISTORICAL HIGHLIGHT

The Russian mathematician Sonya Kovalevsky is regarded as the greatest woman mathematician to have lived before 1900. Since women were barred by law from institutions of higher learning in Russia, Kovalevsky attended Heidelberg University in Germany. Later she was refused admission to the University of Berlin, which also barred women. Even the famous mathematician Karl Weierstrass, who claimed she had "the gift of intuitive genius," was unable to obtain permission for Kovalevsky to attend his lectures. She obtained her doctorate from the University of Göttingen but was without a teaching position for nine years, until the newly formed University of Stockholm broke tradition and appointed her to an academic position. Kovalevsky's prominence as a mathematician reached its peak in 1888, when she received the famous Prix Bordin from the French Académie des Sciences for her research paper "On the Rotation of a Solid about a Fixed Point." The selection committee "recognized in this work not only the power of an expansive and profound mind, but also a great spirit of invention."[†]

[†]D. M. Burton, *The History of Mathematics,* 4th ed. (New York: McGraw-Hill, 1999) pp. 557–560.

PLANES

In two dimensions, the figures (lines, angles, polygons, etc.) all occur in a plane. In three dimensions, there are an infinite number of planes. Each plane partitions space into three disjoint sets: the points on the plane and two **half-spaces.** Portions of a few planes are shown in Figure 9.40. Any two planes either are **parallel,** as in part a, or **intersect** in a line, as in part b.

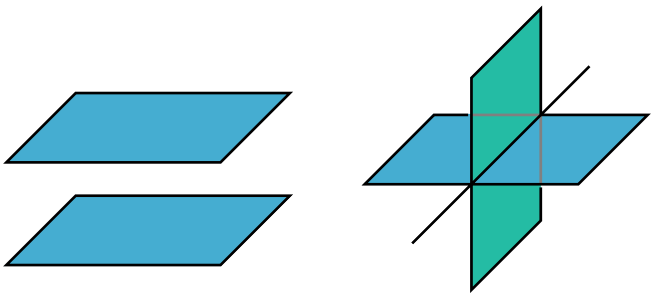


Figure 9.40

When two planes intersect, we call the angle between the planes a **dihedral angle.** Figure 9.41 shows three dihedral angles and their measures. A dihedral angle is measured by measuring the angle whose sides lie in the planes and are perpendicular to the line of intersection of the two planes. Parts a, b, and c of Figure 9.41 show examples of obtuse, right, and acute dihedral angles, respectively.

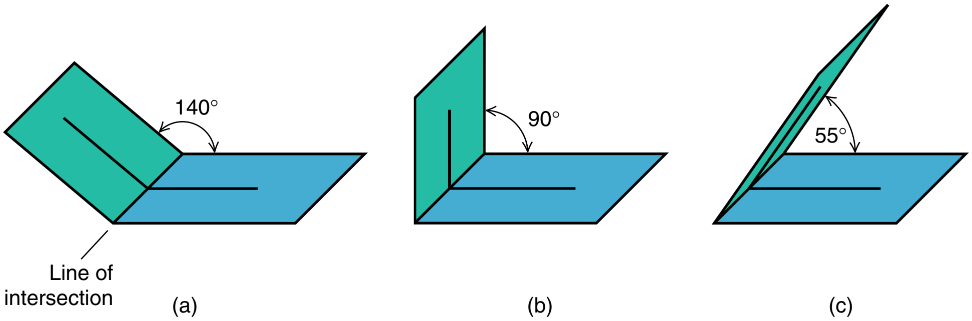


Figure 9.41

When a line *m* in three-dimensional space does not intersect a plane *P*, it is **parallel to the plane,** as in Figure 9.42a. A line *n* is **perpendicular to a plane** *Q* at a point *k* if the line is perpendicular to every line in the plane that contains *K*, as in Figure 9.42b.

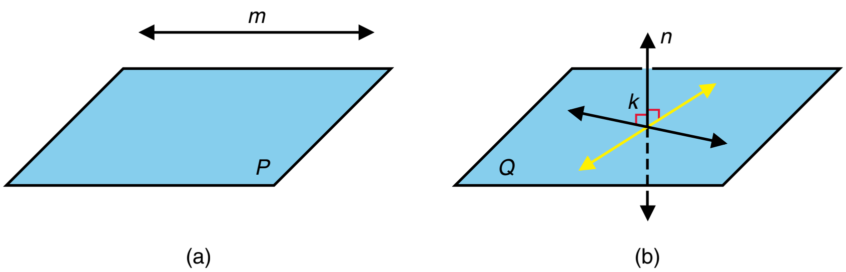


Figure 9.42

POLYHEDRA

The three-dimensional object with flat sides in Figure 9.43 is a crystal of pyrite. Its 12 flat pentagonal sides with their straight edges were not cut by people but were shaped by nature.

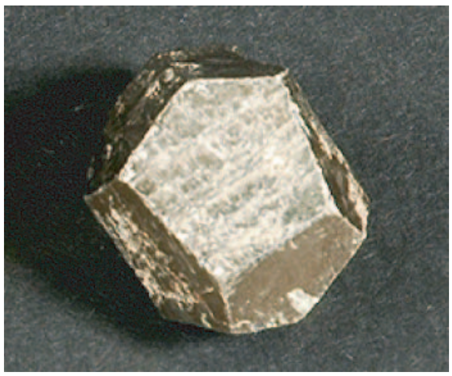


Figure 9.43
Crystal of pyrite

The surface of a figure in space whose sides are polygonal regions, such as the one in Figure 9.43, is called a **polyhedron** (*polyhedra* is the plural). The polygonal regions are called **faces,** and they intersect in the **edges** and **vertices** of the polyhedron. The union of a polyhedron and its interior is called a **solid.** Figure 9.44 shows examples of a polyhedron and two figures that are not polyhedra. The figure in part a is a polyhedron because its faces are polygonal regions. The figures in parts b and c are not polyhedra because one has a curved surface and the other has two faces that are not polygons.

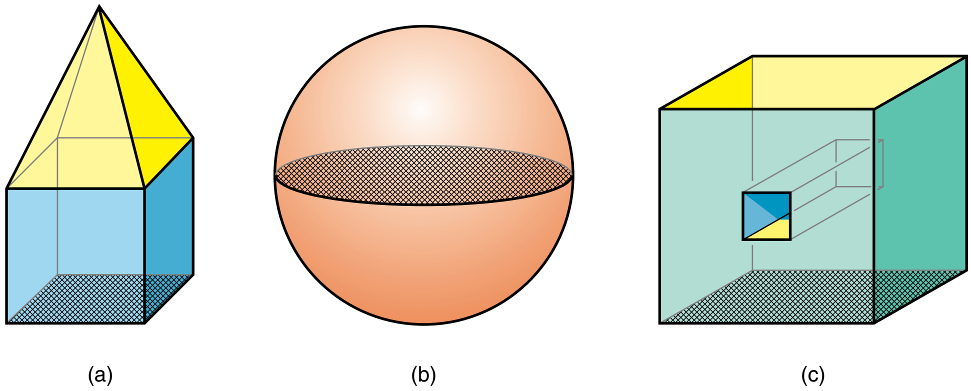
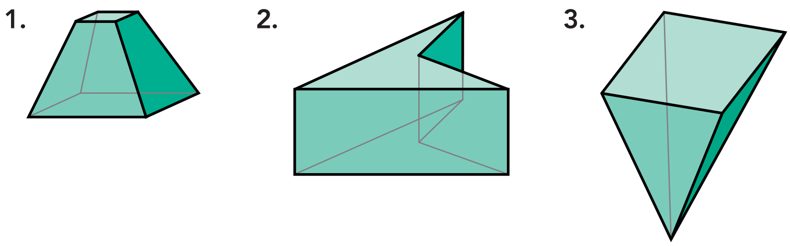


Figure 9.44

A polyhedron is **convex** if the line segment connecting any two of its points is contained inside the polyhedron or on its surface.

EXAMPLE A

Classify the following polyhedra as convex or nonconvex.



Solution Polyhedra 1 and 3 are convex; 2 is nonconvex.

Research Statement

The 6th national mathematics assessment concluded that students need more experiences with concrete models to enhance their visualization skills and more opportunities to see how geometric concepts relate to real-live situations and other mathematical aspects.

Strutchens and Blume 1997

REGULAR POLYHEDRA

The best known of all the polyhedra are the r*egular polyhedra,* or *Platonic solids.* A **regular polyhedron** is a convex polyhedron whose faces are *congruent regular polygons,* the same number of which meet at each vertex. The ancient Greeks proved that there are only five regular polyhedra. Models of these polyhedra are shown in Figure 9.45. The **tetrahedron** has 4 triangles for faces; the **cube** has 6 square faces; the **octahedron** has 8 triangular faces; the **dodecahedron** has 12 pentagons for faces; and the **icosahedron** has 20 triangular faces.

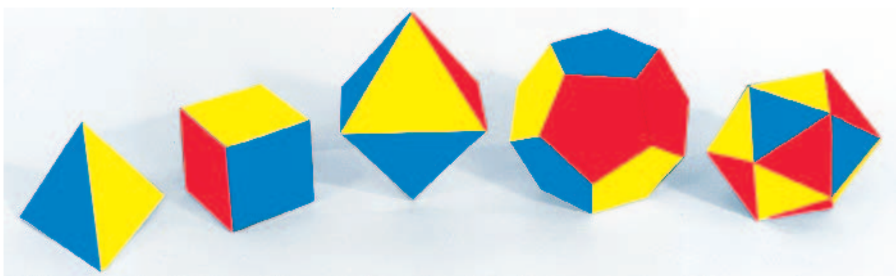


Figure 9.45
From left to right: tetrahedron, cube (hexahedron), octahedron; dodecahedron, icosahedron

The first three of the regular polyhedra shown in Figure 9.45 are found in nature as crystals. The cube and the octahedron occur in the common mineral pyrite, shown in Figure 9.46. The cube, which is embedded in rock, was found in Vermont, and the octahedron is from Peru. The other regular polyhedra, the dodecahedron and the icosahedron, do not occur as crystals but have been found in the skeletons of microscopic sea animals called *radiolarians.*

Figure 9.46
Crystals of pyrite

Semiregular Polyhedra Some polyhedra have two or more different types of regular polygons for faces. The faces of the boracite crystal in Figure 9.47 are squares and equilateral triangles. This crystal, too, developed its flat, regularly shaped faces naturally, without the help of machines or people. Polyhedra whose faces are two or more regular polygons with the same arrangement of polygons around each vertex are called **semiregular polyhedra.** The boracite crystal is one of these. Each of its vertices is surrounded by three squares and one equilateral triangle.

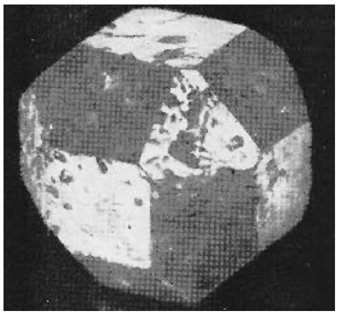


Figure 9.47
Crystal of boracite

Several other semiregular polyhedra are shown in Figure 9.48. You may recognize the combination of hexagons and pentagons in part a as the pattern used on the surface of soccer balls.

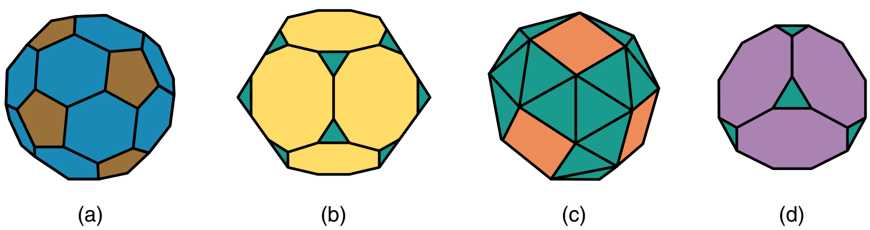


Figure 9.48

EXAMPLE B For each semiregular polyhedron in Figure 9.48, list the polygons in the order in which they occur about any vertex.

Solution Part a: hexagon, hexagon, pentagon; part b: dodecagon, dodecagon, triangle; part c: triangle, triangle, triangle, triangle, square; part d: octagon, octagon, triangle

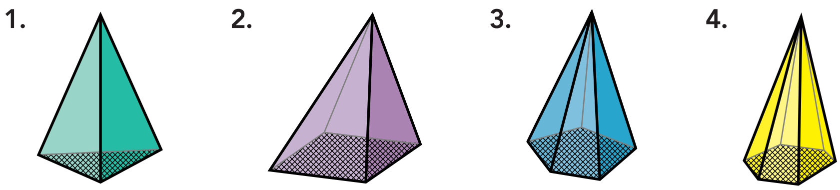
PYRAMIDS AND PRISMS

Chances are that when you hear the word *pyramid,* you think of the monuments built by the ancient Egyptians. Each of the Egyptian pyramids has a square base and triangular sides rising up to the vertex. This is just one type of pyramid. In general, the **base of a pyramid** can be any polygon, but its sides are always triangular. Pyramids are named according to the shape of their bases. Church spires are familiar examples of pyramids. They are usually square, hexagonal, or octagonal pyramids. The spire in the photograph in Figure 9.49 is a hexagonal pyramid that sits on the octagonal roof that is supported by eight columns of the housing for the bell.

Figure 9.49
Community church of Durham, New Hampshire

Several pyramids with different bases are shown in the following example. Pyramids whose sides are isosceles triangles, as in Figures (1), (3), and (4) of Example C, are called **right pyramids.** Otherwise, as in Figure (2) of Example C, the pyramid is called an **oblique pyramid.** The vertex that is not contained in the pyramid's base is called the **apex.**

EXAMPLE C Determine the name of each pyramid.



Solution **1.** Triangular pyramid (also called a tetrahedron) **2.** Oblique square pyramid **3.** Pentagonal pyramid **4.** Hexagonal pyramid

Prisms Prisms are another common type of polyhedron. You probably remember from your science classes that a prism is used to produce the spectrum of colors ranging from violet to red. Because of the angle between the vertical faces of a prism, light directed into one face will be bent when it passes out through the other face (Figure 9.50).

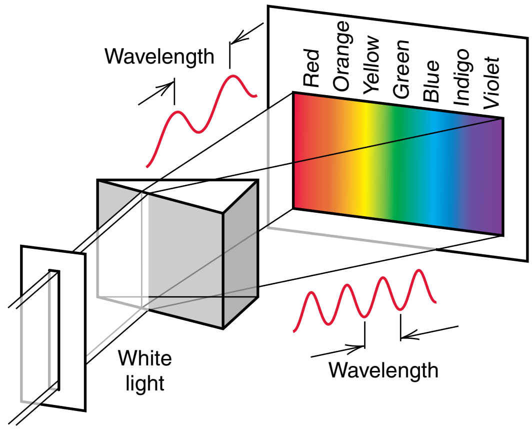


Figure 9.50

A **prism** has two parallel **bases,** upper and lower, which are congruent polygons. Like pyramids, prisms get their names from the shape of their bases. If the lateral sides of a prism are perpendicular to the bases, as in the case of the triangular, quadrilateral, hexagonal, and rectangular prisms in Figure 9.51, they are rectangles. Such a prism is called a **right prism.** A rectangular prism, which is modeled by a box, is the most common type of prism. If some of the lateral faces of a prism are parallelograms that are not rectangles, as in the pentagonal prism, the prism is called an **oblique prism.** The union of a prism and its interior is called a **solid prism.** A rectangular prism that is a solid is sometimes called a **rectangular solid.**

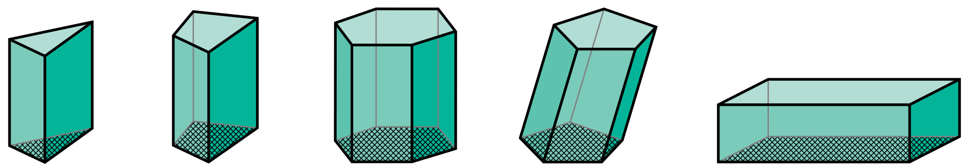
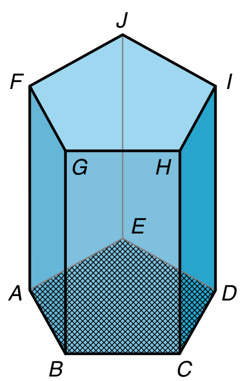


Figure 9.51

EXAMPLE D

The following figure is a right prism with bases that are regular pentagons.



Research Statement

In order to develop a conceptual understanding of geometry, students need to be placed in situations that allow them to apply deductive, inductive, and spatial reasoning.

Geddes and Fortunato 1993

1. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?

2. What is the measure of the dihedral angle between face *GHIJF* and face *CDIH*?

3. Name two faces that are in parallel planes.

Solution **1.** It is the same as the measure of $\measuredangle FGH$, which is 108°. **2.** 90°. Since this is a right prism, the top base is perpendicular to each of the vertical sides. **3.** *ABCDE* and *FGHIJ*

The two oblique hexagonal prisms in Figure 9.52 are crystals that grew with these flat, smooth faces and straight edges. Their lateral faces are parallelograms.

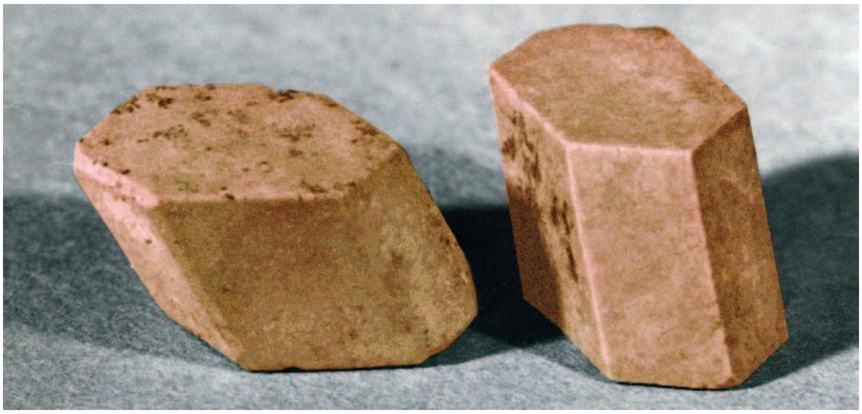


Figure 9.52
Prisms of the crystal orthoclase feldspar

CONES AND CYLINDERS

Cones and cylinders are the circular counterparts of pyramids and prisms. Ice cream cones, paper cups, and party hats are common examples of cones. A cone has a circular region (disk) for a **base** and a lateral surface that slopes to the **vertex (apex).** If the vertex lies directly above the center of the base, the cone is called a **right cone** or usually just a cone; otherwise, it is an **oblique cone** (Figure 9.53).

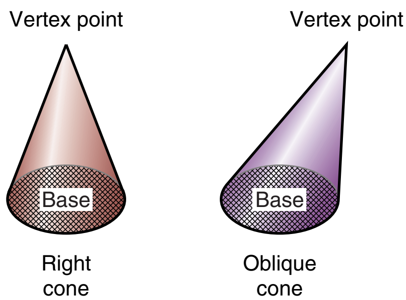


Figure 9.53

Ordinary cans are models of cylinders. A **cylinder** has two parallel circular **bases** (disks) of the same size and a lateral surface that rises from one base to the other. If the centers of the upper base and lower base lie on a line that is perpendicular to each base, the cylinder is called a **right cylinder** or simply a cylinder; otherwise, it is an **oblique cylinder** (Figure 9.54). Almost without exception, the cones and cylinders we use are right cones and right cylinders.

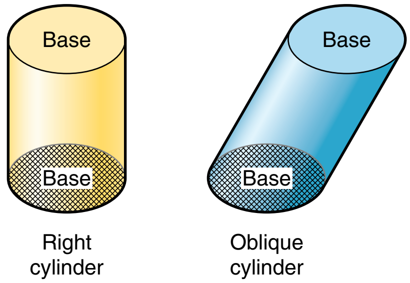


Figure 9.54

SPHERES AND MAPS

The photograph in Figure 9.55 is a view of Earth showing its almost perfect spherical shape. It was photographed from the *Apollo 17* spacecraft during its 1972 lunar mission. The dark regions are water. The Red Sea and the Gulf of Aden are near the top center, and the Arabian Sea and Indian Ocean are on the right.

Figure 9.55
View of Earth as seen by the *Apollo 17* crew traveling toward the Moon. This view extends from the Mediterranean Sea to the Antarctica south polar ice cap. Almost the entire coastline of Africa is visible and the Arabian Peninsula can be seen at the northeastern edge of Africa. The large island off the coast of Africa is the Malagasy Republic and the Asian mainland is on the horizon toward the northeast.

Sphere A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** The union of a sphere and its interior is called a **solid sphere.**

A line segment joining the center of a sphere to a point on the sphere is called a **radius.** The length of such a line segment is also called the **radius of the sphere.** A line segment containing the center of the sphere and whose endpoints are on the sphere is called a **diameter,** and the length of such a line segment is called the **diameter of the sphere.**

The geometry of the sphere is especially important for navigating on the surface of the Earth. You may have noticed that airline maps show curved paths between distant cities. This is because the shortest distance between two points on a sphere is along an arc of a *great circle.* In the drawing of the sphere in Figure 9.56 on page 611, the red arc between

I will identify, describe, and classify 3-dimensional objects.

19.1 3-Dimensional Figures

Learn

I. M. Pei designed the Grand Louvre, the main entrance to the Louvre Museum in Paris, France.

How would you describe this 3-dimensional figure?

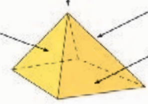
VOCABULARY
See highlighted words.

Parts of a 3-Dimensional Figure

A **3-dimensional figure** is a figure with length, width, and height.

You can describe a 3-dimensional figure by its parts.

A face is a flat side.
A base is a face on which the figure sits.
An edge is where 2 faces meet.
A vertex is where 3 or more faces meet.

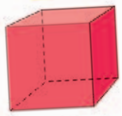


square pyramid

The Grand Louvre is a square pyramid with 5 faces, 8 edges, and 5 vertices.

More 3-Dimensional Figures

cube

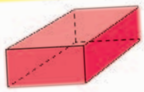


6 faces, 12 edges, 8 vertices

triangular pyramid

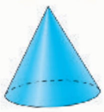
4 faces, 6 edges, 4 vertices

rectangular prism



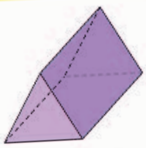
6 faces, 12 edges, 8 vertices

cone



1 circular base

triangular prism



5 faces, 9 edges, 6 vertices

sphere

cylinder

2 circular bases

points X and Y is the arc of a **great circle,** because the center of the red circle is also the center of the sphere. However, the arc of the blue circle between points X and Y is not the arc of a great circle. So the distance between points X and Y along the red arc on a sphere is less than the distance between these points along the blue arc.

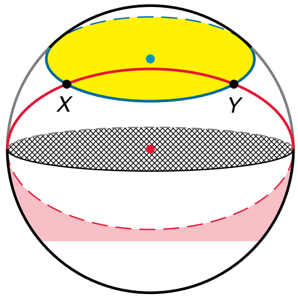


Figure 9.56

Locations on the Earth's surface are often given by naming cities, streets, and buildings. A more general method of describing location uses two systems of circles (Figure 9.57). The circles that are parallel to the equator are called **parallels of latitude** and are shown in part a. Except for the equator, these circles are not great circles. Each parallel of latitude is specified by an angle from 0° to 90°, both north and south of the equator. For example, New York City is at a northern latitude of 41°, and Sydney, Australia, is at a southern latitude of 34°. The second system of circles is shown in part b. These circles pass through the north and south poles and are called **meridians of longitude.** These are great circles, and each is perpendicular to the equator. Since there is no natural point at which to begin numbering the meridians of longitude, the meridian that passes through Greenwich, England, was chosen as the zero meridian. Each meridian of longitude is given by an angle from 0° to 180°, both east and west of the zero meridian. The longitude of New York City is 74° west, and that of Sydney, Australia, is 151° east. These parallels of latitude and meridians of longitude, shown together in part c, form a grid or coordinate system for locating any point on the Earth.

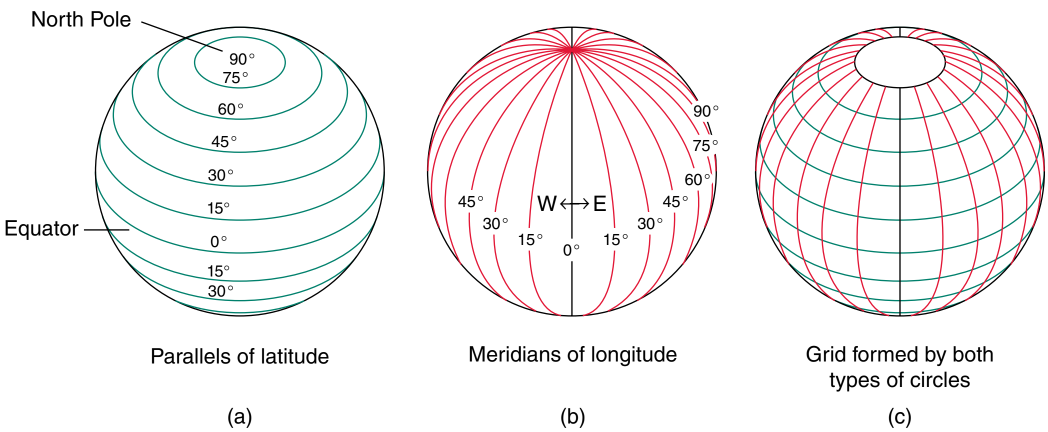
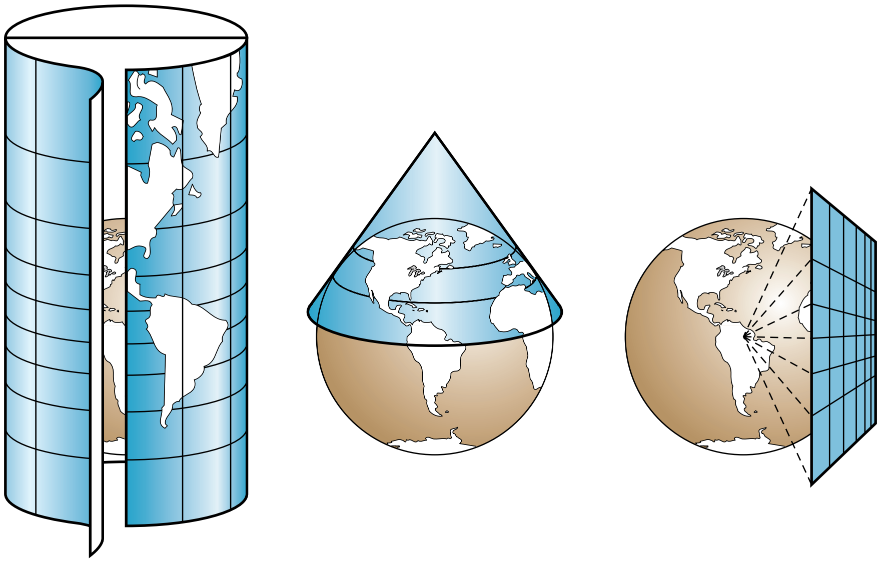


Figure 9.57

Map Projections The globe is a spherical map of the Earth. While such a map accurately represents the Earth's shape and relative distances, we cannot see the whole globe at one time, nor can distances be measured easily. Maps on a flat surface are much more convenient. However, since a sphere cannot be placed flat on a plane without separating or overlapping some of its surface, making flat maps of the earth is a problem. There are three basic solutions: copying the Earth's surface onto a cylinder, a cone, or a plane (Figure 9.58). These methods of copying are called **map projections.** In each case, some distortions of shapes and distances occur.



Cylindrical projection (a) Conic projection (b) Plane projection (c)

Figure 9.58

A **cylindrical projection** (part a), also called a **mercator projection,** is obtained by placing a cylinder around a sphere and copying the surface of the sphere onto the cylinder. The cylinder is then cut to produce a flat map. Regions close to the equator are reproduced most accurately. The closer we get to the poles, the more the map is distorted.

A **conic projection** (part b) is produced by placing a cone with its apex over one of the poles and copying a portion of the surface of a sphere onto the cone. The cone is then cut and laid flat. This type of map construction is commonly used for countries that lie in an east-west direction and are middle latitude countries, as opposed to those near the poles or equator. The maps of the United States that are issued by the American Automobile Association are conical projections.

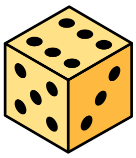
A **plane projection** (part c), also called an **azimuthal projection,** is made by placing a plane next to any point on a sphere and projecting the surface onto the plane. To visualize this process, imagine a light at the center of the sphere, and think of the boundary of a country as being pierced with small holes. The light shining through these holes, as shown by the dashed lines in part c, forms an image of the country on the plane. Less than one-half of the sphere's surface can be copied onto a plane projection, with the greatest distortion taking place at the outer edges of the plane. A plane projection, unlike cylindrical and conical projections, has the advantage that the distortion is uniform from the center of the map to its edges. Plane projections are used for hemispheres and maps of the Arctic and Antarctic. To map the polar regions, a plane is placed perpendicular to the Earth's axis in contact with the north or south pole.

PROBLEM-SOLVING APPLICATION

There is a remarkable formula that relates the numbers of vertices, edges, and faces of a polyhedron. This formula was first stated by René Descartes about 1635. In 1752 it was discovered again by Leonhard Euler and is now referred to as **Euler's formula.** See if you can discover this formula, either before or as you read the parts of the solution presented below.

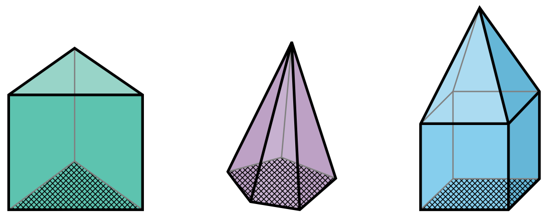
Problem

What is the relationship among the numbers of faces, vertices, and edges of a polyhedron?



Understanding the Problem Euler's formula holds for all polyhedra. Let's look at a specific example. A die is a cube that has six faces. **Question 1:** How many vertices and edges does it have?

Devising a Plan Let's *make a table;* list the numbers of faces, vertices, and edges for several polyhedra; and look for a relationship. **Question 2:** What are the numbers of faces, vertices, and edges for the polyhedra in figures (a), (b), and (c)?

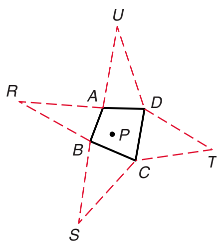


(a) (b) (c)

Laboratory Connection

Pyramid Patterns

Determine how to construct a pattern that will fold to make a pyramid with a given polygonal base and an apex above a given point in the base. Explore this and related questions in this investigation.



Mathematics Investigation
Chapter 9, Section 3
www.mhhe.com/bennett-nelson

Carrying Out the Plan The following table contains the numbers of faces, vertices, and edges for the cube in the margin above and the preceding polyhedra in figures (a) through (c). Using F for the number of faces, V for the number of vertices, and E for the number of edges, we can construct Euler's formula from these data. **Question 3:** What is Euler's formula?

	F	V	E
Cube	6	8	12
Figure (a)	5	6	9
Figure (b)	6	6	10
Figure (c)	9	9	16

Looking Back You may remember that an icosahedron has 20 triangular faces, but may not remember the number of edges or vertices. Altogether, 20 triangles have a total of 60 edges. Since every two edges of a triangle form one edge of an icosahedron, this polyhedron has $60 \div 2 = 30$ edges. Given the numbers of faces and edges for the icosahedron and Euler's formula $F + V - 2 = E$, we can determine the number of vertices. **Question 4:** How many vertices are there?

Answers to Questions 1–4 **1.** 8 vertices and 12 edges **2.** Figure (a): 5 faces, 6 vertices, 9 edges; figure (b): 6 faces, 6 vertices, 10 edges; figure (c): 9 faces, 9 vertices, 16 edges **3.** $F + V - 2 = E$ **4.** 12; $20 + V - 2 = 30$

Leonhard Euler, 1707–1783

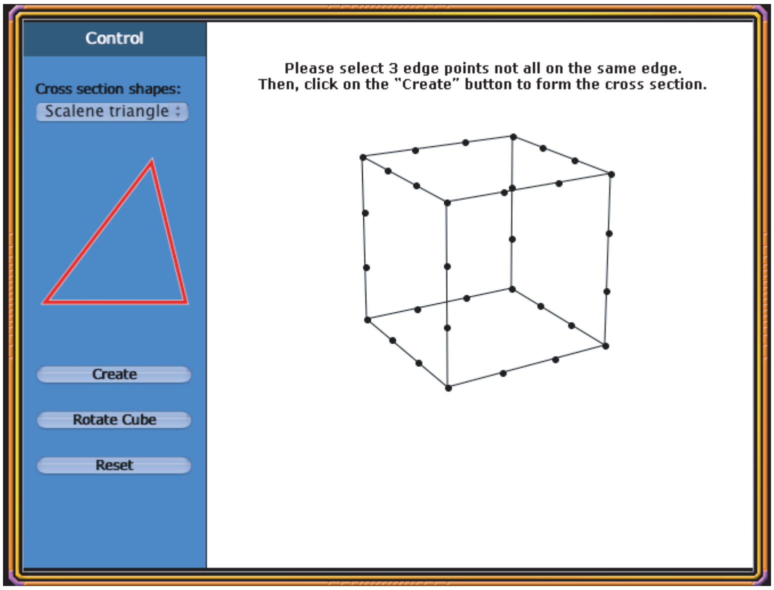
HISTORICAL HIGHLIGHT

Switzerland's Leonhard Euler is considered to be the most prolific writer in the history of mathematics. He published over 850 books and papers, and most branches of mathematics contain his theorems. After he became totally blind at the age of 60, he continued his amazing productivity for 17 years by dictating to a secretary and writing formulas in chalk on a large slate. On the 200th anniversary of his birthday in 1907, a Swiss publisher began reissuing Euler's entire collected works; the collection is expected to run to 75 volumes of about 60 pages each.*

*H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 46–49.

Technology Connection

How would you cut this cube into two parts with one straight slice so that the cross section is a triangle? A trapezoid? This applet lets you select points on the edges of the cube for your slices and then rotate the cube for a better perspective of the resulting cross section.



Cross-Sections of a Cube Applet, Chapter 9, Section 3
www.mhhe.com/bennett-nelson

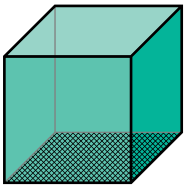
Exercises and Problems 9.3

Crystals of calcite

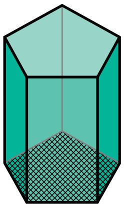
1. The crystals crowded together in the photograph are growing with flat polygonal faces.
   a. What type of polygon is the top face of these crystals?
   b. What type of polyhedron is formed by these crystals?

Which of the figures in exercises 2 and 3 are polyhedra?

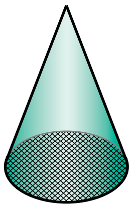
2. a.



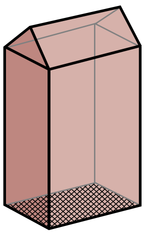
b.



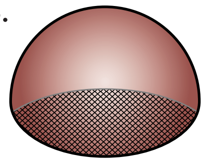
c.



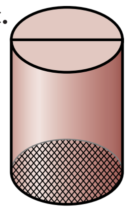
3. a.



b.

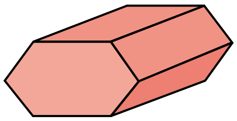


c.

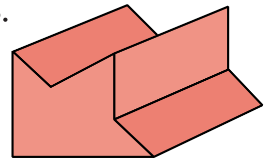


Classify the polyhedra in exercises 4 and 5 as convex or nonconvex.

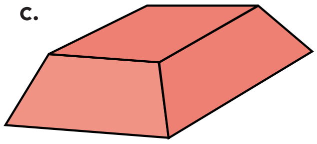
4. a.



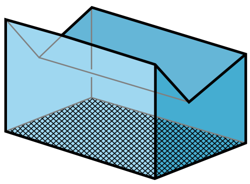
b.



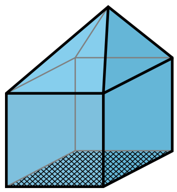
c.



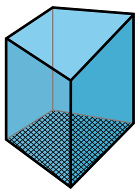
5. a.



b.

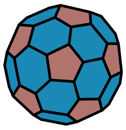


c.



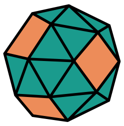
The semiregular polyhedra are classified according to the arrangement of regular polygons around each vertex. Proceeding counterclockwise, list the polygons about a vertex of each polyhedron in exercises 6 and 7.

6. a.



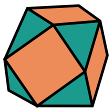
20 hexagons
12 pentagons

b.



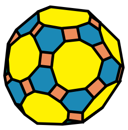
32 triangles
6 squares

7. a.



8 triangles
6 squares

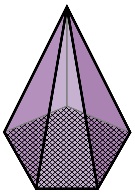
b.



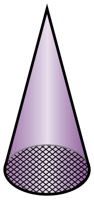
20 hexagons
30 squares
12 decagons

Name each of the figures in exercises 8 and 9.

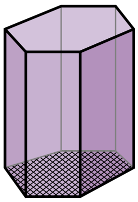
8. a.



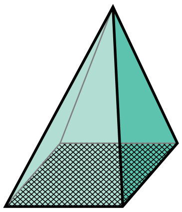
b.



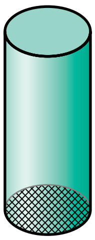
c.



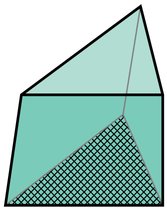
9. a.



b.

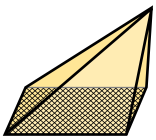


c.

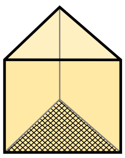


Name the figures in exercises 10 and 11, and state whether they are right or oblique.

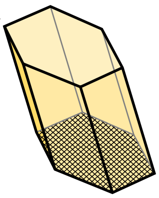
10. a.



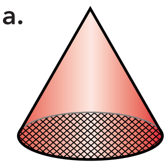
b.



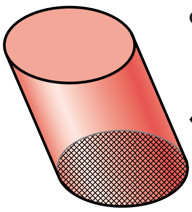
c.



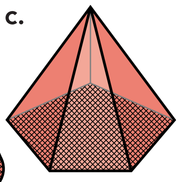
11. a.



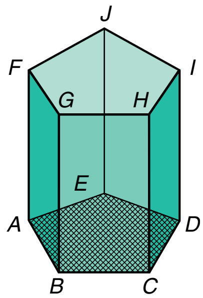
b.



c.

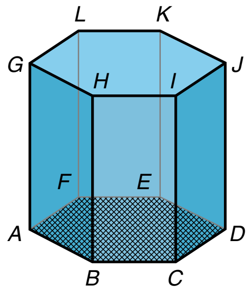


12. The polyhedron below is a right pentagonal prism whose bases are regular polygons.



a. What face is parallel to face *ABCDE*?
b. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?
c. What is the measure of the dihedral angle between face *FGHIJ* and face *EDIJ*?

13. The polyhedron shown here is a right prism, and its bases are regular hexagons.



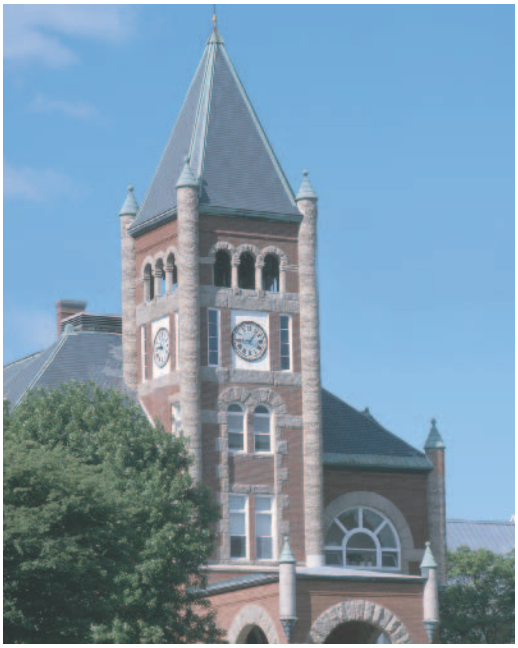
a. What face is parallel to face *GHIJKL*?
b. What face is parallel to face *IJDC*?
c. What is the measure of the dihedral angle between face *ABHG* and face *ABCDEF*?
d. What is the measure of the dihedral angle between face *ABHG* and face *BCIH*?

Which of the three types of projections is best suited for making flat maps of the regions in exercises 14 and 15?

14. a. Australia
b. North, Central, and South America
c. The entire equatorial region between 30° north latitude and 30° south latitude

15. a. Arctic region
b. Western hemisphere between 20° north and 20° south
c. United States

Each of the geometric shapes listed in exercises 16 and 17 can be seen in the photograph. Locate these objects.



Thompson Hall, University of New Hampshire

16. **a.** Cone **b.** Pyramid **c.** Cylinder
d. Sphere **e.** Circle

17. **a.** Obtuse angle **b.** Rectangle **c.** Semicircle
d. Square **e.** Isosceles triangle

18. Use the photo with exercise 19 and your knowledge of the spherical coordinate system to match each of the following cities with its longitude and latitude.

Tokyo	38°N and 120°W
San Francisco	56°N and 4°W
Melbourne	35°N and 140°E
Glasgow	35°S and 20°E
Capetown	38°S and 145°E

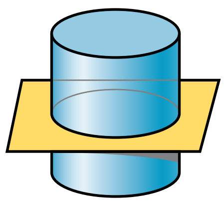
19. Two points on the Earth's surface that are on opposite ends of a line segment through the center of the earth are called **antipodal points.** The coordinates of such points are nicely related. The latitude of one point is as far above the equator as that of the other is below, and the longitudes are supplementary angles (in opposite hemispheres). For example, (30°N, 15°W) is off the west coast of Africa near the Canary Islands, and its antipodal point (30°S, 165°E) is off the eastern coast of Australia.

a. This globe shows that (20°N, 120°W) is a point in the Pacific Ocean just west of Mexico. Its antipodal point is just east of Madagascar. What are the coordinates of this antipodal point?

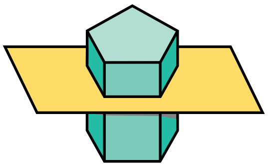
b. The point (30°S, 80°E) is in the Indian Ocean. What are the coordinates of its antipodal point? In what country is it located?

20. China is bounded by latitudes of 20°N and 55°N and by longitudes of 75°E and 135°E. It is playfully assumed that if you could dig a hole straight through the center of the Earth, you would come out in China. For which of the following starting points is this true?
a. Panama (9°N, 80°W)
b. Buenos Aires (35°S, 58°W)
c. New York (41°N, 74°W)

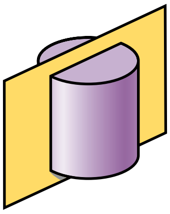
The intersection of a plane and a three-dimensional figure is called a **cross section.** The cross section produced by the intersection of a plane and a right cylinder, where the plane is parallel to the base of the cylinder (see figure), is a circle. Determine the cross sections of the figures in exercises 21 and 22.



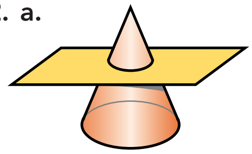
21. a.



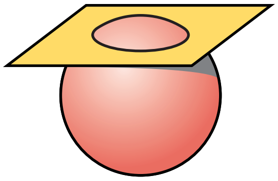
b.



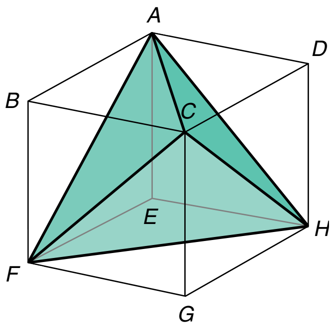
22. a.



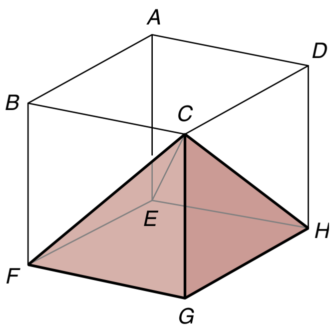
b.



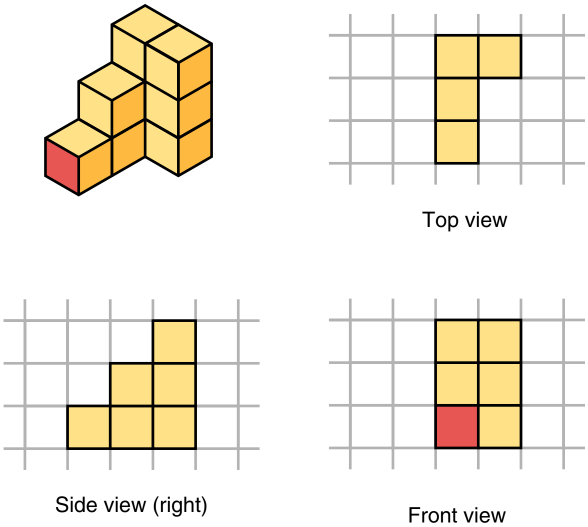
23. A cube can be divided into triangular pyramids in several ways. Pyramid *FHCA* divides this cube into five triangular pyramids. Name the four vertices of each of the other four pyramids.



24. *E*, *F*, *G*, *H*, and *C* are the vertices of a square pyramid inside this cube. Name the five vertices of two more square pyramids that, together with the given pyramid, divide the cube into three pyramids.

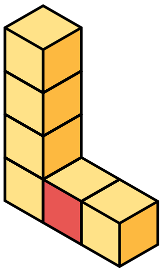


One method of describing a three-dimensional figure is to make a drawing of its different views. There are nine cubes in the following figure (two are hidden), and the top, right, and front views are shown.

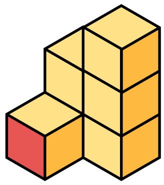


Sketch the top, front, and side views of each of the figures in exercises 25 and 26. (*Note:* Figure 25b has one hidden cube beneath a cube that can be seen, and the color faces of the cubes are part of the front views of the figures.)

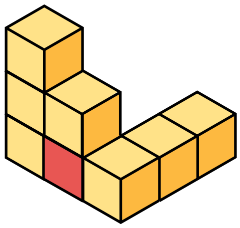
25. a.



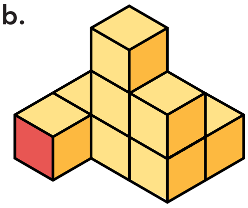
b.



26. a.



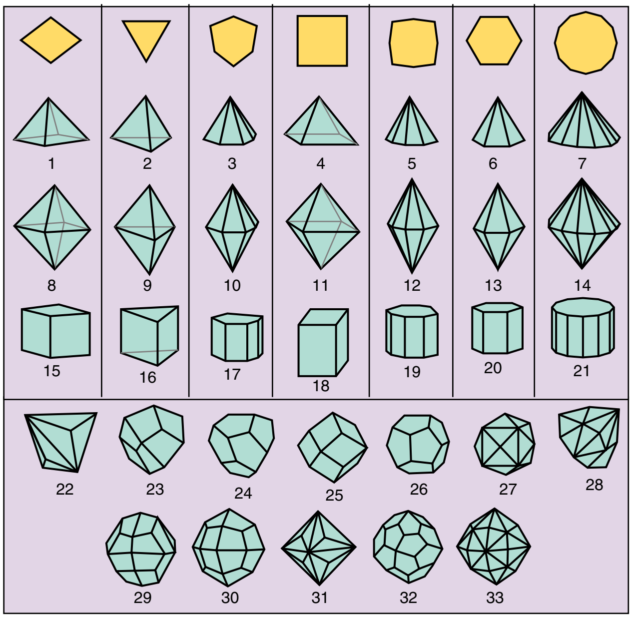
b.



The following table of polyhedra illustrates some of the forms that crystals may take in nature. The polygons at the tops of the columns are the horizontal cross sections of the polyhedra in the columns. Use this table in exercises 27 and 28.

27. a. List the numbers of the polyhedra that are pyramids.
b. Which of the polyhedra is most like a dodecahedron?

28. **a.** List the numbers of the polyhedra that are prisms.
b. Which of the polyhedra is most like an octahedron?

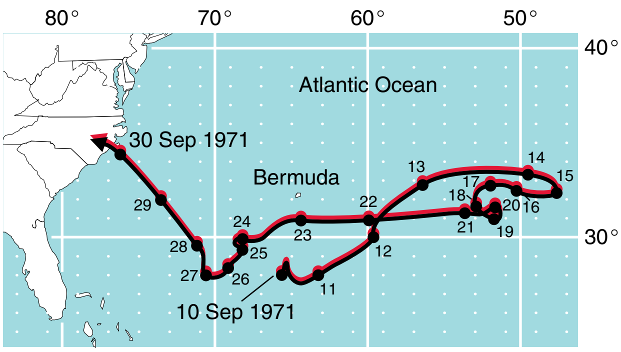


Use Euler's formula in exercises 29 and 30 to determine the missing numbers for each polyhedron. For each set of conditions, find a polyhedron from those numbered from 1 to 21 in the table above that has the given number of faces, vertices, and edges.

29. **a.** 7 faces, 7 vertices, __________ edges
b. 16 faces, __________ vertices, 24 edges
c. __________ faces, 5 vertices, 8 edges

30. **a.** 6 faces, __________ vertices, 9 edges
b. __________ faces, 8 vertices, 12 edges
c. 14 faces, 24 vertices, __________ edges

Hurricane Ginger was christened on September 10, 1971, and became the longest-lived Atlantic hurricane on record. This tropical storm formed approximately 275 miles south of Bermuda and reached the U.S. mainland 20 days later. Use this map in exercises 31 and 32.



Erratic path of Hurricane Ginger

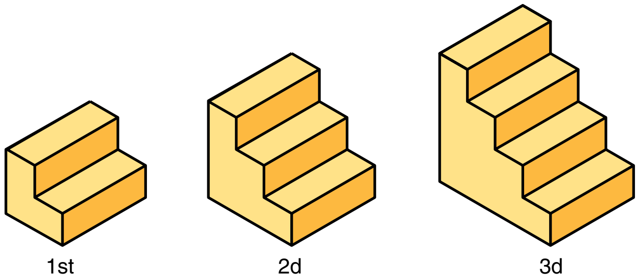
31. The storm's coordinates on September 10 were (28°N, 66°W). What were its coordinates on September 15, September 23, and September 30?

32. At this latitude on the Earth's surface, each degree of longitude spans a distance of approximately 60 miles. About how many miles did this hurricane travel between September 10 and September 30? (*Hint:* Use a piece of string.)

Reasoning and Problem Solving

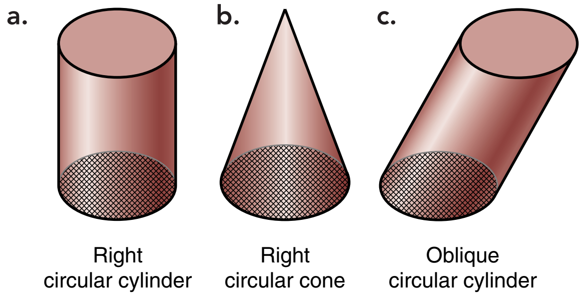
33. Erica is designing a science experiment that requires two different three-dimensional figures such that one fits inside the other and both figures have at least one cross section that is the same for both figures (see exercises 21 and 22). Find such a pair of figures.

34. Here are the first three figures in a staircase pattern. These staircases are polyhedra.



a. The number of faces for the polyhedron in the first figure is 8. How many faces are there for the polyhedron in the 35th figure?
b. The number of edges for the polyhedron in the first figure is 18. How many edges are there for the polyhedron in the 35th figure?
c. The number of vertices for the polyhedron in the first figure is 12. How many vertices are there for the polyhedron in the 35th figure?

35. Sketch and describe how to form a piece of paper into the following figures (without bases).

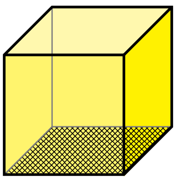


36. Featured Strategies: Making a Drawing and Using a Model The five regular polyhedra and the numbers and shapes of their faces are shown in the following table. Determine the missing numbers of vertices and edges.

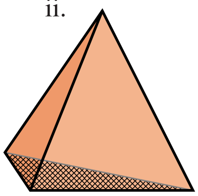
Polyhedron	Vertices	Faces	Edges
Tetrahedron		4 triangles	
Cube	8	6 squares	12
Octahedron		8 triangles	
Dodecahedron		12 pentagons	
Icosahedron		20 triangles	

a. Understanding the Problem The cube is the most familiar of the regular polyhedra. Its 6 faces meet in 12 edges, and its edges meet in 8 vertices (see figure i). How many vertices and edges does a tetrahedron have?

i.



ii.



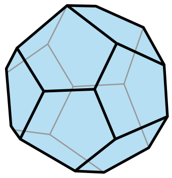
b. Devising a Plan One approach is to use a model or a sketch of the polyhedra and to count the numbers of vertices and edges. Or once we determine either the number of vertices or the number of edges, the missing number can be obtained by using Euler's formula $F + V - 2 = E$.

Another approach that avoids counting is to use the fact that each pair of faces meets in exactly one edge. For example, since a dodecahedron has 12 pentagons for faces and each pair of pentagons shares an edge, the number of edges is $(12 \times 5)/2 = 30$. Using Euler's formula, determine the number of vertices in a dodecahedron.

c. Carrying Out the Plan Continue to find the numbers of edges by multiplying the number of faces by the number of sides on the face and dividing by 2. For example, what is the number of edges in an icosahedron? Fill in the rest of the table above.

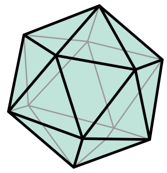
d. Looking Back The number of vertices for each regular polyhedron can also be found directly from the number of edges that meet at each vertex. For example, three edges meet at each vertex of the dodecahedron, as shown in the following Figure iii. Since there are 12 faces and each face has 5 vertex points, the dodecahedron has $(12 \times 5)/3 = 20$ vertex points. Use this approach to determine the number of vertices for the icosahedron in iv.

iii.



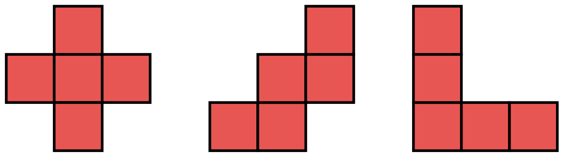
Dodecahedron

iv.



Icosahedron

37. Each of the following polygons contains five squares. There are only 12 such polygons that can be formed in the plane by joining five squares along their edges, and they are called **pentominoes.**



a. Which two of these pentominoes will fold into an open-top box, so that each face of the box is one of the squares?

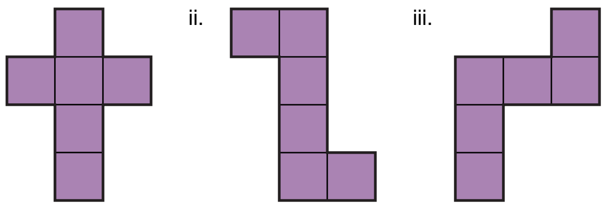
b. Eight of the 12 pentominoes will fold into an open-top box. Find another one of these.

38. The polygons were formed by joining six squares along their edges. There are 35 such polygons, and they are called **hexominoes.**

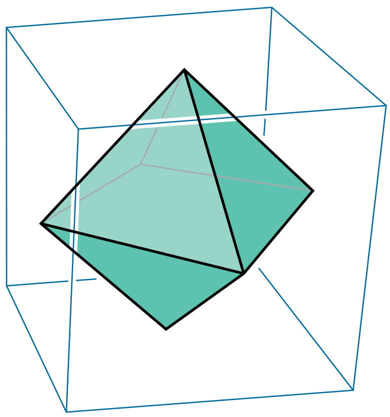
a. Which two of these hexominoes will fold into a cube so that each face of the cube is one of the squares?

b. Eleven of the 35 hexominoes will fold into a cube. Find another such hexomino.

i. ii. iii.



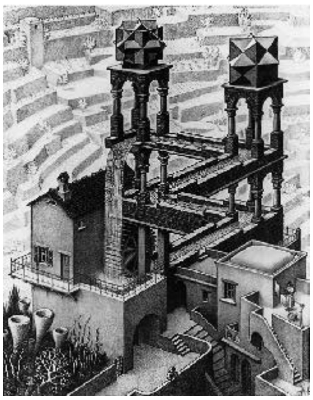
39. The centers of the faces of a cube can be connected to form a regular octahedron. Also, the centers of the faces of an octahedron can be connected to form a cube. Such pairs of polyhedra are called **duals.**



a. How is this dual relationship suggested by the table in exercise 36?
b. Find two other regular polyhedra that are duals of each other.
c. Which regular polyhedron is its own dual?

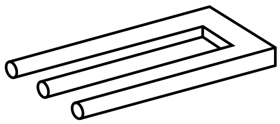
40. There are six categories of illusions.* One category, called *impossible objects,* is produced by drawing three-dimensional figures on two-dimensional surfaces. Find the impossible feature in each of these figures.

a.

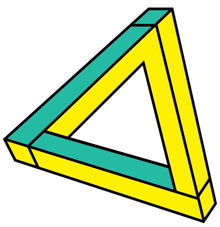


M. C. Escher's "Waterfall"

b.

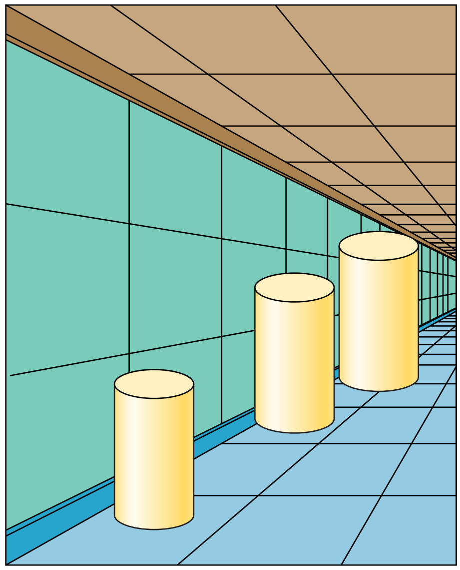


c.

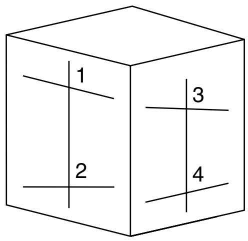


41. A second type of illusion involves depth perception. We have accustomed our eyes to see depth when three-dimensional objects are drawn on two-dimensional surfaces. Answer questions a and b by disregarding the depth illusions.

a. Is one of these cylinders larger than the others?



b. Which of the four numbered angles below is the largest? Which are right angles? (*Hint:* Use a corner of a piece of paper.)



Writing and Discussion

1. Natalie asked her teacher why, when you blow bubbles, they are round like a sphere and not cubes or other shapes. Research the question and then write a response you can give to this student.

2. One of your students wants to make cone-shaped birthday hats and asks you to show him how to do it. Explain, with diagrams, how you would go about this.

3. After using your classroom Earth globe to illustrate a sphere, a student asks how they get the map of the Earth on a flat piece of paper. Research the question and then write a response that would make sense to this student.

*P. A. Rainey, *Illusions* (Hamden, CT: The Shoe String Press, 1973), pp. 18–43.

4. Researchers have concluded that students need more experiences with concrete models. Suppose you are a new teacher going into a school that does not have three-dimensional manipulatives. Compile a list of objects you could acquire, or make, to bring into your classroom to successfully teach spatial concepts and illustrate the three-dimensional objects referred to in this section.

Making Connections

1. The **Standards** statement on page 600 suggests constructing three-dimensional shapes from two-dimensional representations. Hexominoes are polygons formed by joining six squares along their edges. (Three examples are shown in the exercise set for this section.) There are 35 hexominoes and 11 of them will fold into a cube. Identify and diagram the 11 hexominoes that fold into a cube and describe the strategy you used to find them.

2. Repeat the question posed in the **Problem Opener** for this section for a $3 \times 3 \times 3$ cube made up of 27 of the smaller cubes. Repeat for a $4 \times 4 \times 4$ cube and then for an $n \times n \times n$ cube.

3. The **Elementary School Text** example on page 610 shows several common three-dimensional shapes that students are to learn. Make a list of some three-dimensional forms found in your community (buildings, sculptures, etc.) that represent each of the three-dimensional shapes on that page.

4. One of the recommendations in the **Grades 3–5 Standards—***Geometry* (see inside front cover) states: "Identify and build a three-dimensional object from a two-dimensional representation of that object." Explain what this means and give examples of how you would accomplish this.