CHAPTER

4

Exponential and Logarithmic Functions

MOST of the functions we have worked with so far have been polynomial or rational functions, with a few others involving roots. Functions that can be expressed in terms of addition, subtraction, multiplication, division, and roots of variables and constants are called *algebraic functions.* In Chapter 4 we will learn about *exponential and logarithmic functions.* These functions are not algebraic; they belong to the class of *transcendental functions.* Exponential and logarithmic functions are used to model a surprisingly wide variety of real-world phenomena: growth of populations of people, animals, and bacteria; radioactive decay; epidemics; and magnitudes of sounds and earthquakes. These and many other applications will be studied in this chapter.

OUTLINE

4-1 Exponential Functions

Many of the functions we've studied so far have included exponents. But in every case, the exponent was a constant, and the base was often a variable. In this section, we will reverse those roles. In an *exponential function,* the variable appears in an exponent. As we'll see, this has a significant effect on the properties and graphs of these functions. A review of the basic properties of exponents in Appendix A, Section A-2, would be very helpful before moving on.

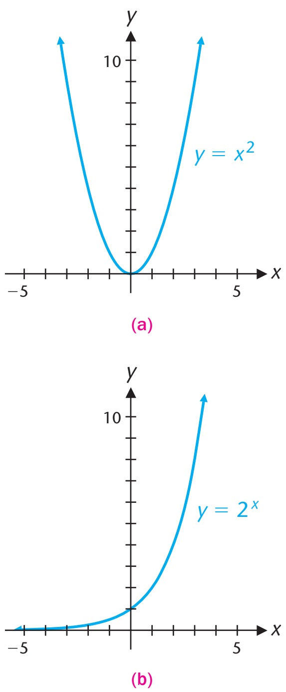


Figure 1

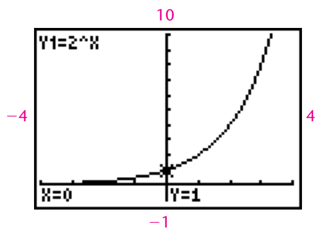


Figure 2

Defining Exponential Functions

Let's start by noting that the functions f and g given by

$$f(x) = 2^x \qquad \text{and} \qquad g(x) = x^2$$

are not the same function. Whether a variable appears as an exponent with a constant base or as a base with a constant exponent makes a big difference. The function g is a quadratic function, which we have already discussed. The function f is an *exponential function*.

The graphs of f and g are shown in Figure 1. As expected, they are very different.

We know how to define the values of 2^x for many types of inputs. For positive integers, it's simply repeated multiplication:

$$2^2 = 2 \cdot 2 = 4; \quad 2^3 = 2 \cdot 2 \cdot 2 = 8; \quad 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

For negative integers, we use properties of negative exponents:

$$2^{-1} = \frac{1}{2}; \quad 2^{-2} = \frac{1}{2^2} = \frac{1}{4}; \quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

For rational numbers, a calculator comes in handy:

$$2^{\frac{1}{2}} = \sqrt{2} \approx 1.4; \quad 2^{\frac{3}{2}} = \sqrt{2^3} \approx 2.8; \quad 2^{\frac{9}{4}} = \sqrt[4]{2^9} \approx 4.8$$

A graphing calculator can be used to obtain the graph in Figure 1 (see Fig. 2).

The only catch is that we don't know how to define 2^x for *all* real numbers. For example, what does

$$2^{\sqrt{2}}$$

mean? Your calculator can give you a decimal approximation, but where does it come from? That question is not easy to answer at this point. In fact, a precise definition of $2^{\sqrt{2}}$ must wait for more advanced courses. For now, we will simply state that for any positive real number b, the expression b^x is defined for all real values of x, and the output is a real number as well. This enables us to draw the continuous graph for $f(x) = 2^x$ in Figure 1. In Problems 85 and 86 in Exercises 4-1, we will explore a method for defining b^x for irrational x values like $\sqrt{2}$.

DEFINITION 1 Exponential Function

The equation

$$f(x) = b^x \qquad b > 0, b \neq 1$$

defines an **exponential function** for each different constant b, called the **base.** The independent variable x may assume any real value.

The domain of f is the set of all real numbers, and it can be shown that the range of f is the set of all positive real numbers. We require the base b to be positive to avoid imaginary numbers such as $(-2)^{1/2}$. Problems 57 and 58 in Exercises 4-1 explore why $b = 0$ and $b = 1$ are excluded.

Analyzing Graphs of Exponential Functions

EXPLORE-DISCUSS 1

Compare the graphs of $f(x) = 3^x$ and $g(x) = 2^x$ by graphing both functions in the same viewing window. Find all points of intersection of the graphs. For which values of x is the graph of f above the graph of g? Below the graph of g? Are the graphs of f and g close together as $x \to \infty$? As $x \to -\infty$? Discuss.

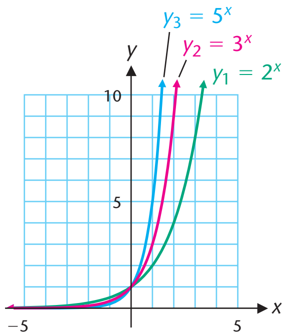


Figure 3 $y = b^x$ for $b = 2, 3, 5$.

The graphs of $y = b^x$ for $b = 2, 3$, and 5 are shown in Figure 3. Note that all three have the same basic shape, and pass through the point (0, 1). Also, the x axis is a horizontal asymptote for each graph, but only as $x \to -\infty$. The main difference between the graphs is their steepness.

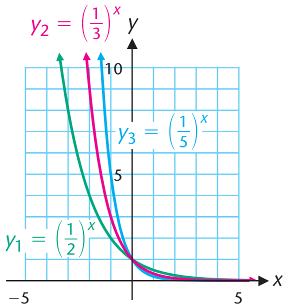


Figure 4 $y = b^x$ for $b = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.

Next, let's look at the graphs of $y = b^x$ for $b = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{5}$ (Fig. 4). Again, all three have the same basic shape, pass through (0, 1), and have a horizontal asymptote $y = 0$, but we can see that for $b < 1$, the asymptote is only as $x \to \infty$. In general, for bases less than 1, the graph is a reflection through the y axis of the graphs for bases greater than 1.

The graphs in Figures 3 and 4 suggest that the graphs of exponential functions have the properties listed in Theorem 1, which we state without proof.

THEOREM 1 Properties of Graphs of Exponential Functions

Let $f(x) = b^x$ be an exponential function, $b > 0$, $b \neq 1$. Then the graph of $f(x)$:

1. Is continuous for all real numbers
2. Has no sharp corners
3. Passes through the point (0, 1)
4. Lies above the x axis, which is a horizontal asymptote either as $x \to \infty$ or $x \to -\infty$, but not both
5. Increases as x increases if $b > 1$; decreases as x increases if $0 < b < 1$
6. Intersects any horizontal line at most once (that is, f is one-to-one)

These properties indicate that the graphs of exponential functions are distinct from the graphs we have already studied. (Actually, Property 4 is enough to ensure that graphs of exponential functions are different from graphs of polynomials and rational functions.) Property 6 is important because it guarantees that exponential functions have inverses. Those inverses, called *logarithmic functions,* are the subject of Section 4-3.

To begin a study of graphing exponentials, it's helpful to sketch a graph or two by hand after plotting points.

EXAMPLE 1 Drawing the Graph of an Exponential Function

Sketch the graph of each function after plotting at least seven points. Then confirm your result with a graphing calculator.

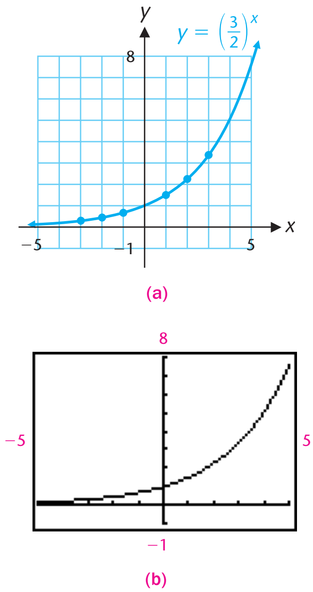
(A) $f(x) = \left(\dfrac{3}{2}\right)^x$ (B) $g(x) = \left(\dfrac{2}{3}\right)^x$

SOLUTIONS

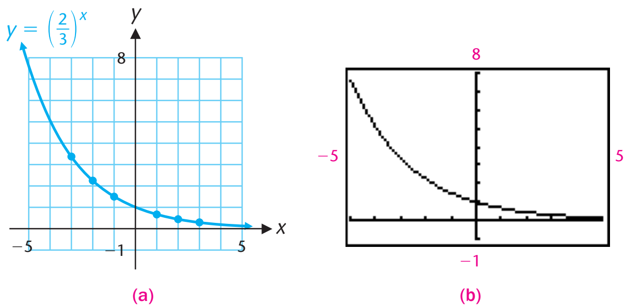
Make a table of values for f and g.

x	$\left(\frac{3}{2}\right)^x$	$\left(\frac{2}{3}\right)^x$
-3	$\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^{3} = \frac{8}{27}$	$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$
-2	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^{2} = \frac{4}{9}$	$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4}$
-1	$\left(\frac{3}{2}\right)^{-1} = \left(\frac{2}{3}\right)^{1} = \frac{2}{3}$	$\left(\frac{2}{3}\right)^{-1} = \left(\frac{3}{2}\right)^{1} = \frac{3}{2}$
0	$\left(\frac{3}{2}\right)^{0} = 1$	$\left(\frac{2}{3}\right)^{0} = 1$
1	$\left(\frac{3}{2}\right)^{1} = \frac{3}{2}$	$\left(\frac{2}{3}\right)^{1} = \frac{2}{3}$
2	$\left(\frac{3}{2}\right)^{2} = \frac{9}{4}$	$\left(\frac{2}{3}\right)^{2} = \frac{4}{9}$
3	$\left(\frac{3}{2}\right)^{3} = \frac{27}{8}$	$\left(\frac{2}{3}\right)^{3} = \frac{8}{27}$

The hand sketches are shown in Figures 5 and 6, along with the corresponding graphing calculator graph.



› **Figure 5**



› **Figure 6**

Notice that the outputs for $x < 0$ in Figure 5 and for $x > 0$ in Figure 6 get so small that it's hard to distinguish the graph from the x axis. Property 4 in Theorem 1 indicates that the graph of an exponential function is always above the x axis, and approaches height zero as $x \to \infty$ or $x \to -\infty$. (Zooming in on the graph, as in Figure 7, illustrates the behavior a bit better.)

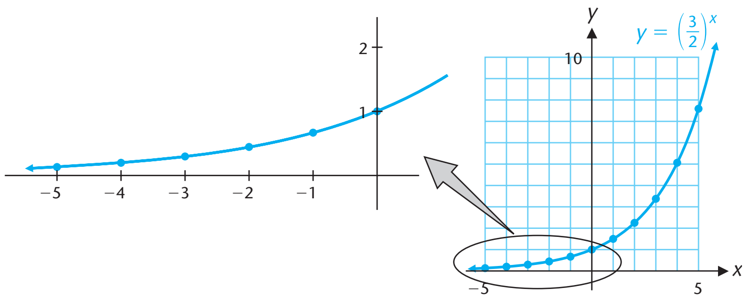


Figure 7

MATCHED PROBLEM 1

Sketch the graph of each function after plotting at least seven points. Then confirm your result with a graphing calculator.

(A) $f(x) = \left(\dfrac{3}{4}\right)^x$ (B) $g(x) = \left(\dfrac{4}{3}\right)^x$

EXPLORE-DISCUSS 2

Examine the graphs of $f(x) = (\frac{3}{2})^x$ and $g(x) = (\frac{2}{3})^x$ from Example 1.

(A) What is the relationship between the graphs?

(B) Rewrite $(\frac{2}{3})^x$ as an exponential with base $\frac{3}{2}$. How does this confirm your answer to part A?

We can also use our knowledge of transformations to draw graphs of more complicated functions involving exponentials.

EXAMPLE 2 Drawing the Graph of an Exponential Function

Use transformations of $y = 2^x$ to graph $f(x) = 2^{x-2} + 4$.

SOLUTION

We start with a graph of $y = 2^x$ [Fig. 8(a)], then shift 2 units right and 4 units up [Fig. 8(b)]. A graphing calculator confirms our result [Fig. 8(c)]. Note that in this case, $y = 4$ is a horizontal asymptote.

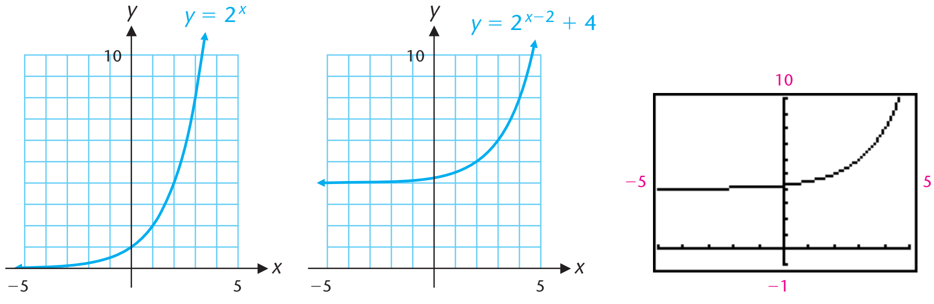


Figure 8

MATCHED PROBLEM 2

Use transformations of $y = (\frac{1}{2})^x$ to graph $f(x) = (\frac{1}{2})^{x+1} - 3$.

Additional Properties of Exponential Functions

The properties of exponents you should be familiar with (see Appendix A, Section A-2) are often stated in terms of exponents that are rational numbers. But we're considering irrational exponents as well in defining exponential functions. Fortunately, these properties still apply. We will summarize the key properties we need in this chapter, and add two other useful properties.

EXPONENTIAL FUNCTION PROPERTIES

For a and b positive, $a \neq 1$, $b \neq 1$, and for any real numbers x and y:

1. Exponent laws:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \frac{2^{5x}}{2^{7x}} \boxed{= 2^{5x-7x}}^{*} = 2^{-2x}$$

2. $a^x = a^y$ if and only if $x = y$. If $6^x = 6^3$, then $x = 3$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$. If $a^4 = 3^4$, then $a = 3$.

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

Property 2 is another way to express the fact that the exponential function $f(x) = a^x$ is one-to-one (see Property 6 of Theorem 1). Because all exponential functions pass through the point (0, 1) (see Property 3 of Theorem 1), property 3 indicates that the graphs of exponential functions with different bases do not intersect at any other points.

EXAMPLE 3 Using Exponential Function Properties

Find all solutions to $2^{10x-1} = 2^{5-2x}$.

SOLUTIONS

Algebraic Solution

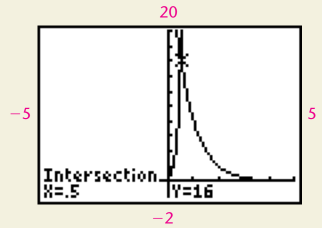
According to Property 2, $2^{10x-1} = 2^{5-2x}$ implies that

$$10x - 1 = 5 - 2x$$
$$12x = 6$$
$$x = \tfrac{6}{12} = \tfrac{1}{2}$$

Check: $2^{10(1/2)-1} = 2^4; \qquad 2^{5-2(1/2)} = 2^4$

Graphical Solution

Graph $y_1 = 2^{10x-1}$ and $y_2 = 2^{5-2x}$, then use the INTERSECT command to obtain $x = 0.5$ (Fig. 9).



› Figure 9

MATCHED PROBLEM 3

Find all solutions to $3^{3+y} = 3^{4y-9}$.

EXAMPLE 4 Using Exponential Function Properties

Find all solutions to $4^{x-3} = 8$.

SOLUTIONS

Algebraic Solution

Notice that the two bases, 4 and 8, can both be written as a power of 2. This will enable us to use Property 2 to equate exponents.

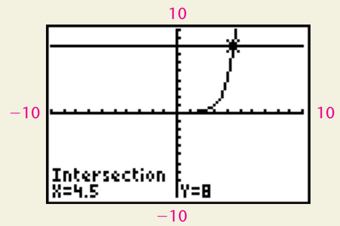
$$4^{x-3} = 8 \quad \text{Express 4 and 8 as powers of 2.}$$
$$(2^2)^{x-3} = 2^3 \quad (a^x)^y = a^{xy}$$
$$2^{x-6} = 2^3 \quad \text{Property 2}$$
$$2x - 6 = 3 \quad \text{Add 6 to both sides.}$$
$$2x = 9 \quad \text{Divide both sides by 2.}$$
$$x = \tfrac{9}{2}$$

CHECK

$$4^{(9/2)-3} = 4^{3/2} = (\sqrt{4})^3 = 2^3 \overset{\checkmark}{=} 8$$

Graphical Solution

Graph $y_1 = 4^{x-3}$ and $y_2 = 8$. Use the INTERSECT command to obtain $x = 4.5$ (Fig. 10).



› Figure 10

MATCHED PROBLEM 4

Solve $27^{x+1} = 9$ for x.

The Exponential Function with Base *e*

Surprisingly, among the exponential functions it is not the function $g(x) = 2^x$ with base 2 or the function $h(x) = 10^x$ with base 10 that is used most frequently in mathematics. Instead, the most commonly used base is a number that you may not be familiar with.

EXPLORE-DISCUSS 3

(A) Calculate the values of $[1 + (1/x)]^x$ for $x = 1, 2, 3, 4,$ and 5. Are the values increasing or decreasing as x gets larger?

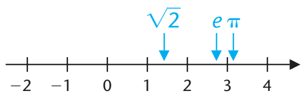
(B) Graph $y = [1 + (1/x)]^x$ and discuss the behavior of the graph as x increases without bound.

Table 1

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.593 74 ...
100	2.704 81 ...
1,000	2.716 92 ...
10,000	2.718 14 ...
100,000	2.718 27 ...
1,000,000	2.718 28 ...

By calculating the value of $[1 + (1/x)]^x$ for larger and larger values of x (Table 1), it looks like $[1 + (1/x)]^x$ approaches a number close to 2.7183. In a calculus course, we can show that as x increases without bound, the value of $[1 + (1/x)]^x$ approaches an irrational number that we call e. Just as irrational numbers such as π and $\sqrt{2}$ have unending, nonrepeating decimal representations, e also has an unending, nonrepeating decimal representation. To 12 decimal places,

$$e = 2.718\,281\,828\,459$$



Don't let the symbol "e" intimidate you! It's just a number.

Exactly who discovered e is still being debated. It is named after the great Swiss mathematician Leonhard Euler (1707–1783), who computed e to 23 decimal places using $[1 + (1/x)]^x$.

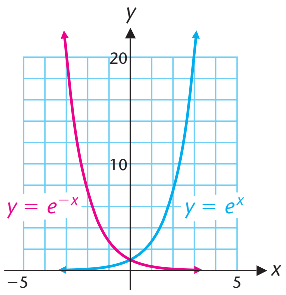
The constant e turns out to be an ideal base for an exponential function because in calculus and higher mathematics many operations take on their simplest form using this base. This is why you will see e used extensively in expressions and formulas that model real-world phenomena.

DEFINITION 2 Exponential Function with Base *e*

For x a real number, the equation

$$f(x) = e^x$$

defines the **exponential function with base *e*.**



› **Figure 11** Exponential functions.

The exponential function with base e is used so frequently that it is often referred to as *the* exponential function. The graphs of $y = e^x$ and $y = e^{-x}$ are shown in Figure 11.

››› EXPLORE-DISCUSS 4

(A) Graph $y_1 = e^x$, $y_2 = e^{0.5x}$, and $y_3 = e^{2x}$ in the same viewing window. How do these graphs compare with the graph of $y = b^x$ for $b > 1$?

(B) Graph $y_1 = e^{-x}$, $y_2 = e^{-0.5x}$, and $y_3 = e^{-2x}$ in the same viewing window. How do these graphs compare with the graph of $y = b^x$ for $0 < b < 1$?

(C) Use the properties of exponential functions to show that all of these functions can be written in the form $y = b^x$.

EXAMPLE 5 Analyzing an Exponential Graph

Describe the graph of $f(x) = 4 - e^{x/2}$, including x and y intercepts, increasing and decreasing properties, and horizontal asymptotes. Round any approximate values to two decimal places.

SOLUTION

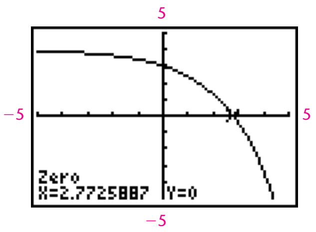
A graphing calculator graph of f is shown in Figure 12(a).

y intercept: $f(0) = 4 - e^0 = 4 - 1 = 3$

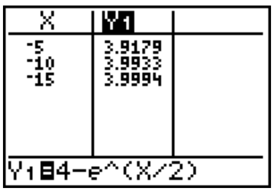
x intercept: $x \approx 2.77$

Graph is decreasing for all x.

Horizontal asymptote: We can write the exponential function $e^{x/2}$ as $(e^{1/2})^x$, and $e^{1/2} \approx 1.65 > 1$, so our earlier study of exponential graphs indicates that $e^{x/2} \to 0$ as $x \to -\infty$. But then, $4 - e^{x/2} \to 4$ as $x \to -\infty$, and $y = 4$ is a horizontal asymptote for the graph of f. The table in Figure 12(b) supports this conclusion.



(a)



(b)

› **Figure 12** $f(x) = 4 - e^{x/2}$.

MATCHED PROBLEM 5

Describe the graph of $f(x) = 2e^{x/2} - 5$, including x and y intercepts, increasing and decreasing properties, and horizontal asymptotes. Round any approximate values to two decimal places.

We will study a wide variety of applications of exponential functions in the next section. For now, it's a good start to examine how exponential functions apply very naturally to the world of finance, something relevant to almost everyone.

› Calculating Compound Interest

The fee paid to use someone else's money is called **interest.** It is usually computed as a percentage, called the **interest rate,** of the original amount (or **principal**) over a given period of time. At the end of the payment period, the interest paid is usually added to the principal amount, so the interest in the next period is earned on both the original amount, as well as the interest previously earned. Interest paid on interest previously earned and reinvested in this manner is called **compound interest.**

Suppose you deposit \$1,000 in a bank that pays 8% interest compounded semiannually. How much will be in your account at the end of 2 years? "Compounded semiannually" means that the interest is paid to your account at the end of each 6-month period, and the interest will in turn earn more interest. To calculate the **interest rate per period,** we take the annual rate r, 8% (or 0.08), and divide by the number m of compounding periods per year, in this case 2. If A_1 represents the amount of money in the account after one compounding period (six months), then

Principal + 4% of principal

$$A_1 = \$1{,}000 + \$1{,}000\left(\frac{0.08}{2}\right) \qquad \text{Factor out \$1,000}$$
$$= \$1{,}000(1 + 0.04)$$

We will next use A_2, A_3, and A_4 to represent the amounts at the end of the second, third, and fourth periods. (Note that the amount we're looking for is A_4.) A_2 is calculated by multiplying the amount at the beginning of the second compounding period (A_1) by 1.04.

$$A_2 = A_1(1 + 0.04) \qquad \text{Substitute in our expression for } A_1.$$
$$= [\$1{,}000(1 + 0.04)](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^2 \qquad P\left(1 + \frac{r}{m}\right)^2$$
$$A_3 = A_2(1 + 0.04) \qquad \text{Substitute in our expression for } A_2.$$
$$= [\$1{,}000(1 + 0.04)^2](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^3 \qquad P\left(1 + \frac{r}{m}\right)^3$$
$$A_4 = A_3(1 + 0.04) \qquad \text{Substitute in our expression for } A_3.$$
$$= [\$1{,}000(1 + 0.04)^3](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^4 \qquad P\left(1 + \frac{r}{m}\right)^4$$
$$= \$1{,}169.86$$

What do you think the savings and loan will owe you at the end of 6 years (12 compounding periods)? If you guessed

$$A = \$1{,}000(1 + 0.04)^{12}$$

you have observed a pattern that is generalized in the following compound interest formula:

COMPOUND INTEREST

If a **principal** P is invested at an annual **rate** r compounded m times a year, then the **amount** A in the account at the end of n compounding periods is given by

$$A = P\left(1 + \frac{r}{m}\right)^n$$

Note that the annual rate r must be expressed in decimal form, and that $n = mt$, where t is years.

EXAMPLE 6 Compound Interest

If you deposit \$5,000 in an account paying 9% compounded daily, how much will you have in the account in 5 years? Compute the answer to the nearest cent.

SOLUTIONS

Interest compounded daily will be compounded 365 times per year.*

Algebraic Solution

We use the compound interest formula with $P = 5{,}000$, $r = 0.09$, $m = 365$, and $n = 5(365) = 1{,}825$:

$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^n \\ &= 5{,}000\left(1 + \frac{0.09}{365}\right)^{1825} \\ &= \$7{,}841.13 \end{aligned}$$

Graphical Solution

Graphing

$$A = 5{,}000\left(1 + \frac{0.09}{365}\right)^x$$

and using the VALUE command (Fig. 13) shows that $A = \$7{,}841.13$.

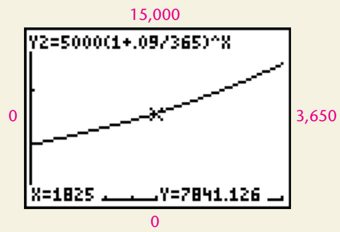


Figure 13

MATCHED PROBLEM 6

If \$1,000 is invested in an account paying 10% compounded monthly, how much will be in the account at the end of 10 years? Compute the answer to the nearest cent.

*In all problems involving interest compounded daily, we assume a 365-day year.

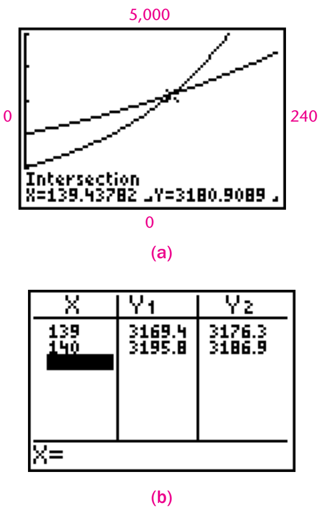
»» **CAUTION** »»

When using the compound interest formula, don't forget to write the interest rate in decimal form.

EXAMPLE 7

Comparing Investments

If \$1,000 is deposited into an account earning 10% compounded monthly and, at the same time, \$2,000 is deposited into an account earning 4% compounded monthly, will the first account ever be worth more than the second? If so, when?



› Figure 14

SOLUTION

Let y_1 and y_2 represent the amounts in the first and second accounts, respectively, then

$$y_1 = 1{,}000(1 + 0.10/12)^x$$
$$y_2 = 2{,}000(1 + 0.04/12)^x$$

where x is the number of compounding periods (months). Using the INTERSECT command to analyze the graphs of y_1 and y_2 [Fig. 14(a)], we see that the graphs intersect at $x \approx 139.438$ months. Because compound interest is paid at the end of each compounding period, we should compare the amount in the accounts after 139 months and after 140 months [Fig. 14(b)]. The table shows that the first account is worth more than the second for $x \geq 140$ months, or 11 years and 8 months.

MATCHED PROBLEM 7

If \$4,000 is deposited into an account earning 10% compounded quarterly and, at the same time, \$5,000 is deposited into an account earning 6% compounded quarterly, when will the first account be worth more than the second?

› Calculating Interest Compounded Continuously

If \$1,000 is deposited in an account that earns compound interest at an annual rate of 8% for 2 years, how will the amount A change if the number of compounding periods is increased? If m is the number of compounding periods per year, then

$$A = 1{,}000\left(1 + \frac{0.08}{m}\right)^{2m}$$

The amount A is computed for several values of m in Table 2. Notice that the largest gain appears in going from annually to semiannually. Then, the gains slow down as m increases. In fact, it appears that A might be approaching something close to \$1,173.50 as m gets larger and larger.

Table 2 Effect of Compounding Frequency

Compounding frequency	m	$A = 100\left(1 + \frac{0.08}{m}\right)^{2m}$
Annually	1	\$1,166.400
Semiannually	2	1,169.859
Quarterly	4	1,171.659
Weekly	52	1,173.367
Daily	365	1,173.490
Hourly	8,760	1,173.501

We now return to the general problem to see if we can determine what happens to $A = P[1 + (r/m)]^{mt}$ as m increases without bound. A little algebraic manipulation of the compound interest formula will lead to an answer and a significant result in the mathematics of finance:

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \qquad \text{Replace } \frac{r}{m} \text{ with } \frac{1}{m/r}, \text{ and } mt \text{ with } \frac{m}{r}\cdot rt.$$

$$= P\left(1 + \frac{1}{m/r}\right)^{(m/r)rt} \qquad \text{Replace } \frac{m}{r} \text{ with variable } x.$$

$$= P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

Does the expression within the square brackets look familiar? Recall from the first part of this section that

$$\left(1 + \frac{1}{x}\right)^{x} \to e \qquad \text{as} \qquad x \to \infty$$

Because the interest rate r is fixed, $x = m/r \to \infty$ as $m \to \infty$. So $(1 + \frac{1}{x})^x \to e$, and

$$P\left(1 + \frac{r}{m}\right)^{mt} = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt} \to Pe^{rt} \qquad \text{as} \qquad m \to \infty$$

This is known as the **continuous compound interest formula,** a very important and widely used formula in business, banking, and economics.

› CONTINUOUS COMPOUND INTEREST FORMULA

If a principal P is invested at an annual rate r compounded continuously, then the amount A in the account at the end of t years is given by

$$A = Pe^{rt}$$

The annual rate r must be expressed as a decimal.

EXAMPLE 8 Interest Compounded Continuously

If \$1,000 is invested at an annual rate of 8% compounded continuously, what amount, to the nearest cent, will be in the account after 2 years?

SOLUTIONS

Algebraic Solution
Use the continuous compound interest formula to find A when $P = \$1{,}000$, $r = 0.08$, and $t = 2$:

$$\begin{aligned} A &= Pe^{rt} \\ &= \$1{,}000e^{(0.08)(2)} \\ &= \$1{,}173.51 \end{aligned}$$

8% is equivalent to $r = 0.08$.

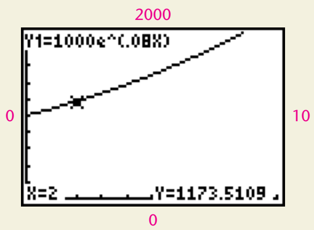
Compare this result with the values calculated in Table 2.

Graphical Solution
Graphing

$$A = 1{,}000e^{0.08x}$$

and using the VALUE command (Fig. 15) shows

$$A = \$1{,}173.51.$$



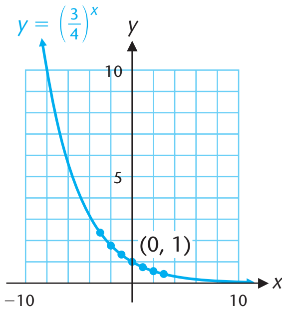
› Figure 15

MATCHED PROBLEM 8

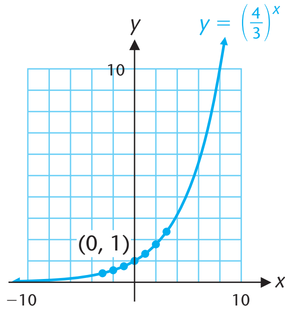
What amount will an account have after 5 years if \$1,000 is invested at an annual rate of 12% compounded annually? Quarterly? Continuously? Compute answers to the nearest cent.

ANSWERS TO MATCHED PROBLEMS

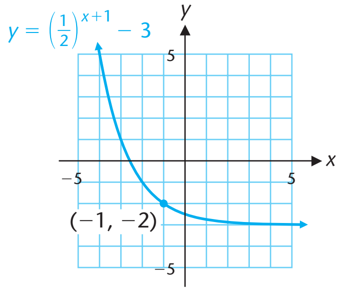
1. (A)



(B)



2.



3. $y = 4$ **4.** $x = -\frac{1}{3}$ **5.** y intercept: -3; x intercept: 1.83; increasing for all x; horizontal asymptote: $y = -5$ **6.** $2,707.04 **7.** After 23 quarters **8.** Annually: $1,762.34; quarterly: $1,806.11; continuously: $1,822.12

4-1 Exercises

**Additional answers can be found in the Instructor Answer Appendix.*

A

1. What is an exponential function?

2. What is the significance of the symbol e in the study of exponential functions?

3. For a function $f(x) = b^x$, explain how you can tell if the graph increases or decreases without looking at the graph.

4. Explain why $f(x) = (1/4)^x$ and $g(x) = 4^{-x}$ are really the same function. Can you use this fact to add to your answer for Question 3?

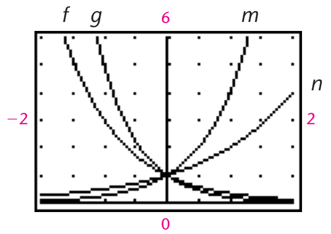
5. How do we know that the equation $e^x = 0$ has no solution?

6. Define the following terms related to compound interest: principal, interest rate, compounding period.

7. Match each equation with the graph of f, g, m, or n in the figure.

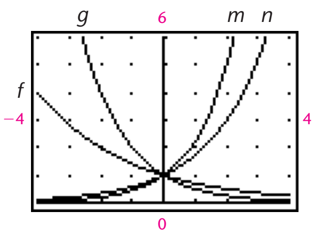
(A) $y = (0.2)^x$ *g* (B) $y = 2^x$ *n*

(C) $y = (\frac{1}{3})^x$ *f* (D) $y = 4^x$ *m*



8. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = e^{-1.2x}$ g (B) $y = e^{0.7x}$ n
(C) $y = e^{-0.4x}$ f (D) $y = e^{1.3x}$ m



In Problems 9–16, compute answers to four significant digits.

9. $5^{\sqrt{3}}$ 16.24 **10.** $3^{-\sqrt{2}}$ 0.2115

11. $e^2 + e^{-2}$ 7.524 **12.** $e - e^{-1}$ 2.350

13. $\sqrt{e}$ 1.649 **14.** $e^{\sqrt{2}}$ 4.113

15. $\dfrac{2^{\pi} + 2^{-\pi}}{2}$ 4.469 **16.** $\dfrac{3^{\pi} - 3^{-\pi}}{2}$ 15.76

In Problems 17–20, sketch the graph of each function after plotting at least six points. Then confirm your result with a graphing calculator.

17. $y = 3^x$ **18.** $y = 5^x$

19. $y = (\frac{1}{3})^x = 3^{-x}$ **20.** $y = (\frac{1}{5})^x = 5^{-x}$

In Problems 21–28, use properties of exponents to simplify.

21. $10^{3x-1}10^{4-x}$ 10^{2x+3} **22.** $(4^{3x})^{2y}$ 4^{6xy} **23.** $\dfrac{3^x}{3^{1-x}}$ 3^{2x-1}

24. $\dfrac{5^{x-3}}{5^{x-4}}$ 5 **25.** $\left(\dfrac{4^x}{5^y}\right)^{3z}$ $\dfrac{4^{3xz}}{5^{3yz}}$ **26.** $(2^x3^y)^z$ $2^{xz}3^{yz}$

27. $\dfrac{e^{5x}}{e^{2x+1}}$ e^{3x-1} **28.** $\dfrac{e^{4-3x}}{e^{2-5x}}$ e^{2x+2}

29. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/x)]^x$: As x gets large, $1 + (1/x)$ approaches 1 because $1/x$ approaches 0, and 1 raised to any power is 1, so $[1 + 1/x]^x$ approaches 1.
(B) Which number does $[1 + (1/x)]^x$ approach as x approaches ∞? e

30. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/x)]^x$: If $b > 1$, then the exponential function b^x approaches ∞ as x approaches ∞, and $1 + (1/x)$ is greater than 1, so $[1 + (1/x)]^x$ approaches infinity as $x \to \infty$.
(B) Which number does $[1 + (1/x)]^x$ approach as x approaches ∞? e

Problem numbers that appear in blue indicate problems that require students to apply their reasoning and writing skills to the solution of the problem.

B *In Problems 31–38, graph each function using transformations of an appropriate function of the form $y = b^x$.*

31. $f(x) = 2^{x-3} - 1$ **32.** $g(x) = 3^{x+1} + 2$

33. $g(x) = \left(\dfrac{1}{3}\right)^{x+5} - 10$ **34.** $f(x) = \left(\dfrac{1}{2}\right)^{x-10} + 5$

35. $g(x) = e^x + 2$ **36.** $g(x) = e^x - 1$

37. $g(x) = 2e^{-(x+2)}$ **38.** $g(x) = 0.5e^{-(x-1)}$

In Problems 39–52, find all solutions to the equation.

39. $5^{3x} = 5^{4x-2}$ $x = 2$ **40.** $10^{2-3x} = 10^{5x-6}$ $x = 1$

41. $7^{x^2} = 7^{2x+3}$ $x = -1, 3$ **42.** $4^{5x-x^2} = 4^{-6}$ $x = 6, -1$

43. $(1 - x)^5 = (2x - 1)^5$ $x = \frac{2}{3}$ **44.** $5^3 = (x + 2)^3$ $x = 3$

45. $9^{x^2} = 3^{3x-1}$ $x = \frac{1}{2}, 1$ **46.** $4^{x^2} = 2^{x+3}$ $x = -1, \frac{3}{2}$

47. $4^{x^2} = 8^x$ $x = 0, \frac{3}{2}$ **48.** $9^{x^2} = 27^{x+3}$ $x = -\frac{3}{2}, 3$

49. $2xe^{-x} = 0$ $x = 0$ **50.** $(x - 3)e^x = 0$ $x = 3$

51. $x^2e^x - 5xe^x = 0$ $x = 0, 5$ **52.** $3xe^{-x} + x^2e^{-x} = 0$ $x = 0, -3$

53. Find all real numbers a such that $a^2 = a^{-2}$. Explain why this does not violate the second exponential function property in the box on page 387. $a = 1$ or $a = -1$

54. Find real numbers a and b such that $a \neq b$ but $a^4 = b^4$. Explain why this does not violate the third exponential function property in the box on page 387.

55. Examine the graph of $y = 1^x$ on a graphing calculator and explain why 1 cannot be the base for an exponential function.

56. Examine the graph of $y = 0^x$ on a graphing calculator and explain why 0 cannot be the base for an exponential function. [*Hint:* Turn the axes off before graphing.]

57. Evaluate $y = 1^x$ for $x = -3, -2, -1, 0, 1, 2,$ and 3. Why is $b = 1$ excluded when defining the exponential function $y = b^x$?

58. Evaluate $y = 0^x$ for $x = -3, -2, -1, 0, 1, 2,$ and 3. Why is $b = 0$ excluded when defining the exponential function $y = b^x$?

59. Explain why the graph of an exponential function cannot be the graph of a polynomial function.

60. Explain why the graph of an exponential function cannot be the graph of a rational function.

In Problems 61–64, simplify.

61. $\dfrac{-2x^1e^{-2x} - 3x^2e^{-2x}}{x^6}$ **62.** $\dfrac{5x^4e^{5x} - 4x^3e^{5x}}{x^8}$

63. $(e^x + e^{-x})^2 + (e^x - e^{-x})^2$

64. $e^x(e^{-x} + 1) - e^{-x}(e^x + 1)$

In Problems 65–76, use a graphing calculator to find local extrema, y intercepts, and x intercepts. Investigate the behavior as $x \to \infty$ and as $x \to -\infty$ and identify any horizontal asymptotes. Round any approximate values to two decimal places.

65. $f(x) = 2 + e^{x-2}$

66. $g(x) = -3 + e^{1+x}$

67. $m(x) = e^{|x|}$

68. $n(x) = e^{-|x|}$

69. $s(x) = e^{-x^2}$

70. $r(x) = e^{x^2}$

71. $F(x) = \dfrac{200}{1 + 3e^{-x}}$

72. $G(x) = \dfrac{100}{1 + e^{-x}}$

73. $m(x) = 2x(3^{-x}) + 2$

74. $h(x) = 3x(2^{-x}) - 1$

75. $f(x) = \dfrac{2^x + 2^{-x}}{2}$

76. $g(x) = \dfrac{3^x + 3^{-x}}{2}$

C

77. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches 0.

78. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches ∞.

It is common practice in many applications of mathematics to approximate nonpolynomial functions with appropriately selected polynomials. For example, the polynomials in Problems 79–82, called ***Taylor polynomials,*** *can be used to approximate the exponential function $f(x) = e^x$. To illustrate this approximation graphically, in each problem graph $f(x) = e^x$ and the indicated polynomial in the same viewing window, $-4 \le x \le 4$ and $-5 \le y \le 50$.*

79. $P_1(x) = 1 + x + \frac{1}{2}x^2$

80. $P_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

81. $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

82. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

83. Investigate the behavior of the functions $f_1(x) = x/e^x$, $f_2(x) = x^2/e^x$, and $f_3(x) = x^3/e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $f_n(x) = x^n/e^x$, where n is any positive integer.

84. Investigate the behavior of the functions $g_1(x) = xe^x$, $g_2(x) = x^2e^x$, and $g_3(x) = x^3e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $g_n(x) = x^ne^x$, where n is any positive integer.

85. The irrational number $\sqrt{2}$ is approximated by 1.414214 to six decimal places. Each of $x = 1.4, 1.41, 1.414, 1.4142, 1.41421$, and 1.414214 is a rational number, so we know how to define 2^x for each. Compute the value of 2^x for each of these x values, and use your results to estimate the value of $2^{\sqrt{2}}$. Then compute $2^{\sqrt{2}}$ using your calculator to check your estimate.

86. The irrational number $\sqrt{3}$ is approximated by 1.732051 to six decimal places. Each of $x = 1.7, 1.73, 1.732, 1.7321$, 1.73205, and 1.732051 is a rational number, so we know how to define 3^x for each. Compute the value of 3^x for each of these x values, and use your results to estimate the value of $3^{\sqrt{3}}$. Then compute $3^{\sqrt{3}}$ using your calculator to check your estimate.

APPLICATIONS*

87. FINANCE Suppose \$4,000 is invested at 11% compounded weekly. How much money will be in the account in
(A) $\frac{1}{2}$ year? \$4,225.92 (B) 10 years? \$12,002.71
Compute answers to the nearest cent.

88. FINANCE Suppose \$2,500 is invested at 7% compounded quarterly. How much money will be in the account in
(A) $\frac{3}{4}$ year? \$2,633.56 (B) 15 years? \$7,079.54
Compute answers to the nearest cent.

89. MONEY GROWTH If you invest \$5,250 in an account paying 11.38% compounded continuously, how much money will be in the account at the end of
(A) 6.25 years? \$10,691.81 (B) 17 years? \$36,336.69

90. MONEY GROWTH If you invest \$7,500 in an account paying 8.35% compounded continuously, how much money will be in the account at the end of
(A) 5.5 years? \$11,871.65 (B) 12 years? \$20,427.93

91. FINANCE If \$3,000 is deposited into an account earning 8% compounded daily and, at the same time, \$5,000 is deposited into an account earning 5% compounded daily, will the first account ever be worth more than the second? If so, when?
Yes, after 6,217 days

92. FINANCE If \$4,000 is deposited into an account earning 9% compounded weekly and, at the same time, \$6,000 is deposited into an account earning 7% compounded weekly, will the first account ever be worth more than the second? If so, when?
Yes, after 1,056 weeks

93. FINANCE Will an investment of \$10,000 at 8.9% compounded daily ever be worth more at the end of a quarter than an investment of \$10,000 at 9% compounded quarterly? Explain.
No

94. FINANCE A sum of \$5,000 is invested at 13% compounded semiannually. Suppose that a second investment of \$5,000 is made at interest rate r compounded daily. For which values of r, to the nearest tenth of a percent, is the second investment better than the first? Discuss. $r \ge 12.6\%$

95. PRESENT VALUE A promissory note will pay \$30,000 at maturity 10 years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 9% compounded continuously? \$12,197.09

*Round monetary amounts to the nearest cent unless specified otherwise. In all problems involving interest that is compounded daily, assume a 365-day year.

96. **PRESENT VALUE** A promissory note will pay \$50,000 at maturity $5\frac{1}{2}$ years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 10% compounded continuously? \$28,847.49

97. **MONEY GROWTH** *Barron's,* a national business and financial weekly, published the following "Top Savings Deposit Yields" for $2\frac{1}{2}$-year certificate of deposit accounts:

Gill Savings	8.30% (CC)
Richardson Savings and Loan	8.40% (CQ)
USA Savings	8.25% (CD)

where CC represents compounded continuously, CQ compounded quarterly, and CD compounded daily. Compute the value of \$1,000 invested in each account at the end of $2\frac{1}{2}$ years.

98. **MONEY GROWTH** Refer to Problem 97. In another issue of *Barron's,* 1-year certificate of deposit accounts included:

Alamo Savings	8.25% (CQ)
Lamar Savings	8.05% (CC)

Compute the value of \$10,000 invested in each account at the end of 1 year.

99. **FINANCE** A couple just had a new child. How much should they invest now at 8.25% compounded daily to have \$100,000 for the child's education 17 years from now? Compute the answer to the nearest dollar.

100. **FINANCE** A person wishes to have \$15,000 cash for a new car 5 years from now. How much should be placed in an account now if the account pays 9.75% compounded weekly? Compute the answer to the nearest dollar.

4-2 Exponential Models

- Modeling Exponential Growth
- Modeling Negative Exponential Growth
- Carbon-14 Dating
- Modeling Limited Growth
- Data Analysis and Regression
- A Comparison of Exponential Growth Phenomena

One of the best reasons for studying exponential functions is the fact that many things that occur naturally in our world can be modeled accurately by these functions. In this section, we will study a wide variety of applications, including growth of populations of people, animals, and bacteria; radioactive decay; spread of epidemics; propagation of rumors; light intensity; atmospheric pressure; and electric circuits. The regression techniques we used in Chapter 2 to construct linear and quadratic models will be extended to construct exponential models.

Modeling Exponential Growth

What sort of function is likely to describe the growth of a population? We will consider this question in Explore-Discuss 1.

››› EXPLORE-DISCUSS 1

A certain species of fruit fly reproduces quickly, with a new generation appearing in about a week.

(A) Suppose that a population starts with 200 flies, and we assume that the population increases by 50 flies each week. Calculate the number of flies after 1, 2, 3, 4, and 5 weeks.

(B) Now suppose that the population increases by 25% of the current population each week. Calculate the number of flies after 1, 2, 3, 4, and 5 weeks.

(C) Which scenario do you think is more realistic? Why?

The population model described in part B of Explore-Discuss 1 is the more realistic one. As a population grows, there are more individuals to reproduce, so the rate of increase grows as well. This sounds an awful lot like compound interest: The percentage added to the population is in effect calculated on both the original amount and the number of new individuals. It should come as no surprise, then, that populations of organisms, from bacteria all the way to human beings, tend to grow exponentially.

One convenient and easily understood measure of growth rate is the **doubling time**—that is, the time it takes for a population to double. Over short periods the **doubling time growth model** is often used to model population growth:

$$A = A_0 2^{t/d}$$

where A = Population at time t

A_0 = Population at time $t = 0$

d = Doubling time

Note that when the amount of time passed is equal to the doubling time ($t = d$),

$$A = A_0 2^{t/d} = A_0 2$$

and the population is double the original, as it should be. We will use this model to solve a population growth problem in Example 1.

EXAMPLE 1 Population Growth

Mexico has a population of around 100 million people, and it is estimated that the population will double in 21 years. If population growth continues at the same rate, what will be the population:

(A) 15 years from now? (B) 30 years from now?

Calculate answers to three significant digits.

SOLUTIONS

Algebraic Solutions

We use the doubling time growth model with $A_0 = 100$ and $d = 21$:

$$A = A_0 2^{t/d} \quad \text{Let } A_0 = 100, d = 21$$

$$A = 100(2^{t/21}) \quad \text{Figure 1}$$

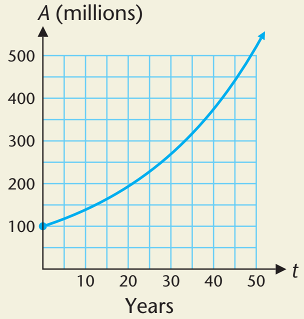
(A) Find A when $t = 15$ years:

$$A = 100(2^{15/21})$$
$$\approx 164 \text{ million people}$$

(B) Find A when $t = 30$ years:

$$A = 100(2^{30/21})$$
$$\approx 269 \text{ million people}$$

Figure 1 $A = 100(2^{t/21})$.



Graphical Solutions

We graph

$$A = 100(2^{x/21})$$

and construct a table of values (Fig. 2).

(A) When $x = 15$ years, $A \approx 164$ million people.

(B) When $x = 30$ years, $A \approx 269$ million people.

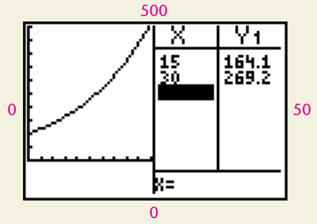


Figure 2

MATCHED PROBLEM 1

The bacterium *Escherichia coli* (*E. coli*) is found naturally in the intestines of many mammals. In a particular laboratory experiment, the doubling time for *E. coli* is found to be 25 minutes. If the experiment starts with a population of 1,000 *E. coli* and there is no change in the doubling time, how many bacteria will be present:

(A) In 10 minutes? (B) In 5 hours?

Write answers to three significant digits.

››› EXPLORE-DISCUSS 2

The doubling time growth model would *not* be expected to give accurate results over long periods. According to the doubling time growth model of Example 1, what was the population of Mexico 500 years ago at the height of Aztec civilization? What will the population of Mexico be 200 years from now? Explain why these results are unrealistic. Discuss factors that affect human populations that are not taken into account by the doubling time growth model.

The doubling time model is not the only one used to model populations. An alternative model based on the continuous compound interest formula will be used in Example 2. In this case, the formula is written as

$$A = A_0 e^{kt}$$

where A = Population at time t

A_0 = Population at time $t = 0$

k = Relative growth rate

The **relative growth rate** is written as a percentage in decimal form. For example, if a population is growing so that at any time the population is increasing at 3% of the current population per year, the relative growth rate k would be 0.03.

EXAMPLE 2 Medicine—Bacteria Growth

Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as modeled by

$$A = A_0 e^{1.386t}$$

where A is the number of bacteria present after t hours and A_0 is the number of bacteria present at $t = 0$. If we start with 1 bacterium, how many bacteria will be present in

(A) 5 hours? (B) 12 hours?

Compute the answers to three significant digits.

SOLUTIONS

Algebraic Solutions

(A) Use $A_0 = 1$ and $t = 5$:

$$\begin{aligned} A &= A_0 e^{1.386t} \\ &= e^{1.386(5)} \\ &= 1{,}020 \end{aligned}$$

(B) Use $A_0 = 1$ and $t = 12$:

$$\begin{aligned} A &= A_0 e^{1.386t} \\ &= e^{1.386(12)} \\ &= 16{,}700{,}000 \end{aligned}$$

Graphical Solutions

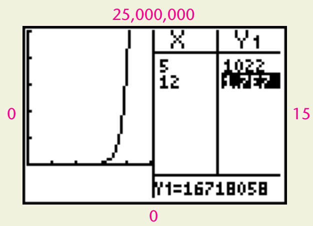
We graph

$$A = e^{1.386x}$$

and construct a table of values (Fig. 3).

(A) When $x = 5$ hours, $A \approx 1{,}020$ bacteria.

(B) When $x = 12$ hours, $A \approx 16{,}700{,}000$ bacteria.



› Figure 3

MATCHED PROBLEM 2

Repeat Example 2 if $A = A_0e^{0.783t}$ and all other information remains the same.

Modeling Negative Exponential Growth

Exponential functions can also be used to model radioactive decay, which is sometimes referred to as **negative growth.** Radioactive materials are used extensively in medical diagnosis and therapy, as power sources in satellites, and as power sources in many countries. If we start with an amount A_0 of a particular radioactive substance, the amount declines exponentially over time. The rate of decay varies depending on the particular radioactive substance. A convenient and easily understood measure of the rate of decay is the **half-life** of the material—that is, the time it takes for half of a particular material to decay. We can use the following **half-life decay model:**

$$\begin{aligned} A &= A_0(\tfrac{1}{2})^{t/h} \\ &= A_0 2^{-t/h} \end{aligned}$$

where A = Amount at time t
A_0 = Amount at time $t = 0$
h = Half-life

Note that when the amount of time passed is equal to the half-life ($t = h$),

$$A = A_0 2^{-h/h} = A_0 2^{-1} = A_0 \cdot \tfrac{1}{2}$$

and the amount of radioactive material is half the original amount, as it should be.

EXAMPLE 3 Radioactive Decay

The radioactive isotope gallium 67 (^{67}Ga), used in the diagnosis of malignant tumors, has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after

(A) 24 hours? (B) 1 week?

Compute answers to three significant digits.

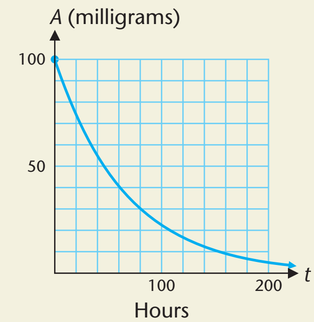
SOLUTIONS

Algebraic Solutions

We use the half-life decay model with $A_0 = 100$ and $h = 46.5$:

$$A = A_0\left(\tfrac{1}{2}\right)^{t/h} = A_0 2^{-t/h} \qquad A_0 = 100,\ h = 46.5$$

$$A = 100(2^{-t/46.5}) \qquad \text{See Figure 4}$$



› **Figure 4** $A = 100(2^{-t/46.5})$.

(A) Find A when $t = 24$ hours:

$$A = 100(2^{-24/46.5})$$
$$= 69.9 \text{ milligrams}$$

(B) Find A when $t = 168$ hours (1 week = 168 hours):

$$A = 100(2^{-168/46.5})$$
$$= 8.17 \text{ milligrams}$$

Graphical Solutions

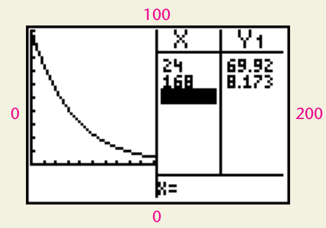
We graph

$$A = 100(2^{-x/46.5})$$

and construct a table of values (Fig. 5).

(A) When $x = 24$ hours, $A \approx 69.9$ milligrams.

(B) When $x = 168$ hours (1 week), $A \approx 8.17$ milligrams.



› **Figure 5**

MATCHED PROBLEM 3

Radioactive gold 198 (^{198}Au), used in imaging the structure of the liver, has a half-life of 2.67 days. If we start with 50 milligrams of the isotope, how many milligrams will be left after:

(A) $\frac{1}{2}$ day? (B) 1 week?

Compute answers to three significant digits.

››› CAUTION ›››

When using exponential models, be aware of the units of time. In Example 3 the half-life was given in hours, so when time was provided in weeks, we had to first convert that into hours before using the half-life formula.

In Example 2, we saw that a base e exponential function can be used as an alternative to the doubling time model. Not surprisingly, the same can be said for the half-life model. In this case, the formula will be

$$A = A_0 e^{-kt}$$

where A = the amount of radioactive material at time t

A_0 = the amount at time $t = 0$

k = a positive constant specific to the type of material

› Carbon-14 Dating

Our atmosphere is constantly being bombarded with cosmic rays. These rays produce neutrons, which in turn react with nitrogen to produce radioactive carbon-14. Radioactive carbon-14 enters all living tissues through carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, carbon-14 is maintained in the living organism at a constant level. Once the organism dies, however, carbon-14 decays according to the equation

$$A = A_0 e^{-0.000124t} \qquad \textbf{Carbon-14 decay equation}$$

where A is the amount of carbon-14 present after t years and A_0 is the amount present at time $t = 0$. This can be used to calculate the approximate age of fossils.

EXAMPLE 4 Carbon-14 Dating

If 1,000 milligrams of carbon-14 are present in the tissue of a recently deceased animal, how many milligrams will be present in

(A) 10,000 years? (B) 50,000 years?

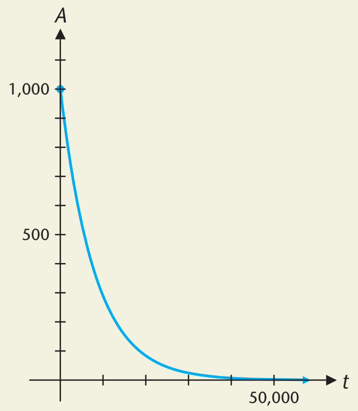
Compute answers to three significant digits.

SOLUTIONS

Algebraic Solutions

Substituting $A_0 = 1{,}000$ in the decay equation, we have

$$A = 1{,}000e^{-0.000124t}$$ Figure 6



› Figure 6

(A) Find A when $t = 10{,}000$:

$$A = 1{,}000e^{-0.000124(10{,}000)}$$
$$= 289 \text{ milligrams}$$

(B) Find A when $t = 50{,}000$:

$$A = 1{,}000e^{-0.000124(50{,}000)}$$
$$= 2.03 \text{ milligrams}$$

Graphical Solutions

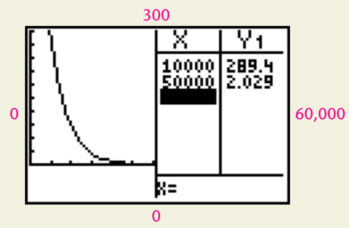
We graph

$$A = 1{,}000e^{-0.000124x}$$

and construct a table of values (Fig. 7).

(A) When $x = 10{,}000$ years, $A \approx 289$ milligrams.

(B) When $x = 50{,}000$ years, $A \approx 2.03$ milligrams.



› Figure 7

We will use the carbon-14 decay equation in Exercise 4-5, where we will be interested in solving for t after being given information about A and A_0.

MATCHED PROBLEM 4

Referring to Example 4, how many milligrams of carbon-14 would have to be present at the beginning to have 10 milligrams present after 20,000 years? Approximate the answer to four significant digits.

› Modeling Limited Growth

One of the problems with using exponential functions to model things like population is that the growth is completely unlimited in the long term. But in real life, there is often some reasonable maximum value, like the largest population that space and resources allow. We can use modified versions of exponential functions to model such phenomena more realistically.

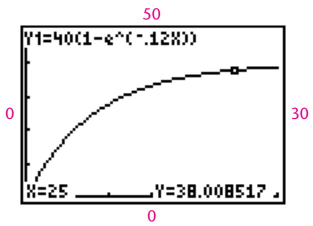
One such type of function is called a *learning curve* since it can be used to model the performance improvement of a person learning a new task. **Learning curves** are functions of the form $A = c(1 - e^{-kt})$, where c and k are positive constants.

EXAMPLE 5 Learning Curve

People assigned to assemble circuit boards for a computer manufacturing company undergo on-the-job training. From past experience, it was found that the learning curve for the average employee is given by

$$A = 40(1 - e^{-0.12t})$$

where A is the number of boards assembled per day after t days of training (Fig. 8).



› **Figure 8** Limited growth.

(A) How many boards can an average employee produce after 3 days of training? After 5 days of training? Round answers to the nearest integer.

(B) How many days of training will it take until an average employee can assemble 25 boards a day? Round answers to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

SOLUTIONS

(A) When $t = 3$,

$$A = 40(1 - e^{-0.12(3)}) = 12 \qquad \text{Rounded to nearest integer}$$

so the average employee can produce 12 boards after 3 days of training. Similarly, when $t = 5$,

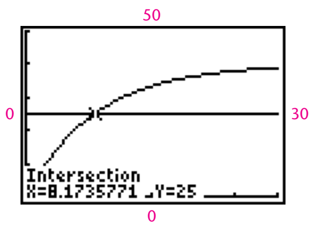
$$A = 40(1 - e^{-0.12(5)}) = 18 \qquad \text{Rounded to nearest integer}$$

so the average employee can produce 18 boards after 5 days of training.

(B) Solve the equation $40(1 - e^{-0.12t}) = 25$ for t by graphing

$$y_1 = 40(1 - e^{-0.12t}) \qquad \text{and} \qquad y_2 = 25$$

and using the INTERSECT command (Fig. 9). It will take about 8 days of training.



› **Figure 9**
$y_1 = 40(1 - e^{-0.12t})$, $y_2 = 25$.

(C) Because $e^{-0.12t}$ approaches 0 as t increases without bound,

$$A = 40(1 - e^{-0.12t}) \rightarrow 40(1 - 0) = 40$$

So the limiting value of A is 40 boards per day. (This can be supported by the graph.)

MATCHED PROBLEM 5

A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million potential viewers. A model for the number of people A, in millions, who are aware of the product after t days of advertising was found to be

$$A = 2(1 - e^{-0.037t})$$

(A) How many viewers are aware of the product after 2 days? After 10 days? Express answers as integers, rounded to three significant digits.

(B) How many days will it take until half of the potential viewers will become aware of the product? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

Another limited-growth model is useful for phenomena such as the spread of an epidemic or the propagation of a rumor. It is called the *logistic equation,* and is given by

$$A = \frac{M}{(1 + ce^{-kt})}$$

where M, c, and k are positive constants. Logistic growth, illustrated in Example 6, also approaches a limiting value as t increases without bound.

EXAMPLE 6 Logistic Growth in an Epidemic

A certain community consists of 1,000 people. One individual who has just returned from another community has a particularly contagious strain of influenza. Assume the community has not had influenza shots and all are susceptible. The spread of the disease in the community is predicted to be given by the logistic curve

$$A(t) = \frac{1{,}000}{1 + 999e^{-0.3t}}$$

where A is the number of people who have contracted influenza after t days.

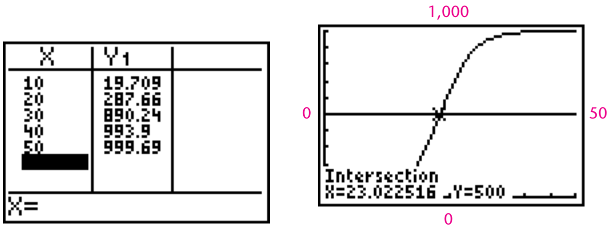
(A) How many people have contracted influenza after 10 days? After 20 days?

(B) How many days will it take until half the community has contracted influenza? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

SOLUTIONS

(A) Enter $y_1 = 1{,}000/(1 + 999e^{-0.3t})$ into a graphing calculator. The table in Figure 10(a) shows that $A(10) \approx 20$ individuals and $A(20) \approx 288$ individuals.



(a) (b)

› **Figure 10** Logistic growth.

(B) Figure 10(b) shows that the graph of $A(t)$ intersects the line $y = 500$ after approximately 23 days.

(C) The values in Figure 10(a) and the graph in Figure 10(b) both indicate that A approaches 1,000 as t increases without bound. We can confirm this algebraically by noting that because $999e^{-0.3t} \to 0$ as t increases without bound,

$$A(t) = \frac{1{,}000}{1 + 999e^{-0.3t}} \to \frac{1{,}000}{1 + 0} = 1{,}000$$

Thus, the upper limit on the growth of A is 1,000, the total number of people in the community.

MATCHED PROBLEM 6

A group of 400 parents, relatives, and friends are waiting anxiously at Kennedy Airport for a charter flight returning students after a year in Europe. It is stormy and the plane is late. A particular parent thought he had heard that the plane's radio had gone out and related this news to some friends, who in turn passed it on to others. The propagation of this rumor is predicted to be given by

$$A(t) = \frac{400}{1 + 399e^{-0.4t}}$$

where A is the number of people who have heard the rumor after t minutes.

(A) How many people have heard the rumor after 10 minutes? After 20 minutes?

(B) How many minutes will it take until half the group has heard the rumor? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

Data Analysis and Regression

Many graphing calculators with regression commands have options for exponential regression. We can use exponential regression to fit a function of the form $y = ab^x$ to a set of data points, and logistic regression to fit a function of the form

$$y = \frac{c}{1 + ae^{-bx}}$$

to a set of data points. The techniques are similar to those introduced in Chapter 2 for linear and quadratic functions.

EXAMPLE 7

Infectious Diseases

Table 1 Reported Cases of Infectious Diseases

Year	Mumps	Rubella
1970	104,953	56,552
1980	8,576	3,904
1990	5,292	1,125
1995	906	128
2000	323	152

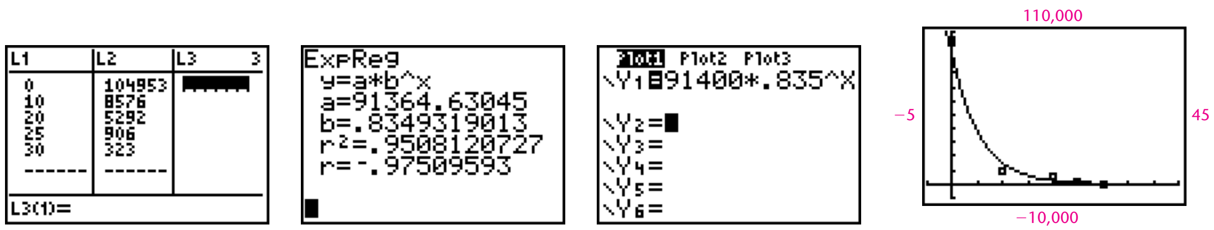
The U.S. Department of Health and Human Services published the data in Table 1.

(A) Let x represent time in years with $x = 0$ representing 1970, and let y represent the corresponding number of reported cases of mumps. Use regression analysis on a graphing calculator to find an exponential function of the form $y = ab^x$ that models the data. (Round the constants a and b to three significant digits.)

(B) Use the exponential regression function to predict the number of reported cases of mumps in 2010.

SOLUTIONS

(A) Figure 11 shows the details of constructing the model on a graphing calculator.



(a) Data

(b) Regression equation

(c) Regression equation entered in equation editor

(d) Graph of data and regression equation

Figure 11

(B) Evaluating $y_1 = 91{,}400(0.835)^x$ at $x = 40$ gives a prediction of 67 cases of mumps in 2010.

MATCHED PROBLEM 7

Repeat Example 7 for reported cases of rubella.

EXAMPLE 8 AIDS Cases and Deaths

Table 2 Acquired Immunodeficiency Syndrome (AIDS) Cases and Deaths in the United States

Year	Cases diagnosed to date	Known deaths to date
1988	107,755	62,468
1991	261,259	159,294
1994	493,713	296,507
1997	672,970	406,179
2000	774,467	447,648
2005	956,665	550,394

The U.S. Department of Health and Human Services published the data in Table 2.

(A) Let x represent time in years with $x = 0$ representing 1988, and let y represent the corresponding number of AIDS cases diagnosed to date. Use regression analysis on a graphing utility to find a logistic function of the form

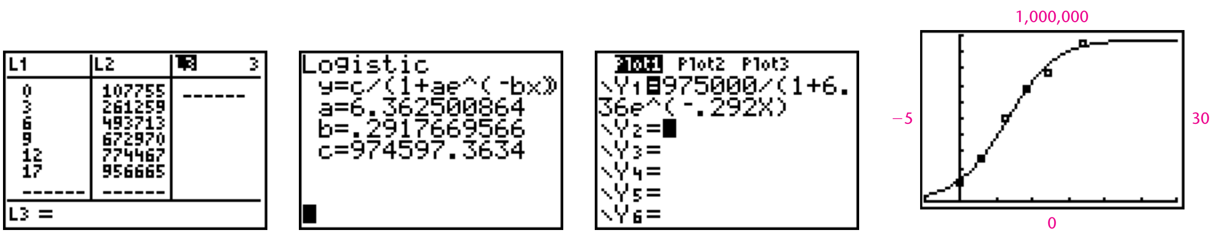
$$y = \frac{c}{1 + ae^{-bx}}$$

that models the data. (Round the constants a, b, and c to three significant digits.)

(B) Use the logistic regression function to predict the number of cases of AIDS diagnosed by 2015.

SOLUTIONS

(A) Figure 12 shows the details of constructing the model on a graphing calculator.



(a) Data

(b) Regression equation

(c) Regression equation entered in equation editor

(d) Graph of data and regression equation

Figure 12

(B) Evaluating

$$y_1 = \frac{975{,}000}{1 + 6.36e^{-0.292x}}$$

at $x = 27$ gives a prediction of approximately 973,000 cases of AIDS diagnosed by 2015.

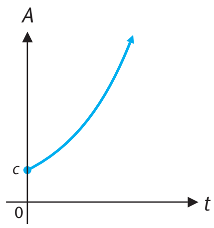
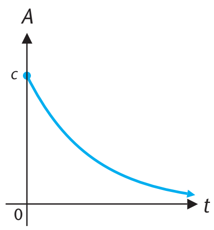
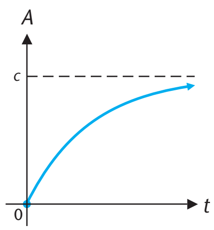
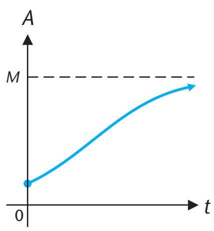
MATCHED PROBLEM 8

Repeat Example 8 for known deaths from AIDS to date.

› A Comparison of Exponential Growth Phenomena

The equations and graphs given in Table 3 compare several widely used growth models. These are divided basically into two groups: unlimited growth and limited growth. Following each equation and graph is a short, incomplete list of areas in which the models are used. We have only touched on a subject that has been extensively developed and that you are likely to study in greater depth in the future.

Table 3 Exponential Growth and Decay

Description	Equation	Graph	Short list of uses
Unlimited growth	$A = A_0e^{kt}$ $k > 0$		Short-term population growth (people, bacteria, etc.); growth of money at continuous compound interest
Exponential decay	$A = A_0e^{-kt}$ $k > 0$		Radioactive decay; light absorption in water, glass, and the like; atmospheric pressure; electric circuits
Limited growth	$A = c(1 - e^{-kt})$ $c, k > 0$		Learning skills; sales fads; company growth; electric circuits
Logistic growth	$A = \dfrac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth; epidemics; sales of new products; spread of rumors; company growth

ANSWERS TO MATCHED PROBLEMS

1. (A) 1,320 bacteria (B) 4,100,000 bacteria
2. (A) 50 bacteria (B) 12,000 bacteria **3.** (A) 43.9 milligrams (B) 8.12 milligrams
4. 119.4 milligrams **5.** (A) 143,000 viewers; 619,000 viewers (B) 19 days
(C) A approaches an upper limit of 2 million, the number of potential viewers
6. (A) 48 individuals; 353 individuals (B) 15 minutes (C) A approaches an upper limit of 400, the number of people in the entire group.
7. (A) $y = 44{,}500(0.815)^x$ (B) 12 cases
8. (A) $y = \dfrac{549{,}000}{1 + 6.14e^{-0.311x}}$ (B) 548,000 known deaths

4-2 Exercises

**Additional answers can be found in the Instructor Answer Appendix.*

1. Define the terms "doubling time" and "half-life" in your own words.

2. One of the models below represents positive growth, and the other represents negative growth. Classify each, and explain how you decided on your answer. (Assume that $k > 0$.)

$$A = A_0e^{-kt} \qquad A = A_0e^{kt}$$

3. Explain the difference between exponential growth and limited growth.

4. Explain why a limited growth model would be more accurate than regular exponential growth in modeling the long-term population of birds on an island in Lake Erie.

In Problems 5–8, write an exponential equation describing the given population at any time t.

5. Initial population 200; doubling time 5 months $A = 200(2)^{t/5}$

6. Initial population 5,000; doubling time 3 years $A = 5{,}000(2)^{t/3}$

7. Initial population 2,000; continuous growth at 2% per year $A = 2{,}000e^{0.02t}$

8. Initial population 500; continuous growth at 3% per week $A = 500e^{0.03t}$

In Problems 9–12, write an exponential equation describing the amount of radioactive material present at any time t.

9. Initial amount 100 grams; half-life 6 hours

10. Initial amount 5 pounds; half-life 1,300 years

11. Initial amount 4 kilograms; continuous decay at 12.4% per year $A = 4e^{-0.124t}$

12. Initial amount 50 milligrams; continuous decay at 0.03% per year $A = 50e^{-0.03t}$

APPLICATIONS

13. GAMING A person bets on red and black on a roulette wheel using a *Martingale strategy*. That is, a $2 bet is placed on red, and the bet is doubled each time until a win occurs. The process is then repeated. If black occurs n times in a row, then $L = 2^n$ dollars is lost on the nth bet. Graph this function for $1 \le n \le 10$. Although the function is defined only for positive integers, points on this type of graph are usually joined with a smooth curve as a visual aid.

14. BACTERIAL GROWTH If bacteria in a certain culture double every $\frac{1}{2}$ hour, write an equation that gives the number of bacteria N in the culture after t hours, assuming the culture has 100 bacteria at the start. Graph the equation for $0 \le t \le 5$.

15. POPULATION GROWTH Because of its short life span and frequent breeding, the fruit fly *Drosophila* is used in some genetic studies. Raymond Pearl of Johns Hopkins University, for example, studied 300 successive generations of descendants of a single pair of *Drosophila* flies. In a laboratory situation with ample food supply and space, the doubling time for a particular population is 2.4 days. If we start with 5 male and 5 female flies, how many flies should we expect to have in
(A) 1 week? 76 flies (B) 2 weeks? 570 flies

16. POPULATION GROWTH If Kenya has a population of about 34,000,000 people and a doubling time of 27 years and if the growth continues at the same rate, find the population in
(A) 10 years 44,000,000 (B) 30 years 73,000,000
Compute answers to two significant digits.

17. COMPUTER DESIGN In 1965, Gordon Moore, founder of Intel, predicted that the number of transistors that could be

placed on a computer chip would double every 2 years. This has come to be known as *Moore's law*. In 1970, 2,200 transistors could be placed on a chip. Use Moore's law to predict the number of transistors in
(A) 1990 2,252,800 (B) 2005 407,800,360

18. HISTORY OF TECHNOLOGY The earliest mechanical clocks appeared around 1350 in Europe, and would gain or lose an average of 30 minutes per day. After that, accuracy roughly doubled every 30 years. Find the predicted accuracy of clocks in
(A) 1700 (B) 2000

19. INSECTICIDES The use of the insecticide DDT is no longer allowed in many countries because of its long-term adverse effects. If a farmer uses 25 pounds of active DDT, assuming its half-life is 12 years, how much will still be active after
(A) 5 years? 19 pounds (B) 20 years? 7.9 pounds
Compute answers to two significant digits.

20. RADIOACTIVE TRACERS The radioactive isotope technetium-99m ($^{99\text{m}}$Tc) is used in imaging the brain. The isotope has a half-life of 6 hours. If 12 milligrams are used, how much will be present after
(A) 3 hours? 8.49 mg (B) 24 hours? 0.750 mg
Compute answers to three significant digits.

21. POPULATION GROWTH If the world population is about 6.5 billion people now and if the population grows continuously at a relative growth rate of 1.14%, what will the population be in 10 years? Compute the answer to two significant digits. 7.3 billion

22. POPULATION GROWTH If the population of Mexico is around 106 million people now and if the population grows continuously at a relative growth rate of 1.17%, what will the population be in 8 years? Compute the answer to three significant digits. 116 million

23. POPULATION GROWTH In 2005 the population of Russia was 143 million and the population of Nigeria was 129 million. If the populations of Russia and Nigeria grow continuously at relative growth rates of −0.37% and 2.56%, respectively, in what year will Nigeria have a greater population than Russia? 2008

24. POPULATION GROWTH In 2005 the population of Germany was 82 million and the population of Egypt was 78 million. If the populations of Germany and Egypt grow continuously at relative growth rates of 0% and 1.78%, respectively, in what year will Egypt have a greater population than Germany? 2007

25. SPACE SCIENCE Radioactive isotopes, as well as solar cells, are used to supply power to space vehicles. The isotopes gradually lose power because of radioactive decay. On a particular space vehicle the nuclear energy source has a power output of P watts after t days of use as given by

$$P = 75e^{-0.0035t}$$

Graph this function for $0 \le t \le 100$.

26. EARTH SCIENCE The atmospheric pressure P, in pounds per square inch, decreases exponentially with altitude h, in miles above sea level, as given by

$$P = 14.7e^{-0.21h}$$

Graph this function for $0 \le h \le 10$.

27. MARINE BIOLOGY Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity I relative to depth d, in feet, for one of the clearest bodies of water in the world, the Sargasso Sea in the West Indies, can be approximated by

$$I = I_0e^{-0.00942d}$$

where I_0 is the intensity of light at the surface. To the nearest percent, what percentage of the surface light will reach a depth of
(A) 50 feet? 62% (B) 100 feet? 39%

28. MARINE BIOLOGY Refer to Problem 27. In some waters with a great deal of sediment, the photic zone may go down only 15 to 20 feet. In some murky harbors, the intensity of light d feet below the surface is given approximately by

$$I = I_0e^{-0.23d}$$

What percentage of the surface light will reach a depth of
(A) 10 feet? 10% (B) 20 feet? 1%

29. AIDS EPIDEMIC The World Health Organization estimated that 39.4 million people worldwide were living with HIV in 2004. Assuming that number continues to increase at a relative growth rate of 3.2% compounded continuously, estimate the number of people living with HIV in
(A) 2010 47.7 million people (B) 2015 56.0 million people

30. AIDS EPIDEMIC The World Health Organization estimated that there were 3.1 million deaths worldwide from HIV/AIDS during the year 2004. Assuming that number continues to increase at a relative growth rate of 4.3% compounded continuously, estimate the number of deaths from HIV/AIDS during the year
(A) 2008 3.7 million (B) 2012 4.4 million

31. NEWTON'S LAW OF COOLING This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

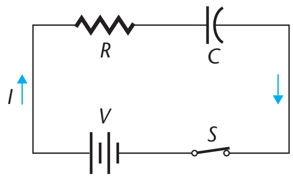
where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in the refrigerator to cool before a dinner party. If the temperature in the refrigerator is kept at 40°F and $k = 0.4$, find the temperature of the wine, to the nearest degree, after 3 hours. (In Exercise 4-5 we will find out how to determine k.) $T = 50°\text{F}$

32. NEWTON'S LAW OF COOLING Refer to Problem 31. What is the temperature, to the nearest degree, of the wine after 5 hours in the refrigerator? $T = 44°F$

33. PHOTOGRAPHY An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered, and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time*. For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

Find the value that q approaches as t increases without bound and interpret.



34. MEDICINE An electronic heart pacemaker uses the same type of circuit as the flash unit in Problem 33, but it is designed so that the capacitor discharges 72 times a minute. For a particular pacemaker, the charge on the capacitor t seconds after it starts recharging is given by

$$q = 0.000\,008(1 - e^{-2t})$$

Find the value that q approaches as t increases without bound and interpret.

35. WILDLIFE MANAGEMENT A herd of 20 white-tailed deer is introduced to a coastal island where there had been no deer before. Their population is predicted to increase according to the logistic curve

$$A = \frac{100}{1 + 4e^{-0.14t}}$$

where A is the number of deer expected in the herd after t years.
(A) How many deer will be present after 2 years? After 6 years? Round answers to the nearest integer. 25 deer, 37 deer
(B) How many years will it take for the herd to grow to 50 deer? Round answer to the nearest integer. 10 years
(C) Does A approach a limiting value as t increases without bound? Explain.

36. TRAINING A trainee is hired by a computer manufacturing company to learn to test a particular model of a personal computer after it comes off the assembly line. The learning curve for an average trainee is given by

$$A = \frac{200}{4 + 21e^{-0.1t}}$$

where A is the number of computers an average trainee can test per day after t days of training.
(A) How many computers can an average trainee be expected to test after 3 days of training? After 6 days? Round answers to the nearest integer. 10 computers, 13 computers
(B) How many days will it take until an average trainee can test 30 computers per day? Round answer to the nearest integer. 21 days
(C) Does A approach a limiting value as t increases without bound? Explain.

Problems 37–40 require a graphing calculator or a computer that can calculate exponential and logistic regression models for a given data set.

37. DEPRECIATION Table 4 gives the market value of a minivan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Round to four significant digits. Estimate the purchase price of the van. Estimate the value of the van 10 years after its purchase. Round answers to the nearest dollar.

Table 4

x	Value ($)
1	12,575
2	9,455
3	8,115
4	6,845
5	5,225
6	4,485

Source: Kelley Blue Book

38. DEPRECIATION Table 5 gives the market value of a luxury sedan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the sedan. Estimate the value of the sedan 10 years after its purchase. Round answers to the nearest dollar.

Table 5

x	Value ($)
1	23,125
2	19,050
3	15,625
4	11,875
5	9,450
6	7,125

Source: Kelley Blue Book

39. NUCLEAR POWER Table 6 gives data on nuclear power generation by region for the years 1980–1999.

Table 6 Nuclear Power Generation

	(Billion kilowatt-hours)	
Year	North America	Central and South America
1980	287.0	2.2
1985	440.8	8.4
1990	649.0	9.0
1995	774.4	9.5
1998	750.2	10.3
1999	807.5	10.5

Source: U.S. Energy Information Administration

(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model $(y = \frac{c}{1 + ae^{-bx}})$ for the generation of nuclear power in North America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in North America in 2010.
836.0 billion kilowatt-hours

40. NUCLEAR POWER Refer to Table 6.
(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model $(y = \frac{c}{1 + ae^{-bx}})$ for the generation of nuclear power in Central and South America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in Central and South America in 2010.
9.9 billion kilowatt-hours

4-3 Logarithmic Functions

- Defining Logarithmic Functions
- Converting Between Logarithmic Form and Exponential Form
- Properties of Logarithmic Functions
- Common and Natural Logarithms
- The Change-of-Base Formula

Solving an equation like $3^x = 9$ is easy: We know that $3^2 = 9$, so $x = 2$ is the solution. But what about an equation like $3^x = 20$? There probably is an exponent x between 2 and 3 for which 3^x is 20, but its exact value is not at all clear.

Compare this situation to an equation like $x^2 = 9$. This is easy to solve because we know that 3^2 and $(-3)^2$ are both 9. But what about $x^2 = 20$? To solve this equation, we needed to introduce a new function to be the opposite of the squaring function. This, of course, is the function $f(x) = \sqrt{x}$.

In this section, we will do something very similar with exponential functions. In the first section of this chapter, we learned that exponential functions are one-to-one, so we can define their inverses. These are known as the *logarithmic functions*.

Defining Logarithmic Functions

The exponential function $f(x) = b^x$ for $b > 0$, $b \neq 1$, is a one-to-one function, and therefore has an inverse. Its inverse, denoted $f^{-1}(x) = \log_b x$ (read "log to the base b of x") is called the *logarithmic function with base b*. Just like exponentials, there are different logarithmic functions for each positive base other than 1. A point (x, y)

is on the graph of $f^{-1} = \log_b x$ if and only if the point (y, x) is on the graph of $f = b^x$. In other words,

$$y = \log_b x \text{ if and only if } x = b^y$$

In a specific example,

$$y = \log_2 x \text{ if and only if } x = 2^y, \text{ and}$$

$\log_2 x$ is the power to which 2 must be raised to obtain x: $2^{\log_2 x} = 2^y = x$.

We can use this fact to learn some things about the logarithmic functions from our knowledge of exponential functions. For example, the graph of $f^{-1}(x) = \log_b x$ is the graph of $f(x) = b^x$ reflected through the line $y = x$. Also, the domain of $f^{-1}(x) = \log_b x$ is the range of $f(x) = b^x$, and vice versa.

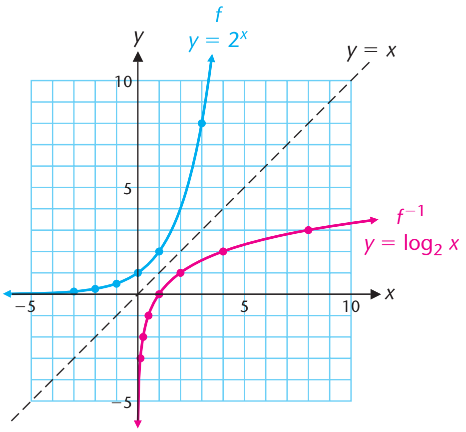
In Example 1, we will use information about $f(x) = 2^x$ to graph its inverse, $f^{-1}(x) = \log_2 x$.

EXAMPLE 1 Graphing a Logarithmic Function

Make a table of values for $f(x) = 2^x$ and reverse the ordered pairs to obtain a table of values for $f^{-1}(x) = \log_2 x$. Then use both tables to graph $f(x)$ and $f^{-1}(x)$ on the same set of axes.

SOLUTION

We chose to evaluate f for integer values from -3 to 3. The tables are shown here, along with the graph (Fig. 1). Note the important comments about domain and range below the graph.



DOMAIN of $f = (-\infty, \infty)$ = RANGE of f^{-1}
RANGE of $f = (0, \infty)$ = DOMAIN of f^{-1}

› **Figure 1** Logarithmic function with base 2.

f: x	$y = 2^x$	f^{-1}: x	$y = \log_2 x$
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

MATCHED PROBLEM 1

Repeat Example 1 for $f(x) = (\frac{1}{2})^x$ and $f^{-1}(x) = \log_{1/2} x$.

DEFINITION 1 Logarithmic Function

For $b > 0, b \neq 1$, the inverse of $f(x) = b^x$, denoted $f^{-1}(x) = \log_b x$, is the **logarithmic function** with base b.

Logarithmic form		Exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

The log to the base b of x is the exponent to which b must be raised to obtain x. For example,

$$y = \log_{10} x \quad \text{is equivalent to} \quad x = 10^y$$

$$y = \log_e x \quad \text{is equivalent to} \quad x = e^y$$

Remember: A logarithm is an exponent.

It is very important to remember that the equations $y = \log_b x$ and $x = b^y$ define the same function, and as such can be used interchangeably.

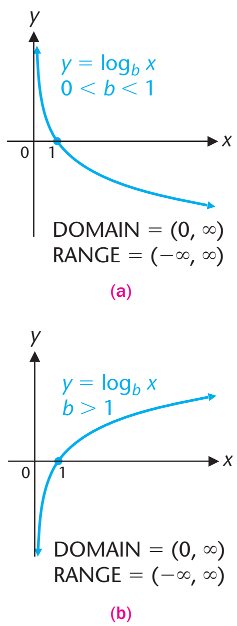


Figure 2 Typical logarithmic graphs.

Because the domain of an exponential function includes all real numbers and its range is the set of positive real numbers, the domain of a logarithmic function is the set of all positive real numbers and its range is the set of all real numbers. Thus, $\log_{10} 3$ is defined, but $\log_{10} 0$ and $\log_{10}(-5)$ are not defined.

In short, the function $y = \log_b x$ for any b is only defined for positive x values. Typical logarithmic curves are shown in Figure 2. Notice that in each case, the y axis is a vertical asymptote for the graph.

The graphs in Example 1 and Figure 2 suggest that logarithmic graphs share some common properties. Several of these properties are listed in Theorem 1. It might be helpful in understanding them to review Theorem 1 in Section 4-1. Each of these properties is a consequence of a corresponding property of exponential graphs.

THEOREM 1 Properties of Graphs of Logarithmic Functions

Let $f(x) = \log_b x$ be a logarithmic function, $b > 0, b \neq 1$. Then the graph of $f(x)$:

1. Is continuous on its domain $(0, \infty)$
2. Has no sharp corners
3. Passes through the point $(1, 0)$
4. Lies to the right of the y axis, which is a vertical asymptote
5. Is increasing as x increases if $b > 1$; is decreasing as x increases if $0 < b < 1$
6. Intersects any horizontal line exactly once, so is one-to-one

››› EXPLORE-DISCUSS 1

For the exponential function $f(x) = (\frac{2}{3})^x$, graph f and $y = x$ on the same coordinate system. Then sketch the graph of f^{-1}. Use the DRAW INVERSE command on a graphing calculator to check your work. Discuss the domains and ranges of f and its inverse. By what other name is f^{-1} known?

› Converting Between Logarithmic Form and Exponential Form

We now look into the matter of converting logarithmic forms to equivalent exponential forms, and vice versa. Throughout the remainder of the chapter, it will be useful to sometimes convert a logarithmic expression into the equivalent exponential form. At other times, it will be useful to do the reverse.

EXAMPLE 2 Logarithmic–Exponential Conversions

Change each logarithmic form to an equivalent exponential form.

(A) $\log_2 8 = 3$ (B) $\log_{25} 5 = \frac{1}{2}$ (C) $\log_2 (\frac{1}{4}) = -2$

SOLUTIONS

(A) $\log_2 8 = 3$ is equivalent to $8 = 2^3$.
(B) $\log_{25} 5 = \frac{1}{2}$ is equivalent to $5 = 25^{1/2}$.
(C) $\log_2 (\frac{1}{4}) = -2$ is equivalent to $\frac{1}{4} = 2^{-2}$.

Note that in each case, the base of the logarithm matches the base of the corresponding exponent.

MATCHED PROBLEM 2

Change each logarithmic form to an equivalent exponential form.

(A) $\log_3 27 = 3$ (B) $\log_{36} 6 = \frac{1}{2}$ (C) $\log_3 (\frac{1}{9}) = -2$

EXAMPLE 3 Logarithmic–Exponential Conversions

Change each exponential form to an equivalent logarithmic form.

(A) $49 = 7^2$ (B) $3 = \sqrt{9}$ (C) $\frac{1}{5} = 5^{-1}$

SOLUTIONS

(A) $49 = 7^2$ is equivalent to $\log_7 49 = 2$.
(B) $3 = \sqrt{9}$ is equivalent to $\log_9 3 = \frac{1}{2}$.
(C) $\frac{1}{5} = 5^{-1}$ is equivalent to $\log_5 (\frac{1}{5}) = -1$.

Again, the bases match.

MATCHED PROBLEM 3

Change each exponential form to an equivalent logarithmic form.

(A) $64 = 4^3$ (B) $2 = \sqrt[3]{8}$ (C) $\frac{1}{16} = 4^{-2}$

To gain a little deeper understanding of logarithmic functions and their relationship to the exponential functions, we will consider a few problems where we want to find x, b, or y in $y = \log_b x$, given the other two values. All values were chosen so that the problems can be solved without a calculator. In each case, converting to the equivalent exponential form is useful.

EXAMPLE 4 Solutions of the Equation $y = \log_b x$

Find x, b, or y as indicated.

(A) Find y: $y = \log_4 8$. (B) Find x: $\log_3 x = -2$. (C) Find b: $\log_b 81 = 4$.

SOLUTIONS

(A) Write $y = \log_4 8$ in equivalent exponential form.

$$8 = 4^y \quad \text{Write each number to the same base 2.}$$
$$2^3 = 2^{2y} \quad \text{Recall that } b^m = b^n \text{ if and only if } m = n.$$
$$2y = 3$$
$$y = \frac{3}{2}$$

We conclude that $\frac{3}{2} = \log_4 8$.

(B) Write $\log_3 x = -2$ in equivalent exponential form.

$$x = 3^{-2}$$
$$= \frac{1}{3^2} = \frac{1}{9}$$

We conclude that $\log_3 \left(\frac{1}{9}\right) = -2$.

(C) Write $\log_b 81 = 4$ in equivalent exponential form:

$81 = b^4$ Write 81 as a fourth power.

$3^4 = b^4$ b could be 3 or -3, but the base of a logarithm must be positive.

$b = 3$

We conclude that $\log_3 81 = 4$.

MATCHED PROBLEM 4

Find x, b, or y as indicated.

(A) Find y: $y = \log_9 27$. (B) Find x: $\log_2 x = -3$. (C) Find b: $\log_b 100 = 2$.

Properties of Logarithmic Functions

Some of the properties of exponential functions that we studied in Section 4-1 can be used to develop corresponding properties of logarithmic functions. Several of these important properties of logarithmic functions are listed in Theorem 1. We will justify them individually.

THEOREM 2 Properties of Logarithmic Functions

If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b M = \log_b N$ if and only if $M = N$
6. $\log_b MN = \log_b M + \log_b N$
7. $\log_b \frac{M}{N} = \log_b M - \log_b N$
8. $\log_b M^p = p \log_b M$

CAUTION

1. In Properties 3 and 4, it's essential that the base of the exponential and the base of the logarithm are the same.
2. Properties 6 and 7 are often misinterpreted, so you should examine them carefully.

$$\frac{\log_b M}{\log_b N} \neq \log_b M - \log_b N$$

$\log_b M - \log_b N = \log_b \frac{M}{N}$; $\frac{\log_b M}{\log_b N}$ cannot be simplified.

$$\log_b (M + N) \neq \log_b M + \log_b N$$

$\log_b M + \log_b N = \log_b MN$; $\log_b (M + N)$ cannot be simplified.

Now we will justify properties in Theorem 2.

1. $\log_b 1 = 0$ because $b^0 = 1$
2. $\log_b b = 1$ because $b^1 = b$
3. and 4. These are simply another way to state that $f(x) = b^x$ and $f^{-1}(x) = \log_b x$ are inverse functions. Property 3 can be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. Property 4 can be written as $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. This matches our characterization of inverse functions in Theorem 5, Section 1-6. Collectively, these properties say that if you apply an exponential function and a logarithmic function with the same base consecutively (in either order) you end up with the same value you started with.
5. This follows from the fact that logarithmic functions are one-to-one.

Properties 6, 7, and 8 are used often in manipulating logarithmic expressions. We will justify them in Problems 111 and 112 in Exercises 4-3, and Problem 68 in the Chapter 4 Review Exercises.

EXAMPLE 5 Using Logarithmic Properties

Simplify, using the properties in Theorem 2.

(A) $\log_e 1$ (B) $\log_{10} 10$ (C) $\log_e e^{2x+1}$
(D) $\log_{10} 0.01$ (E) $10^{\log_{10} 7}$ (F) $e^{\log_e x^2}$

SOLUTIONS

(A) $\log_e 1 = 0$ Property 1
(B) $\log_{10} 10 = 1$ Property 2
(C) $\log_e e^{2x+1} = 2x + 1$ Property 3
(D) $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$ Property 3
(E) $10^{\log_{10} 7} = 7$ Property 4
(F) $e^{\log_e x^2} = x^2$ Property 4

MATCHED PROBLEM 5

Simplify, using the properties in Theorem 2.

(A) $\log_{10} 10^{-5}$ (B) $\log_5 25$ (C) $\log_{10} 1$
(D) $\log_e e^{m+n}$ (E) $10^{\log_{10} 4}$ (F) $e^{\log_e (x^4+1)}$

Common and Natural Logarithms

To work with logarithms effectively, we will need to be able to calculate (or at least approximate) the logarithms of any positive number to a variety of bases. Historically, tables were used for this purpose, but now calculators are used because they are faster and can find far more values than any table can possibly include.

Of all possible bases, there are two that are used most often. **Common logarithms** are logarithms with base 10. **Natural logarithms** are logarithms with base e. Most calculators have a function key labeled "log" and a function key labeled "ln." The former represents the common logarithmic function and the latter the natural logarithmic function. In fact, "log" and "ln" are both used in most math books, and whenever you see either used in this book without a base indicated, they should be interpreted as follows:

LOGARITHMIC FUNCTIONS

$y = \log x = \log_{10} x$ Common logarithmic function

$y = \ln x = \log_e x$ Natural logarithmic function

EXPLORE-DISCUSS 2

(A) Sketch the graph of $y = 10^x$, $y = \log x$, and $y = x$ in the same coordinate system and state the domain and range of the common logarithmic function.

(B) Sketch the graph of $y = e^x$, $y = \ln x$, and $y = x$ in the same coordinate system and state the domain and range of the natural logarithmic function.

EXAMPLE 6 Calculator Evaluation of Logarithms

Use a calculator to evaluate each to six decimal places.

(A) log 3,184 (B) ln 0.000 349 (C) log (−3.24)

SOLUTIONS

(A) log 3,184 = 3.502 973 (B) ln 0.000 349 = −7.960 439

(C) log (−3.24) = Error

Why is an error indicated in part C? Because −3.24 is not in the domain of the log function. [*Note:* Calculators display error messages in various ways. Some calculators use a more advanced definition of logarithmic functions that involves complex numbers. They will display an ordered pair, representing a complex number, as the value of log (−3.24), rather than an error message. You should interpret such a display as indicating that the number entered is not in the domain of the logarithmic function as we have defined it.]

MATCHED PROBLEM 6

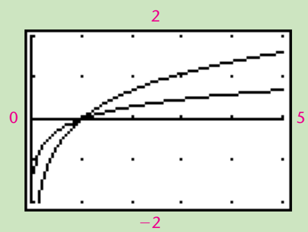
Use a calculator to evaluate each to six decimal places.

(A) log 0.013 529 (B) ln 28.693 28 (C) ln (−0.438)

When working with common and natural logarithms, we will follow the common practice of using the equal sign "=" where it might be technically correct to use the approximately equal sign "≈." No harm is done as long as we keep in mind that in a statement such as log 3.184 = 0.503, the number on the right is only assumed accurate to three decimal places and is not exact.

››› EXPLORE-DISCUSS 3

Graphs of the functions $f(x) = \log x$ and $g(x) = \ln x$ are shown in the graphing calculator display of Figure 3. Which graph belongs to which function? It appears from the display that one of the functions may be a constant multiple of the other. Is that true? Find and discuss the evidence for your answer.



› Figure 3

EXAMPLE 7 Calculator Evaluation of Logarithms

Use a calculator to evaluate each expression to three decimal places.

(A) $\dfrac{\log 2}{\log 1.1}$ (B) $\log \dfrac{2}{1.1}$ (C) $\log 2 - \log 1.1$

SOLUTIONS

(A) $\dfrac{\log 2}{\log 1.1} = 7.273$ Enter as (log 2) ÷ (log 1.1).

(B) $\log \dfrac{2}{1.1} = 0.260$ Enter as log (2 ÷ 1.1).

(C) $\log 2 - \log 1.1 = 0.260$. Note that $\dfrac{\log 2}{\log 1.1} \neq \log 2 - \log 1.1$, but $\log \dfrac{2}{1.1} = \log 2 - \log 1.1$ (see Theorem 2).

MATCHED PROBLEM 7

Use a calculator to evaluate each to three decimal places.

(A) $\dfrac{\ln 3}{\ln 1.08}$ (B) $\ln \dfrac{3}{1.08}$ (C) $\ln 3 - \ln 1.08$

We now turn to the second problem: Given the logarithm of a number, find the number. To solve this problem, we make direct use of the logarithmic–exponential relationships, and change logarithmic expressions into exponential form.

› LOGARITHMIC–EXPONENTIAL RELATIONSHIPS

$\log x = y$ is equivalent to $x = 10^y$

$\ln x = y$ is equivalent to $x = e^y$

EXAMPLE 8 Solving $\log_b x = y$ for x

Find x to three significant digits, given the indicated logarithms.

(A) $\log x = -9.315$ (B) $\ln x = 2.386$

SOLUTIONS

(A) $\log x = -9.315$ Change to exponential form (Definition 1).

$$x = 10^{-9.315}$$
$$= 4.84 \times 10^{-10}$$

Notice that the answer is displayed in scientific notation in the calculator.

(B) $\ln x = 2.386$ Change to exponential form (Definition 1).

$$x = e^{2.386}$$
$$= 10.9$$

MATCHED PROBLEM 8

Find x to four significant digits, given the indicated logarithms.

(A) $\ln x = -5.062$ (B) $\log x = 12.0821$

››› EXPLORE-DISCUSS 4

Example 8 was solved algebraically using logarithmic–exponential relationships. Use the INTERSECT command on a graphing calculator to solve this problem graphically. Discuss the relative merits of the two approaches.

› The Change-of-Base Formula

How would you find the logarithm of a positive number to a base other than 10 or e? For example, how would you find $\log_3 5.2$? In Example 9 we evaluate this logarithm using several properties of logarithms. Then we develop a change-of-base formula to find such logarithms more easily.

EXAMPLE 9 **Evaluating a Base 3 Logarithm**

Evaluate $\log_3 5.2$ to four decimal places.

SOLUTION

Let $y = \log_3 5.2$ and proceed as follows:

$$\log_3 5.2 = y \quad \text{Change to exponential form.}$$
$$5.2 = 3^y \quad \text{Apply the natural log (or common log) to each side.}$$
$$\ln 5.2 = \ln 3^y \quad \text{use } \log_b M^p = p \log_b M$$
$$\ln 5.2 = y \ln 3 \quad \text{Solve for } y.$$
$$y = \frac{\ln 5.2}{\ln 3}$$

Replace y with $\log_3 5.2$ from the first step, and use a calculator to evaluate the right side:

$$\log_3 5.2 = \frac{\ln 5.2}{\ln 3} = 1.5007$$

MATCHED PROBLEM 9

Evaluate $\log_{0.5} 0.0372$ to four decimal places.

If we repeat the process we used in Example 9 on a generic logarithm, something interesting happens. The goal is to evaluate $\log_b N$, where b is any acceptable base, and N is any positive real number. As in Example 9, let $y = \log_b N$.

$$\begin{aligned} \log_b N &= y && \text{Write in exponential form.} \\ N &= b^y && \text{Apply natural log to each side.} \\ \ln N &= \ln b^y && \text{use } \ln b^y = y \ln b \text{ (Property 8, Theorem 2).} \\ \ln N &= y \ln b && \text{Solve for } y. \\ y &= \frac{\ln N}{\ln b} \end{aligned}$$

This provides a formula for evaluating a logarithm to any base by using natural log:

$$\log_b N = \frac{\ln N}{\ln b}$$

We could also have used log base 10 rather than natural log, and developed an alternative formula:

$$\log_b N = \frac{\log N}{\log b}$$

In fact, the same approach would enable us to rewrite $\log_b N$ in terms of a logarithm with any base we choose!

› THE CHANGE-OF-BASE FORMULA

For any $b > 0$, $b \neq 1$, and any positive real number N,

$$\log_b N = \frac{\log_a N}{\log_a b}$$

where a is any positive number other than 1.

››› EXPLORE-DISCUSS 5

If b is any positive real number different from 1, the change-of-base formula implies that the function $y = \log_b x$ is a constant multiple of the natural logarithmic function; that is, $\log_b x = k \ln x$ for some k.

(A) Graph the functions $y = \ln x$, $y = 2 \ln x$, $y = 0.5 \ln x$, and $y = -3 \ln x$.

(B) Write each function of part A in the form $y = \log_b x$ by finding the base b to two decimal places.

(C) Is every exponential function $y = b^x$ a constant multiple of $y = e^x$? Explain.

ANSWERS TO MATCHED PROBLEMS

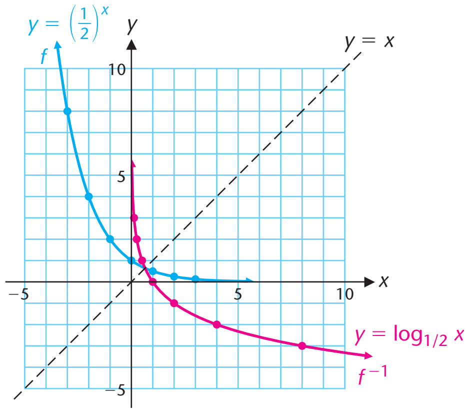
1.

f

x	$y = \left(\frac{1}{2}\right)^x$
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

f^{-1}

x	$y = \log_{1/2} x$
8	−3
4	−2
2	−1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{8}$	3



2. (A) $27 = 3^3$ (B) $6 = 36^{1/2}$ (C) $\frac{1}{9} = 3^{-2}$
3. (A) $\log_4 64 = 3$ (B) $\log_8 2 = \frac{1}{3}$ (C) $\log_4 (\frac{1}{16}) = -2$
4. (A) $y = \frac{3}{2}$ (B) $x = \frac{1}{8}$ (C) $b = 10$
5. (A) −5 (B) 2 (C) 0 (D) $m + n$ (E) 4 (F) $x^4 + 1$
6. (A) −1.868 734 (B) 3.356 663 (C) Not possible
7. (A) 14.275 (B) 1.022 (C) 1.022
8. (A) $x = 0.006\,333$ (B) $x = 1.208 \times 10^{12}$ **9.** 4.7486

4-3 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

1. Describe the relationship between logarithmic functions and exponential functions in your own words.

2. Explain why there are infinitely many different logarithmic functions.

3. Why are logarithmic functions undefined for zero and negative inputs?

4. Why is $\log_b 1 = 0$ for any base?

5. Explain how to calculate $\log_5 3$ on a calculator that only has log buttons for base 10 and base e.

6. Using the word "inverse," explain why $\log_b b^x = x$ for any x and any acceptable base b.

Rewrite Problems 7–12 in equivalent exponential form.

7. $\log_3 81 = 4$ $81 = 3^4$

8. $\log_5 125 = 3$ $125 = 5^3$

9. $\log_{10} 0.001 = -3$ $0.001 = 10^{-3}$

10. $\log_{10} 1{,}000 = 3$ $1{,}000 = 10^3$

11. $\log_6 \frac{1}{36} = -2$ $\frac{1}{36} = 6^{-2}$

12. $\log_2 \frac{1}{64} = -6$ $\frac{1}{64} = 2^{-6}$

Rewrite Problems 13–18 in equivalent logarithmic form.

13. $8 = 4^{3/2}$ $\log_4 8 = \frac{3}{2}$

14. $9 = 27^{2/3}$ $\log_{27} 9 = \frac{2}{3}$

15. $\frac{1}{2} = 32^{-1/5}$ $\log_{32} \frac{1}{2} = -\frac{1}{5}$

16. $\frac{1}{8} = 2^{-3}$ $\log_2 \frac{1}{8} = -3$

17. $(\frac{2}{3})^3 = \frac{8}{27}$ $\log_{2/3} \frac{8}{27} = 3$

18. $(\frac{5}{2})^{-2} = 0.16$ $\log_{5/2} 0.16 = -2$

In Problems 19–22, make a table of values similar to the one in Example 1, then use it to graph both functions by hand.

19. $f(x) = 3^x$ $f^{-1}(x) = \log_3 x$

20. $f(x) = (\frac{1}{3})^x$ $f^{-1}(x) = \log_{1/3} x$

21. $f(x) = (\frac{2}{3})^x$ $f^{-1}(x) = \log_{2/3} x$

22. $f(x) = 10^x$ $f^{-1}(x) = \log x$

In Problems 23–38, simplify each expression using Theorem 2.

23. $\log_{16} 1$ 0

24. $\log_{25} 1$ 0

25. $\log_{0.5} 0.5$ 1

26. $\log_7 7$ 1

27. $\log_e e^4$ 4

28. $\log_{10} 10^5$ 5

29. $\log_{10} 0.01$ −2

30. $\log_{10} 100$ 2

31. $\log_3 27$ 3

32. $\log_4 256$ 4

33. $\log_{1/2} 2$ −1

34. $\log_{1/5} (\frac{1}{25})$ 2

35. $e^{\log_e 5}$ 5

36. $e^{\log_e 10}$ 10

37. $\log_5 \sqrt[3]{5}$ $\frac{1}{3}$

38. $\log_2 \sqrt{8}$ $\frac{3}{2}$

In Problems 39–46, evaluate to four decimal places.

39. $\log 49{,}236$ 4.6923

40. $\log 691{,}450$ 5.8398

41. $\ln 54.081$ 3.9905

42. $\ln 19.722$ 2.9817

43. $\log_7 13$ 1.3181

44. $\log_9 78$ 1.9828

45. $\log_5 120.24$ 2.9759

46. $\log_{17} 304.66$ 2.0186

In Problems 47–54, evaluate x to four significant digits.

47. $\log x = 5.3027$ 200,800

48. $\log x = 1.9168$ 82.57

49. $\log x = -3.1773$ 0.0006648

50. $\log x = -2.0411$ 0.009097

51. $\ln x = 3.8655$ 47.73

52. $\ln x = 5.0884$ 162.1

53. $\ln x = -0.3916$ 0.6760

54. $\ln x = -4.1083$ 0.01644

B *Find x, y, or b, as indicated in Problems 55–72.*

55. $\log_2 x = 2$ $x = 4$

56. $\log_3 x = 3$ $x = 27$

57. $\log_4 16 = y$ $y = 2$

58. $\log_8 64 = y$ $y = 2$

59. $\log_b 16 = 2$ $b = 4$

60. $\log_b 10^{-3} = -3$ $b = 10$

61. $\log_b 1 = 0$ b is any positive real number except 1.

62. $\log_b b = 1$ b is any positive real number except 1.

63. $\log_4 x = \frac{1}{2}$ $x = 2$

64. $\log_8 x = \frac{1}{3}$ $x = 2$

65. $\log_{1/3} 9 = y$ $y = -2$

66. $\log_{49} (\frac{1}{7}) = y$ $y = -\frac{1}{2}$

67. $\log_b 1{,}000 = \frac{3}{2}$ $b = 100$

68. $\log_b 4 = \frac{2}{3}$ $b = 8$

69. $\log_8 x = -\frac{4}{3}$ $x = \frac{1}{16}$

70. $\log_{25} x = -\frac{3}{2}$ $x = \frac{1}{125}$

71. $\log_{16} 8 = y$ $y = \frac{3}{4}$

72. $\log_9 27 = y$ $y = \frac{3}{2}$

In Problems 73–78, evaluate to three decimal places.

73. $\dfrac{\log 2}{\log 1.15}$ 4.959

74. $\dfrac{\log 2}{\log 1.12}$ 6.116

75. $\dfrac{\ln 3}{\ln 1.15}$ 7.861

76. $\dfrac{\ln 4}{\ln 1.2}$ 7.604

77. $\dfrac{\ln 150}{2 \ln 3}$ 2.280

78. $\dfrac{\log 200}{3 \log 2}$ 2.548

In Problems 79–82, rewrite the expression in terms of log x and log y.

79. $\log \left(\dfrac{x}{y}\right)$

80. $\log (xy)$

81. $\log (x^4 y^3)$

82. $\log \left(\dfrac{x^2}{\sqrt{y}}\right)$

In Problems 83–86, rewrite the expression as a single log.

83. $\ln x - \ln y$ **84.** $\log_3 x + \log_3 y$

85. $2 \ln x + 5 \ln y - \ln z$ **86.** $\log a - 2 \log b + 3 \log c$

In Problems 87–90, given that $\log x = -2$ and $\log y = 3$, find:

87. $\log (xy)$ **88.** $\log\left(\frac{x}{y}\right)$ **89.** $\log\left(\frac{\sqrt{x}}{y^3}\right)$ **90.** $\log (x^5 y^3)$

C *In Problems 91–98, use transformations to explain how the graph of g is related to the graph of the given logarithmic function f. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.*

91. $g(x) = 3 + \log_2 x; f(x) = \log_2 x$

92. $g(x) = -4 + \log_3 x; f(x) = \log_3 x$

93. $g(x) = \log_{1/3} (x - 2); f(x) = \log_{1/3} x$

94. $g(x) = \log_{1/2} (x + 3); f(x) = \log_{1/2} x$

95. $g(x) = -1 - \log x; f(x) = \log x$

96. $g(x) = 2 - \log x; f(x) = \log x$

97. $g(x) = 5 - 3 \ln x; f(x) = \ln x$

98. $g(x) = -3 - 2 \ln x; f(x) = \ln x$

In Problems 99–102, find f^{-1}.

99. $f(x) = \log_5 x$ $f^{-1}(x) = 5^x$ **100.** $f(x) = \log_{1/3} x$ $f^{-1}(x) = 3^{-x}$

101. $f(x) = 4 \log_3 (x + 3)$ $f^{-1}(x) = 3^{x/4} - 3$ **102.** $f(x) = 2 \log_2 (x - 5)$ $f^{-1}(x) = 2^{x/2} + 5$

103. Let $f(x) = \log_3 (2 - x)$.
(A) Find f^{-1}. (B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

104. Let $f(x) = \log_2 (-3 - x)$.
(A) Find f^{-1}. (B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

105. What is wrong with the following "proof" that 3 is less than 2?

$$
\begin{aligned}
1 &< 3 && \text{Divide both sides by 27.}\\
\tfrac{1}{27} &< \tfrac{3}{27}\\
\tfrac{1}{27} &< \tfrac{1}{9}\\
(\tfrac{1}{3})^3 &< (\tfrac{1}{3})^2\\
\log (\tfrac{1}{3})^3 &< \log (\tfrac{1}{3})^2\\
3 \log \tfrac{1}{3} &< 2 \log \tfrac{1}{3} && \text{Divide both sides by } \log \tfrac{1}{3}.\\
3 &< 2
\end{aligned}
$$

106. What is wrong with the following "proof" that 1 is greater than 2?

$$
\begin{aligned}
3 &> 2 && \text{Multiply both sides by } \log \tfrac{1}{2}.\\
3 \log \tfrac{1}{2} &> 2 \log \tfrac{1}{2}\\
\log (\tfrac{1}{2})^3 &> \log (\tfrac{1}{2})^2\\
(\tfrac{1}{2})^3 &> (\tfrac{1}{2})^2\\
\tfrac{1}{8} &> \tfrac{1}{4} && \text{Multiply both sides by 8.}\\
1 &> 2
\end{aligned}
$$

The polynomials in Problems 107–110, called ***Taylor polynomials,*** *can be used to approximate the function $g(x) = \ln(1 + x)$. To illustrate this approximation graphically, in each problem, graph $g(x) = \ln(1 + x)$ and the indicated polynomial in the same viewing window, $-1 \le x \le 3$ and $-2 \le y \le 2$.*

107. $P_1(x) = x - \frac{1}{2}x^2$ **108.** $P_2(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

109. $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

110. $P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

111. Prove that for any positive M, N, and b $(b \ne 1)$, $\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$. (*Hint:* Start by writing $u = \log_b M$ and $v = \log_b N$ and changing each to exponential form.)

112. Prove that for any positive integer p and any positive b and M $(b \ne 1)$, $\log_b M^p = p \log_b M$. [*Hint:* Write M^p as $M \cdot M \cdots M$ (p factors).]

4-4 Logarithmic Models

› Logarithmic Scales
› Data Analysis and Regression

Logarithmic functions occur naturally as the inverses of exponential functions. But that's not to say that they are not useful in their own right. Some of these uses are probably familiar to you, but you might not have realized that they involved logarithmic functions.

In this section, we will study logarithmic scales that are used to compare the intensity of sounds, the severity of earthquakes, and the brightness of distant stars. We will also look at using regression to model data with a logarithmic function, and discuss what sort of data is likely to fit such a model.

› Logarithmic Scales

SOUND INTENSITY: The human ear is able to hear sound over an incredible range of intensities. The loudest sound a healthy person can hear without damage to the eardrum has an intensity 1 trillion (1,000,000,000,000) times that of the softest sound a person can hear. If we were to use these intensities as a scale for measuring volume, we would be stuck using numbers from zero all the way to the trillions, which seems cumbersome, if not downright silly. In the last section, we saw that logarithmic functions increase very slowly. We can take advantage of this to create a scale for sound intensity that is much more condensed, and therefore more manageable.

The decibel scale for sound intensity is an example of such a scale. The **decibel,** named after the inventor of the telephone, Alexander Graham Bell (1847–1922), is defined as follows:

$$D = 10 \log \frac{I}{I_0} \qquad \text{Decibel scale} \qquad (1)$$

where D is the **decibel level** of the sound, I is the **intensity** of the sound measured in watts per square meter (W/m^2), and I_0 is the intensity of the least audible sound that an average healthy young person can hear. The latter is standardized to be $I_0 = 10^{-12}$ watts per square meter. Table 1 lists some typical sound intensities from familiar sources. In Example 1 and problems 1 and 2 in Exercises 4, we will calculate the decibel levels for these sounds.

Table 1 Typical Sound Intensities

Sound intensity (W/m^2)	Sound
1.0×10^{-12}	Threshold of hearing
5.2×10^{-10}	Whisper
3.2×10^{-6}	Normal conversation
8.5×10^{-4}	Heavy traffic
3.2×10^{-3}	Jackhammer
1.0×10^{0}	Threshold of pain
8.3×10^{2}	Jet plane with afterburner

EXAMPLE 1 Sound Intensity

(A) Find the number of decibels from a whisper with sound intensity 5.20×10^{-10} watts per square meter, then from heavy traffic at 8.5×10^{-4} watts per square meter. Round your answers to two decimal places.

(B) How many times larger is the sound intensity of heavy traffic compared to a whisper?

SOLUTIONS

(A) We can use the decibel formula (1) with $I_0 = 10^{-12}$. First, we use $I = 5.2 \times 10^{-10}$:

$$\begin{aligned} D &= 10 \log \frac{I}{I_0} && \text{Substitute } I = 5.2 \times 10^{-10}, I_0 = 10^{-12}. \\ &= 10 \log \frac{5.2 \times 10^{-10}}{10^{-12}} && \text{Simplify the fraction.} \\ &= 10 \log 520 \\ &= 27.16 \text{ decibels} \end{aligned}$$

Next, for heavy traffic:

$$\begin{aligned} D &= 10 \log \frac{I}{I_0} && \text{Substitute } I = 8.5 \times 10^{-4}, I_0 = 10^{-12}. \\ &= 10 \log \frac{8.5 \times 10^{-4}}{10^{-12}} && \text{Simplify the fraction.} \\ &= 10 \log 850{,}000{,}000 \\ &= 89.29 \text{ decibels} \end{aligned}$$

(B) Dividing the larger intensity by the smaller,

$$\frac{8.5 \times 10^{-4}}{5.2 \times 10^{-10}} = 1{,}634{,}615.4$$

we see that the sound intensity of heavy traffic is more than 1.6 million times as great as the intensity of a whisper!

MATCHED PROBLEM 1

Find the number of decibels from a jackhammer with sound intensity 3.2×10^{-3} watts per square meter. Compute the answer to two decimal places.

››› EXPLORE-DISCUSS 1

Suppose that you are asked to draw a graph of the data in Table 1, with sound intensities on the x axis, and the corresponding decibel levels on the y axis.

(A) What would be the coordinates of the point corresponding to a jack-hammer (see Matched Problem 1)?

(B) Suppose the axes of this graph are labeled as follows: Each tick mark on the x axis corresponds to the intensity of the least audible sound (10^{-12} watts per square meter), and each tick mark on the y axis corresponds to 1 decibel. If there is $\frac{1}{8}$ inch between all tick marks, how far away from the x axis is the point you found in part A? From the y axis? (Give the first answer in inches and the second in miles!) Discuss your result.

EARTHQUAKE INTENSITY: The energy released by the largest earthquake recorded, measured in joules, is about 100 billion (100,000,000,000) times the energy released by a small earthquake that is barely felt. Over the past 150 years several people from various countries have devised different types of measures of earthquake magnitudes so that their severity could be compared without using tremendously large numbers. In 1935 the California seismologist Charles Richter devised a logarithmic scale that bears his name and is still widely used in the United States. The **magnitude** of an earthquake M on the **Richter scale*** is given as follows:

$$M = \frac{2}{3} \log \frac{E}{E_0} \quad \text{Richter scale} \tag{2}$$

where E is the energy released by the earthquake, measured in joules, and E_0 is the energy released by a very small reference earthquake, which has been standardized to be

$$E_0 = 10^{4.40} \text{ joules}$$

The destructive power of earthquakes relative to magnitudes on the Richter scale is indicated in Table 2.

Table 2 The Richter Scale

Magnitude on Richter scale	Destructive power
$M < 4.5$	Small
$4.5 < M < 5.5$	Moderate
$5.5 < M < 6.5$	Large
$6.5 < M < 7.5$	Major
$7.5 < M$	Greatest

*Originally, Richter defined the magnitude of an earthquake in terms of logarithms of the maximum seismic wave amplitude, in thousandths of a millimeter, measured on a standard seismograph. Formula (2) gives essentially the same magnitude that Richter obtained for a given earthquake but in terms of logarithms of the energy released by the earthquake.

EXAMPLE 2 Earthquake Intensity

The 1906 San Francisco earthquake released approximately 5.96×10^{16} joules of energy. Another quake struck the Bay Area just before game 3 of the 1989 World Series, releasing 1.12×10^{15} joules of energy.

(A) Find the magnitude of each earthquake on the Richter scale. Round your answers to two decimal places.

(B) How many times more energy did the 1906 earthquake release than the one in 1989?

SOLUTIONS

(A) We can use the magnitude formula (2) with $E_0 = 10^{4.40}$. First, for the 1906 earthquake, we use $E = 5.96 \times 10^{16}$:

$$
\begin{aligned}
M &= \frac{2}{3} \log \frac{E}{E_0} && \text{Substitute } E = 5.96 \times 10^{16}, E_0 = 10^{4.40}. \\
&= \frac{2}{3} \log \frac{5.96 \times 10^{16}}{10^{4.40}} \\
&= 8.25
\end{aligned}
$$

Next, for the 1989 earthquake:

$$
\begin{aligned}
M &= \frac{2}{3} \log \frac{E}{E_0} && \text{Substitute } E = 1.12 \times 10^{15}, E_0 = 10^{4.40}. \\
&= \frac{2}{3} \log \frac{1.12 \times 10^{15}}{10^{4.40}} \\
&= 7.1
\end{aligned}
$$

(B) Dividing the larger energy release by the smaller,

$$
\frac{5.96 \times 10^{16}}{1.12 \times 10^{15}} = 53.2
$$

we see that the 1906 earthquake released 53.2 times as much energy as the 1989 quake.

MATCHED PROBLEM 2

The 1985 earthquake in central Chile released approximately 1.26×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to two decimal places.

EXAMPLE 3 Earthquake Intensity

If the energy release of one earthquake is 1,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

SOLUTION

Let

$$M_1 = \frac{2}{3}\log\frac{E_1}{E_0} \qquad \text{and} \qquad M_2 = \frac{2}{3}\log\frac{E_2}{E_0}$$

be the Richter equations for the smaller and larger earthquakes, respectively. Since the larger earthquake released 1,000 times as much energy, we can write $E_2 = 1{,}000E_1$.

$$M_2 = \frac{2}{3}\log\frac{E_2}{E_0}$$ Substitute $1{,}000E_1$ for E_2.

$$= \frac{2}{3}\log\frac{1{,}000E_1}{E_0}$$ Use $\log(MN) = \log M + \log N$.

$$= \frac{2}{3}\left(\log 1{,}000 + \log\frac{E_1}{E_0}\right)$$ $\log 1{,}000 = \log 10^3 = 3$

$$= \frac{2}{3}\left(3 + \log\frac{E_1}{E_0}\right)$$ Distribute.

$$= \frac{2}{3}(3) + \frac{2}{3}\log\frac{E_1}{E_0}$$ $\frac{2}{3}\log\frac{E_1}{E_0}$ is M_1!

$$= 2 + M_1$$

Thus, an earthquake with 1,000 times the energy of another has a Richter scale reading of 2 more than the other.

MATCHED PROBLEM 3

If the energy release of one earthquake is 10,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

ROCKET FLIGHT: The theory of rocket flight uses advanced mathematics and physics to show that the **velocity** v of a rocket at burnout (depletion of fuel supply) is given by

$$v = c\ln\frac{W_t}{W_b} \qquad \textbf{Rocket equation} \qquad (3)$$

where c is the exhaust velocity of the rocket engine, W_t is the takeoff weight (fuel, structure, and payload), and W_b is the burnout weight (structure and payload).

Because of the Earth's atmospheric resistance, a launch vehicle velocity of at least 9.0 kilometers per second is required to achieve the minimum altitude needed for a stable orbit. Formula (3) indicates that to increase velocity v, either the weight ratio W_t/W_b must be increased or the exhaust velocity c must be increased. The weight ratio can be increased by the use of solid fuels, and the exhaust velocity can be increased by improving the fuels, solid or liquid.

EXAMPLE 4

Rocket Flight Theory

A typical single-stage, solid-fuel rocket may have a weight ratio $W_t/W_b = 18.7$ and an exhaust velocity $c = 2.38$ kilometers per second. Would this rocket reach a launch velocity of 9.0 kilometers per second?

SOLUTION

We can use the rocket equation (3) with $c = 2.38$ and $\frac{W_t}{W_b} = 18.7$:

$$\begin{aligned} v &= c \ln \frac{W_t}{W_b} \\ &= 2.38 \ln 18.7 \\ &= 6.97 \text{ kilometers per second} \end{aligned}$$

The velocity of the launch vehicle is far short of the 9.0 kilometers per second required to achieve orbit. This is why multiple-stage launchers are used—the deadweight from a preceding stage can be jettisoned into the ocean when the next stage takes over.

MATCHED PROBLEM 4

A launch vehicle using liquid fuel, such as a mixture of liquid hydrogen and liquid oxygen, can produce an exhaust velocity of $c = 4.7$ kilometers per second. However, the weight ratio W_t/W_b must be low—around 5.5 for some vehicles—because of the increased structural weight to accommodate the liquid fuel. How much more or less than the 9.0 kilometers per second required to reach orbit will be achieved by this vehicle?

› Data Analysis and Regression

Based on the logarithmic graphs we studied in the last section, when a quantity increases relatively rapidly at first, but then levels off and increases very slowly, it might be a good candidate to be modeled by a logarithmic function. Most graphing calculators with regression commands can fit functions of the form $y = a + b \ln x$ to a set of data points using the same techniques we used earlier for other types of regression.

EXAMPLE 5 Home Ownership Rates

Table 3 Home Ownership Rates

Year	Home ownership rate (%)
1940	43.6
1950	55.0
1960	61.9
1970	62.9
1980	64.4
1990	64.2
2000	67.4

The U.S. Census Bureau published the data in Table 3 on home ownership rates.

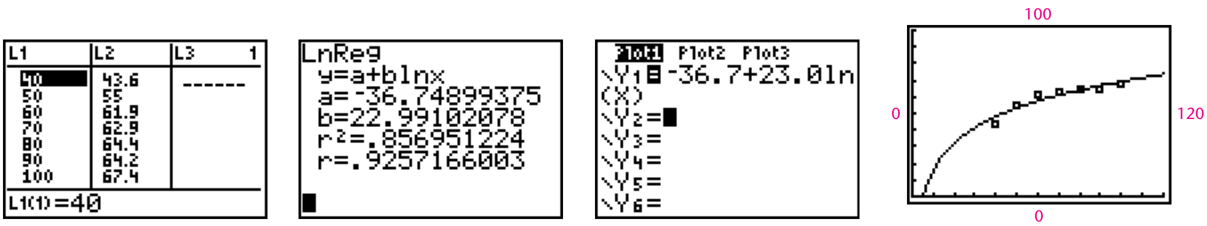
(A) Let x represent time in years with $x = 0$ representing 1900, and let y represent the corresponding home ownership rate. Use regression analysis on a graphing calculator to find a logarithmic function of the form $y = a + b \ln x$ that models the data. (Round the constants a and b to three significant digits.)

(B) Use your logarithmic function to predict the home ownership rate in 2010.

SOLUTIONS

(A) Figure 1 shows the details of constructing the model on a graphing calculator.

(B) The year 2010 corresponds to $x = 110$. Evaluating $y_1 = -36.7 + 23.0 \ln x$ at $x = 110$ predicts a home ownership rate of 71.4% in 2010.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

Figure 1

MATCHED PROBLEM 5

Refer to Example 5. The home ownership rate in 1995 was 64.7%.

(A) Find a logarithmic regression equation for the expanded data set.

(B) Predict the home ownership rate in 2010.

ANSWERS TO MATCHED PROBLEMS

1. 95.05 decibels **2.** 7.80 **3.** 2.67 **4.** 1 kilometer per second less

5. (A) $-31.5 + 21.7 \ln x$ (B) 70.5%

4-4 Exercises

**Additional answers can be found in the Instructor Answer Appendix.*

1. Describe the decibel scale in your own words.

2. Describe the Richter scale in your own words.

3. Explain why logarithms are a good choice for describing sound intensity and earthquake magnitude.

4. Think of a real-life quantity that is likely to be modeled well by a logarithmic function, and explain your reasoning.

APPLICATIONS

5. SOUND What is the decibel level of
(A) The threshold of hearing, 1.0×10^{-12} watts per square meter? 0 decibels
(B) The threshold of pain, 1.0 watt per square meter?
Compute answers to two significant digits. 120 decibels

6. SOUND What is the decibel level of
(A) A normal conversation, 3.2×10^{-6} watts per square meter? 65 decibels
(B) A jet plane with an afterburner, 8.3×10^{2} watts per square meter? 150 decibels
Compute answers to two significant digits.

7. SOUND If the intensity of a sound from one source is 1,000 times that of another, how much more is the decibel level of the louder sound than the quieter one? 30 decibels

8. SOUND If the intensity of a sound from one source is 10,000 times that of another, how much more is the decibel level of the louder sound than the quieter one? 40 decibels

9. EARTHQUAKES One of the strongest recorded earthquakes to date was in Colombia in 1906, with an energy release of 1.99×10^{17} joules. What was its magnitude on the Richter scale? Compute the answer to one decimal place. 8.6

10. EARTHQUAKES Anchorage, Alaska, had a major earthquake in 1964 that released 7.08×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to one decimal place. 8.3

11. EARTHQUAKES The 1933 Long Beach, California, earthquake had a Richter scale reading of 6.3, and the 1964 Anchorage, Alaska, earthquake had a Richter scale reading of 8.3. How many times more powerful was the Anchorage earthquake than the Long Beach earthquake? 1,000 times as powerful

12. EARTHQUAKES Generally, an earthquake requires a magnitude of over 5.6 on the Richter scale to inflict serious damage. How many times more powerful than this was the great 1906 Colombia earthquake, which registered a magnitude of 8.6 on the Richter scale? 32,000 times as powerful

13. EXPLOSIVE ENERGY The atomic bomb dropped on Nagasaki, Japan, on August 9, 1945, released about 1.34×10^{14} joules of energy. What would be the magnitude of an earthquake that released that much energy? 6.5

14. EXPLOSIVE ENERGY The largest and most powerful nuclear weapon ever detonated was tested by the Soviet Union on October 30, 1961, on an island in the Arctic Sea. The blast was so powerful there were reports of windows breaking in Finland, over 700 miles away. The detonation released about 2.1×10^{17} joules of energy. What would be the magnitude of an earthquake that released that much energy? 8.6

15. ASTRONOMY A moderate-size solar flare observed on the sun on July 9, 1996, released enough energy to power the United States for almost 23,000 years at 2001 consumption levels, 2.38×10^{21} joules. What would be the magnitude of an earthquake that released that much energy? 11.3

16. CONSTRUCTION The energy released by a typical construction site explosion is about 7.94×10^{5} joules. What would be the magnitude of an earthquake that released that much energy? 1.0

17. SPACE VEHICLES A new solid-fuel rocket has a weight ratio $W_t/W_b = 19.8$ and an exhaust velocity $c = 2.57$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places. 7.67 km/s

18. SPACE VEHICLES A liquid-fuel rocket has a weight ratio $W_t/W_b = 6.2$ and an exhaust velocity $c = 5.2$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places. 9.49 km/s

19. CHEMISTRY The hydrogen ion concentration of a substance is related to its acidity and basicity. Because hydrogen ion concentrations vary over a very wide range, logarithms are used to create a compressed **pH scale,** which is defined as follows:

$$\text{pH} = -\log\,[\text{H}^+]$$

where $[\text{H}^+]$ is the hydrogen ion concentration, in moles per liter. Pure water has a pH of 7, which means it is neutral. Substances with a pH less than 7 are acidic, and those with a pH

greater than 7 are basic. Compute the pH of each substance listed, given the indicated hydrogen ion concentration. Also, indicate whether each substance is acidic or basic. Compute answers to one decimal place.
(A) Seawater, 4.63×10^{-9} 8.3, basic
(B) Vinegar, 9.32×10^{-4} 3.0, acidic

20. CHEMISTRY Refer to Problem 19. Compute the pH of each substance below, given the indicated hydrogen ion concentration. Also, indicate whether it is acidic or basic. Compute answers to one decimal place.
(A) Milk, 2.83×10^{-7} 6.5 acidic
(B) Garden mulch, 3.78×10^{-6} 5.4 acidic

21. ECOLOGY Refer to Problem 19. Many lakes in Canada and the United States will no longer sustain some forms of wildlife because of the increase in acidity of the water from acid rain and snow caused by sulfur dioxide emissions from industry. If the pH of a sample of rainwater is 5.2, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits. 6.3×10^{-6} moles per liter

22. ECOLOGY Refer to Problem 19. If normal rainwater has a pH of 5.7, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits. 2×10^{-6} moles per liter

23. ASTRONOMY The brightness of stars is expressed in terms of magnitudes on a numerical scale that increases as the brightness decreases. The magnitude m is given by the formula

$$m = 6 - 2.5 \log \frac{L}{L_0}$$

where L is the light flux of the star and L_0 is the light flux of the dimmest stars visible to the naked eye.
(A) What is the magnitude of the dimmest stars visible to the naked eye? $m = 6$
(B) How many times brighter is a star of magnitude 1 than a star of magnitude 6? 100 times brighter

24. ASTRONOMY An optical instrument is required to observe stars beyond the sixth magnitude, the limit of ordinary vision. However, even optical instruments have their limitations. The limiting magnitude L of any optical telescope with lens diameter D, in inches, is given by

$$L = 8.8 + 5.1 \log D$$

(A) Find the limiting magnitude for a homemade 6-inch reflecting telescope. 12.8
(B) Find the diameter of a lens that would have a limiting magnitude of 20.6. 206 in.
Compute answers to three significant digits.

25. AGRICULTURE Table 4 shows the yield (bushels per acre) and the total production (millions of bushels) for corn in the United States for selected years since 1950. Let x represent years since 1900.

Table 4 United States Corn Production

Year	Yield (bushels per acre)	Total production (million bushels)
1950	37.6	2,782
1960	55.6	3,479
1970	81.4	4,802
1980	97.7	6,867
1990	115.6	7,802
2000	137.0	9,915

Source: U.S. Department of Agriculture

(A) Find a logarithmic regression model ($y = a + b \ln x$) for the yield. Estimate (to one decimal place) the yield in 2003 and in 2010.
(B) The actual yield in 2003 was 142 bushels per acre. How does this compare with the estimated yield in part A? What effect with this additional 2003 information have on the estimate for 2010? Explain.

26. AGRICULTURE Refer to Table 4.
(A) Find a logarithmic regression model ($y = a + b \ln x$) for the total production. Estimate (to the nearest million) the production in 2003 and in 2010.
(B) The actual production in 2003 was 10,114 million bushels. How does this compare with the estimated production in part A? What effect will this 2003 production information have on the estimate for 2010? Explain.

4-5 Exponential and Logarithmic Equations

- Solving Exponential Equations
- Solving Logarithmic Equations

When quantities are modeled by exponential or logarithmic functions, it's not a surprise that solving equations involving expressions of these types is useful in studying those quantities.

Equations involving exponential and logarithmic functions, such as

$$2^{3x-2} = 5 \qquad \text{and} \qquad \log(x + 3) + \log x = 1$$

are called **exponential** and **logarithmic equations,** respectively. The properties of logarithms that we studied in Section 4-3 play a central role in their solution. Of course, a graphing calculator can be used to find approximate solutions for many exponential and logarithmic equations. However, there are situations in which the algebraic solution is necessary. In this section, we will emphasize algebraic solutions, but will still consider graphical solutions in many cases.

Solving Exponential Equations

The distinguishing feature of exponential equations is that the variable appears in an exponent. Before defining logarithms, we didn't have a reliable method for removing variables from an exponent: Now we do. To illustrate the idea, we return to the equation we considered at the beginning of Section 4-3, $3^x = 20$.

EXAMPLE 1 Solving an Exponential Equation

Solve $3^x = 20$. Round your answer to four decimal places.

SOLUTIONS

Algebraic Solution

The key is to apply a logarithmic function to each side, then use one of the properties of logs from Section 4-3.

$3^x = 20$	Apply common or natural log to both sides.
$\ln 3^x = \ln 20$	Use $\log_b N^p = p \log_b N$.
$x \ln 3 = \ln 20$	Solve for x.
$x = \dfrac{\ln 20}{\ln 3} \approx 2.7268$	

Notice that the solution is between 2 and 3, as we surmised at the beginning of Section 4-3 (since $3^2 = 9$ and $3^3 = 27$).

Graphical Solution

Graph $y_1 = 3^x$ and $y_2 = 20$ and use the INTERSECT command (Fig. 1).

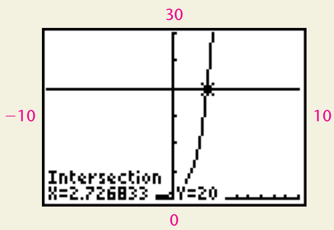


Figure 1 $y_1 = 3^x, y_2 = 20$.

The solution is $x = 2.7268$ to four decimal places.

MATCHED PROBLEM 1

Solve $5^x = 30$. Round your answer to four decimal places.

In Example 1, the choice of natural log to apply to both sides of the equation was unimportant. We could have chosen common log, or really log with any base. We'll usually choose either natural or common log because those are easiest to compute using a calculator.

In Example 2, we will use the technique of Example 1 on a slightly more complicated equation.

EXAMPLE 2 Solving an Exponential Equation

Solve $2^{3x-2} = 5$ for x to four decimal places.

SOLUTIONS

Algebraic Solution

Again, we will use logs to get x out of the exponent.

$$2^{3x-2} = 5 \quad \text{Take the common or natural log of both sides.}$$

$$\log 2^{3x-2} = \log 5 \quad \text{Use } \log_b N^p = p \log_b N \text{ to get } 3x-2 \text{ out of the exponent position.}$$

$$(3x-2)\log 2 = \log 5 \quad \text{Solve.}$$

$$3x - 2 = \frac{\log 5}{\log 2} \quad \textit{Remember: } \frac{\log 5}{\log 2} \neq \log 5 - \log 2.$$

$$3x = 2 + \frac{\log 5}{\log 2} \quad \text{Multiply both sides by } \tfrac{1}{3}.$$

$$x = \frac{1}{3}\left(2 + \frac{\log 5}{\log 2}\right)$$

$$= 1.4406 \quad \text{To four decimal places.}$$

Graphical Solution

Graph $y_1 = 2^{3x-2}$ and $y_2 = 5$ and use the INTERSECT command (Fig. 2).

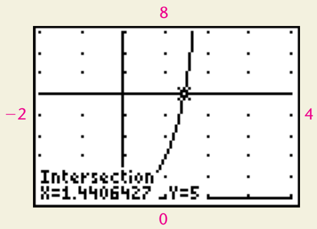


Figure 2 $y_1 = 2^{3x-2}, y_2 = 5.$

MATCHED PROBLEM 2

Solve $35^{1-2x} = 7$ for x to four decimal places.

Being able to solve exponential equations comes in handy when working with quantities that can be modeled with exponential functions.

EXAMPLE 3 Compound Interest

A certain amount of money P (principal) is invested at an annual rate r compounded annually. The amount of money A in the account after t years, assuming no withdrawals, is given by

$$A = P\left(1 + \frac{r}{m}\right)^n = P(1 + r)^n \quad m = 1 \text{ for annual compounding}$$

How many years to the nearest year will it take the money to double if it is invested at 6% compounded annually?

SOLUTIONS

Algebraic Solution
We don't know the original amount, so we'll have to just use P to represent it. We can substitute $r = 0.6$ to get

$$A = P(1.06)^n$$

We are asked to find the number of years (n) when the amount (A) equals twice the original amount ($2P$). So we substitute $2P$ for A and solve for n.

$$2P = P(1.06)^n \quad \text{Divide both sides by } P.$$

$$2 = 1.06^n \quad \text{Take the common or natural log of both sides.}$$

$$\log 2 = \log 1.06^n \quad \text{Note how log properties are used to get } n \text{ out of the exponent position.}$$

$$\log 2 = n \log 1.06 \quad \text{Solve for } n.$$

$$n = \frac{\log 2}{\log 1.06}$$

$$= 12 \text{ years} \quad \text{To the nearest year.}$$

Graphical Solution
From the first part of the algebraic solution, we need to solve the equation $2 = 1.06^n$.

Graph $y_1 = 1.06^x$ and $y_2 = 2$ and use the INTERSECT command (Fig. 3).

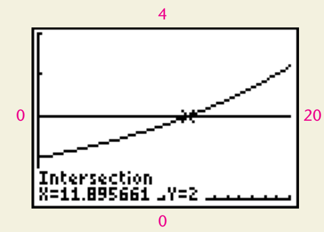


Figure 3 $y_1 = 1.06^x$, $y_2 = 2$.

The solution (rounded to the nearest year) is 12.

MATCHED PROBLEM 3

Repeat Example 3, changing the interest rate to 9% compounded annually.

››› CAUTION ›››

When solving exponential equations, it is crucial to first isolate the exponential expression before applying a log function to each side. [In Example 3, this entailed dividing both sides by P to isolate the exponential expression $(1.06)^n$.]

EXAMPLE 4 Atmospheric Pressure

The atmospheric pressure P, in pounds per square inch, at x miles above sea level is given approximately by

$$P = 14.7e^{-0.21x}$$

At what height will the atmospheric pressure be half the sea-level pressure? Compute the answer to two significant digits.

SOLUTIONS

Algebraic Solution

Since x represents miles above sea level, sea-level pressure is the pressure at $x = 0$:

$$P = 14.7e^{0} = 14.7$$

One-half of sea-level pressure is $14.7/2 = 7.35$. Now our problem is to find x so that $P = 7.35$; that is, we solve $7.35 = 14.7e^{-0.21x}$ for x:

$$7.35 = 14.7e^{-0.21x}$$ Divide both sides by 14.7 to isolate the exponential expression.

$$0.5 = e^{-0.21x}$$ Because the base is e, take the natural log of both sides.

$$\ln 0.5 = \ln e^{-0.21x}$$ Use the property $\ln e^{a} = a$.

$$\ln 0.5 = -0.21x$$ Solve for x.

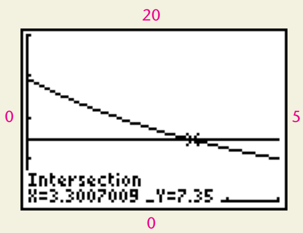
$$x = \frac{\ln 0.5}{-0.21}$$

$$= 3.3 \text{ miles}$$ To two significant digits.

Graphical Solution

From the first part of the algebraic solution, we need to solve $7.35 = 14.7e^{-0.21x}$.

Graph $y_1 = 14.7e^{-0.21x}$ and $y_2 = 7.35$ and use the INTERSECT command (Fig. 4).



› **Figure 4** $y_1 = 14.7e^{-0.21x}$, $y_2 = 7.35$.

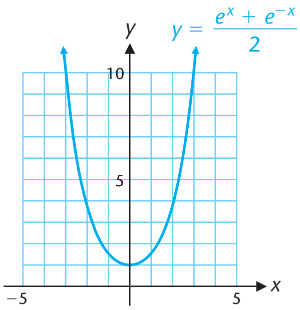
MATCHED PROBLEM 4

Using the formula in Example 4, find the altitude in miles so that the atmospheric pressure will be one-eighth that at sea level. Compute the answer to two significant digits.

Many people assume that a cable hanging between two fixed points (think of utility wires between two poles) are parabolas, but actually they are not. Instead, they follow the shape of the graph in Figure 5, known as a **catenary.** Catenaries are important in engineering and architecture, and are often studied in calculus. The graph of the equation

$$y = \frac{e^{x} + e^{-x}}{2} \tag{1}$$

is an example of a catenary.



› **Figure 5** Catenary.

EXAMPLE 5 Solving an Exponential Equation

Given equation (1), find x for $y = 2.5$. Compute the answer to four decimal places.

SOLUTIONS

Algebraic Solution

$$y = \frac{e^x + e^{-x}}{2}$$ Substitute $y = 2.5$.

$$2.5 = \frac{e^x + e^{-x}}{2}$$ Multiply both sides by 2 to clear fractions.

$$5 = e^x + e^{-x}$$ Multiply both sides by e^x to eliminate negative exponents.

$$5e^x = e^{2x} + 1$$ Rearrange so that zero is on one side.

$$e^{2x} - 5e^x + 1 = 0$$ Use the quadratic formula.

Let $u = e^x$, then

$$u^2 - 5u + 1 = 0$$

$$u = \frac{5 \pm \sqrt{25 - 4(1)(1)}}{2}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$ Replace u with e^x and solve for x.

$$e^x = \frac{5 \pm \sqrt{21}}{2}$$ Take the natural log of both sides (both values on the right are positive).

$$\ln e^x = \ln \frac{5 \pm \sqrt{21}}{2}$$ Use $\ln e^x = x$.

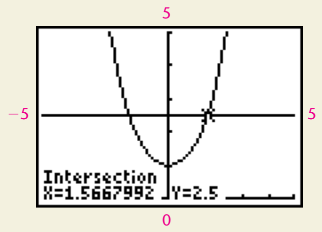
$$x = \ln \frac{5 \pm \sqrt{21}}{2}$$

$$= -1.5668, 1.5668$$

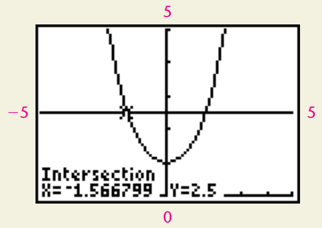
Note that the algebraic method produces exact solutions, an important consideration in certain calculus applications (see Problems 69–72 in Exercises 4-5).

Graphical Solution

Graph $y_1 = (e^x + e^{-x})/2$ and $y_2 = 2.5$ and use the INTERSECT command (Fig. 6).



(a)



(b)

Figure 6 $y_1 = \frac{e^x + e^{-x}}{2}$, $y_2 = 2.5$.

The two solutions are $x = -1.5668$ and $x = 1.5668$ to four decimal places.

MATCHED PROBLEM 5

Given $y = (e^x - e^{-x})/2$, find x for $y = 1.5$. Compute the answer to three decimal places.

››› EXPLORE-DISCUSS 1

Let $y = e^{2x} + 3e^{x} + e^{-x}$

(A) Try to find x when $y = 7$ using the method of Example 5. Explain the difficulty that arises.

(B) Use a graphing calculator to find x when $y = 7$.

› Solving Logarithmic Equations

We will begin our study of solving logarithmic equations with a key observation. For equations of the form

$$\log_b x = a$$

changing to exponential form solves the equation, as in Example 6.

EXAMPLE 6

Solve $\log_5 x = 3$.

SOLUTION

Change to exponential form:

$$5^3 = x$$
$$x = 125$$

MATCHED PROBLEM 6

Solve $\log_2 x = -4$.

Obviously, this is a very simple example, but it provides some valuable insight in solving logarithmic equations. If we can reduce an equation to the form $\log_b(\text{expression}) = a$, where "expression" is something involving the variable, then changing to exponential form should result in an equation we already know how to solve.

EXAMPLE 7 Solving a Logarithmic Equation

Solve $\log(x + 3) + \log x = 1$, and check.

SOLUTIONS

Algebraic Solution

First use properties of logarithms to express the left side as a single logarithm, then convert to exponential form and solve for x, as in Example 6.

$\log(x + 3) + \log x = 1$	Combine left side using $\log M + \log N = \log MN$.
$\log[x(x + 3)] = 1$	Change to equivalent exponential form.
$x(x + 3) = 10^1$	Write in $ax^2 + bx + c = 0$ form and solve.
$x^2 + 3x - 10 = 0$	Factor.
$(x + 5)(x - 2) = 0$	
$x = -5, 2$	

CHECK

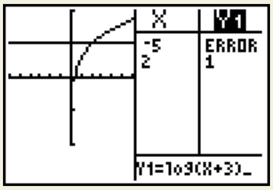
$x = -5$: $\log(-5 + 3) + \log(-5)$ is not defined because the domain of the log function is $(0, \infty)$.

$x = 2$: $\log(2 + 3) + \log 2 = \log 5 + \log 2$
$= \log(5 \cdot 2) = \log 10 \overset{\checkmark}{=} 1$

The only solution to the original equation is $x = 2$. Remember, solutions should be checked in the original equation to see whether any should be discarded.

Graphical Solution

Graph $y_1 = \log(x + 3) + \log x$ and $y_2 = 1$ and use the INTERSECT command. Figure 7 shows that $x = 2$ is a solution, and also shows that y_1 (the left side of the original equation) is not defined at $x = -5$, the extraneous solution produced by the algebraic method.



› Figure 7
$y_1 = \log(x + 3) + \log x$,
$y_2 = 1$.

MATCHED PROBLEM 7

Solve $\log(x - 15) = 2 - \log x$, and check.

››› CAUTION ›››

It's important to check your answer when solving logarithmic equations. Because log functions are undefined for negative inputs, extraneous solutions are common.

EXAMPLE 8 Solving a Logarithmic Equation

Solve $(\ln x)^2 = \ln x^2$.

SOLUTIONS

Algebraic Solution

There are no logarithmic properties for simplifying $(\ln x)^2$. However, we can simplify $\ln x^2$, obtaining an equation involving $\ln x$ and $(\ln x)^2$.

$$(\ln x)^2 = \ln x^2 \quad \text{Use } \log_b N^p = p \log_b N.$$
$$(\ln x)^2 = 2 \ln x \quad \text{Rearrange so that zero is on one side.}$$
$$(\ln x)^2 - 2 \ln x = 0 \quad \text{Factor out } \ln x.$$
$$(\ln x)(\ln x - 2) = 0 \quad \text{Set each factor equal to zero.}$$
$$\ln x = 0 \quad \text{or} \quad \ln x - 2 = 0$$
$$\ln x = 0 \quad \text{or} \quad \ln x = 2 \quad \text{Change to exponential form. Recall that } \ln x = \log_e x.$$
$$e^0 = x \quad \text{or} \quad e^2 = x$$
$$x = 1, e^2$$

Checking that both $x = 1$ and $x = e^2$ are solutions to the original equation is left to you. Don't let us down.

Graphical Solution

Graph $y_1 = (\ln x)^2$ and $y_2 = \ln x^2$ and use the INTERSECT command to obtain the solutions $x = 1$ and $x = 7.3890561$ (Fig. 8). The second solution is not exact; it is an approximation to e^2.

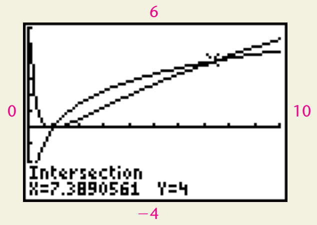


Figure 8

MATCHED PROBLEM 8

Solve $\log x^2 = (\log x)^2$.

CAUTION

Note that

$$(\log_b x)^2 \neq \log_b x^2 \qquad (\log_b x)^2 = (\log_b x)(\log_b x) \quad \log_b x^2 = 2 \log_b x$$

You might find it helpful to keep these straight by writing $\log_b x^2$ as $\log_b (x^2)$.

EXAMPLE 9 Earthquake Intensity

Recall from the last section that the magnitude of an earthquake on the Richter scale is given by

$$M = \frac{2}{3} \log \frac{E}{E_0}$$

Solve for E in terms of the other symbols.

SOLUTION

$$M = \frac{2}{3} \log \frac{E}{E_0} \qquad \text{Multiply both sides by } \tfrac{3}{2}.$$

$$\log \frac{E}{E_0} = \frac{3M}{2} \qquad \text{Change to exponential form.}$$

$$\frac{E}{E_0} = 10^{3M/2} \qquad \text{Multiply both sides by } E_0.$$

$$E = E_0 10^{3M/2}$$

MATCHED PROBLEM 9

Solve the rocket equation from the last section for W_b in terms of the other symbols:

$$v = c \ln \frac{W_t}{W_b}$$

ANSWERS TO MATCHED PROBLEMS

1. 2.1133 **2.** $x = 0.2263$
3. More than double in 9 years, but not quite double in 8 years
4. 9.9 miles **5.** $x = 1.195$ **6.** $x = \frac{1}{16}$
7. $x = 20$ **8.** $x = 1{,}100$ **9.** $W_b = W_t e^{-v/c}$

4-5 Exercises

**Additional answers can be found in the Instructor Answer Appendix.*

A

1. Which property of logarithms do you think is most useful in solving exponential equations? Explain.

2. Which properties of logarithms do you think are most useful in solving equations with more than one logarithm? Explain.

3. If u and v represent expressions with variable x, how can you solve equations of the form $\log_b u = \log_b v$ for x? Explain why this works.

4. Why is it especially important to check answers when solving logarithmic equations?

5. Explain the difference between $(\ln x)^2$ and $\ln x^2$.

6. When solving logarithmic and exponential equations, what is the advantage of solving algebraically, rather than graphically?

Solve Problems 7–22 algebraically and graphically. Round answers to three significant digits.

7. $10^{-x} = 0.0347$ $x = 1.46$

8. $10^{x} = 14.3$ $x = 1.16$

9. $10^{3x+1} = 92$ $x = 0.321$

10. $10^{5x-2} = 348$ $x = 0.908$

11. $e^{x} = 3.65$ $x = 1.29$

12. $e^{-x} = 0.0142$ $x = 4.25$

13. $e^{2x-1} = 405$ $x = 3.50$

14. $e^{3x+5} = 23.8$ $x = -0.610$

15. $5^{x} = 18$ $x = 1.80$

16. $3^{x} = 4$ $x = 1.26$

17. $2^{-x} = 0.238$ $x = 2.07$

18. $3^{-x} = 0.074$ $x = 2.37$

19. $\log_5 (2x - 7) = 2$ $x = 16$

20. $\log_2 (4 - x) = 4$ $x = -12$

21. $\log_3 (x^2 - 8x) = 2$ $x = -1, 9$

22. $\log_2 (x^2 + 5) = 3$ $x = \pm\sqrt{3}$

Solve Problems 23–32 exactly.

23. $\log 5 + \log x = 2$ $x = 20$

24. $\log x - \log 8 = 1$ $x = 80$

25. $\log x + \log (x - 3) = 1$ $x = 5$

26. $\log (x - 9) + \log 100x = 3$ $x = 10$

27. $\log (x + 1) - \log (x - 1) = 1$ $x = \frac{11}{9}$

28. $\log (2x + 1) = 1 + \log (x - 2)$ $x = \frac{21}{8}$

29. $\ln (4x - 3) = \ln (x + 1)$ $x = \frac{4}{3}$

30. $\log_5 (2 - x) = \log_5 (3x + 8)$ $x = -\frac{3}{2}$

31. $\log_2 (x^2 - 2x) = \log_2 (3x - 6)$ $x = 3$

32. $\log_7 (x + 1) = \log_7 (2x^2 - x - 3)$ $x = 2$

B *Solve Problems 33–44 algebraically and graphically. Round answers to three significant digits.*

33. $2 = 1.05^{x}$ $x = 14.2$

34. $3 = 1.06^{x}$ $x = 18.9$

35. $e^{-1.4x} = 13$ $x = -1.83$

36. $e^{0.32x} = 632$ $x = 20.2$

37. $5 + 3^{x} = 10$ $x = 1.46$

38. $-3 = (\frac{1}{2})^{x} - 12$ $x = -3.17$

39. $10^{2x+5} - 7 = 13$ $x = -1.85$

40. $3 - 4^{7-x} = -16$ $x = 4.88$

41. $123 = 500e^{-0.12x}$ $x = 11.7$

42. $438 = 200e^{0.25x}$ $x = 3.14$

43. $e^{-x^2} = 0.23$ $x = \pm 1.21$

44. $e^{x^2} = 125$ $x = \pm 2.20$

Solve Problems 45–56 exactly.

45. $\log x - \log 5 = \log 2 - \log (x - 3)$ $x = 5$

46. $\log (6x + 5) - \log 3 = \log 2 - \log x$ $x = \frac{2}{3}$

47. $\ln x = \ln (2x - 1) - \ln (x - 2)$ $x = 2 + \sqrt{3}$

48. $\ln (x + 1) = \ln (3x + 1) - \ln x$ $x = 1 + \sqrt{2}$

49. $\log (2x + 1) = 1 - \log (x - 1)$ $x = \frac{1 + \sqrt{89}}{4}$

50. $1 - \log (x - 2) = \log (3x + 1)$ $x = 3$

51. $(\ln x)^3 = \ln x^4$ $x = 1, e^2, e^{-2}$

52. $(\log x)^3 = \log x^4$ $x = 1, x = 10^{\pm 2}$

53. $\ln (\ln x) = 1$ $x = e^{e}$

54. $\log (\log x) = 1$ $x = 10^{10}$

55. $x^{\log x} = 100x$ $x = 100, 0.1$

56. $3^{\log x} = 3x$ $x = 10^{(\log 3)/(\log 3 - 1)}$

In Problems 57–60,
(A) Explain the difficulty in solving the equation exactly.
(B) Determine the number of solutions by graphing the functions on each side of the equation.

57. $e^{x/2} = 5 \ln x$ (B) 2

58. $\ln (\ln x) + \ln x = 2$ (B) 1

59. $3^{x} + 2 = 7 + x - e^{-x}$ (B) 2

60. $e^{x/4} = 5 \log x + 4 \ln x$ (B) 2

C *Solve Problems 61–68 for the indicated variable in terms of the remaining symbols. Use the natural log for solving exponential equations.*

61. $A = Pe^{rt}$ for r (finance) $r = \frac{1}{t} \ln \frac{A}{P}$

62. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ for t (finance) $t = \frac{\ln \frac{A}{P}}{n \ln (1 + \frac{r}{n})}$

63. $D = 10 \log \frac{I}{I_0}$ for I (sound) $I = I_0(10^{D/10})$

64. $t = \frac{-1}{k}(\ln A - \ln A_0)$ for A (decay) $A = A_0e^{-kt}$

65. $M = 6 - 2.5 \log \frac{I}{I_0}$ for I (astronomy) $I = I_0[10^{(6-M)/2.5}]$

66. $L = 8.8 + 5.1 \log D$ for D (astronomy) $D = 10^{(L-8.8)/5.1}$

67. $I = \frac{E}{R}(1 - e^{-Rt/L})$ for t (circuitry) $t = -\frac{L}{R}\ln\left(1 - \frac{RI}{E}\right)$

68. $S = R\frac{(1+i)^n - 1}{i}$ for n (annuity) $n = \frac{\ln(\frac{Si}{R} + 1)}{\ln(1+i)}$

The following combinations of exponential functions define four of six ***hyperbolic functions,*** *an important class of functions in calculus and higher mathematics. Solve Problems 69–72 for x in terms of y. The results are used to define* ***inverse hyperbolic functions,*** *another important class of functions in calculus and higher mathematics.*

69. $y = \frac{e^x + e^{-x}$}{2}$

70. $y = \frac{e^x - e^{-x}}{2}$

71. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

72. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

In Problems 73–84, use a graphing calculator to approximate to two decimal places any solutions of the equation in the interval $0 \le x \le 1$. None of these equations can be solved exactly using any step-by-step algebraic process.

73. $2^{-x} - 2x = 0$ $x = 0.38$

74. $3^{-x} - 3x = 0$ $x = 0.25$

75. $x3^x - 1 = 0$ $x = 0.55$

76. $x2^x - 1 = 0$ $x = 0.64$

77. $e^{-x} - x = 0$ $x = 0.57$

78. $xe^{2x} - 1 = 0$ $x = 0.43$

79. $xe^x - 2 = 0$ $x = 0.85$

80. $e^{-x} - 2x = 0$ $x = 0.35$

81. $\ln x + 2x = 0$ $x = 0.43$

82. $\ln x + x^2 = 0$ $x = 0.65$

83. $\ln x + e^x = 0$ $x = 0.27$

84. $\ln x + x = 0$ $x = 0.57$

APPLICATIONS

85. COMPOUND INTEREST How many years, to the nearest year, will it take a sum of money to double if it is invested at 15% compounded annually? 5 years to the nearest year

86. COMPOUND INTEREST How many years, to the nearest year, will it take money to quadruple if it is invested at 20% compounded annually? 8 years to the nearest year

87. COMPOUND INTEREST At what annual rate compounded continuously will \$1,000 have to be invested to grow to \$2,500 in 10 years? Compute the answer to three significant digits. $r = 0.0916$ or 9.16%

88. COMPOUND INTEREST How many years will it take \$5,000 to grow to \$8,000 if it is invested at an annual rate of 9% compounded continuously? Compute the answer to three significant digits. 5.22 years

89. IMMIGRATION According to the U.S. Office of Immigration Statistics, there were 10.5 million illegal immigrants in the United States in May 2005, and that number had grown to 11.3 million by May 2007.
(A) Find the relative growth rate if we use the $P = P_0e^{rt}$ model for population growth. Round to three significant digits. 3.67% per year
(B) Use your answer from part A to write a function describing the illegal immigrant population in millions in terms of years after May 2005, and use it to predict when the illegal immigrant population should reach 20 million. $P = 10.5e^{0.0367t}$; in 2022

90. POPULATION GROWTH According to U.S. Census Bureau estimates, the population of the United States was 227.2 million on July 1, 1980, and 249.5 million on July 1, 1990.
(A) Find the relative growth rate if we use the $P = P_0e^{rt}$ model for population growth. Round to three significant digits.
(B) Use your answer from part A to write a function describing the population of the United States in millions in terms of years after July 1980, and use it to predict when the population should reach 400 million. $P = 227.2e^{0.00936t}$; in 2040
(C) Use your function from part B to estimate the population of the United States today, then compare your estimate to the one found at www.census.gov/population/www/popclockus.html.

91. WORLD POPULATION A mathematical model for world population growth over short periods is given by

$$P = P_0e^{rt}$$

where P is the population after t years, P_0 is the population at $t = 0$, and the population is assumed to grow continuously at the annual rate r. How many years, to the nearest year, will it take the world population to double if it grows continuously at an annual rate of 1.14%? 61 years

92. WORLD POPULATION Refer to Problem 91. Starting with a world population of 6.5 billion people and assuming that the population grows continuously at an annual rate of 1.14%, how many years, to the nearest year, will it be before there is only 1 square yard of land per person? Earth contains approximately 1.7×10^{14} square yards of land. 892 years

93. MEDICAL RESEARCH A medical researcher is testing a radioactive isotope for use in a new imaging process. She finds that an original sample of 5 grams decays to 1 gram in 6 hours. Find the half-life of the sample to three significant digits. [Recall that the half-life model is $A = A_0(\frac{1}{2})^{t/h}$, where A_0 is the original amount and h is the half-life.] 2.58 hours

94. CARBON-14 DATING If 90% of a sample of carbon-14 remains after 866 years, what is the half-life of carbon-14? (See Problem 93 for the half-life model.) 5,697 years

As long as a plant or animal remains alive, carbon-14 is maintained in a constant amount in its tissues. Once dead, however, the plant or animal ceases taking in carbon, and carbon-14 diminishes by radioactive decay. The amount remaining can be modeled by the equation $A = A_0e^{-0.000124t}$, where A is the amount after t years, and A_0 is the amount at time $t = 0$. Use this model to solve Problems 95–98.

95. CARBON-14 DATING In 2003, Japanese scientists announced the beginning of an effort to bring the long-extinct woolly mammoth back to life using modern cloning techniques. Their efforts were focused on an especially well-preserved specimen discovered frozen in the Siberian ice. Nearby samples of plant material were found to have 28.9% of the amount of carbon-14 in a living sample. What was the approximate age of these samples? 10,010 years

96. CARBON-14 DATING In 2004, archaeologist Al Goodyear discovered a site in South Carolina that contains evidence of the earliest human settlement in North America. Carbon dating of burned plant material indicated 0.2% of the amount of carbon-14 in a live sample. How old was that sample? 50,118 years

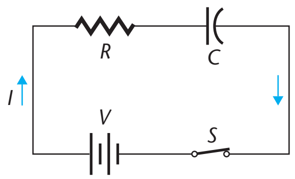
97. CARBON-14 DATING Many scholars believe that the earliest nonnative settlers of North America were Vikings who sailed from Iceland. If a fragment of a wooden tool found and dated in 2004 had 88.3% of the amount of carbon-14 in a living sample, when was this tool made? The year 1001

98. CARBON-14 DATING In 1998, the Shroud of Turin was examined by researchers, who found that plant fibers in the fabric had 92.1% of the amount of carbon-14 in a living sample. If this is accurate, when was the fabric made? 1334

99. PHOTOGRAPHY An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time.* For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

How many seconds will it take the capacitor to reach a charge of 0.0007 coulomb? Compute the answer to three significant digits. 7.52 seconds



100. ADVERTISING A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million possible viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

How many days, to the nearest day, will the advertising campaign have to last so that 80% of the possible viewers will be aware of the product? 43 days

101. NEWTON'S LAW OF COOLING This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in a refrigerator at 40°F to cool before a dinner party. After an hour the temperature of the wine is found to be 61.5°F. Find the constant k, to two decimal places, and the time, to one decimal place, it will take the wine to cool from 72 to 50°F. $k = 0.40$, 2.9 hours

102. MARINE BIOLOGY Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity is reduced according to the exponential function

$$I = I_0e^{-kd}$$

where I is the intensity d feet below the surface and I_0 is the intensity at the surface. The constant k is called the *coefficient of extinction*. At Crystal Lake in Wisconsin it was found that half the surface light remained at a depth of 14.3 feet. Find k, and find the depth of the photic zone. Compute answers to three significant digits. $k = 0.0485$, 95.0 ft

Problems 103–106 are based on the Richter scale equation from Section 4-4, $M = \frac{2}{3}\log\frac{E}{10^{4.40}}$, where M is the magnitude and E is the amount of energy in joules released by the earthquake. Round all calculations to three significant digits.

103. EARTHQUAKES There were 11 earthquakes recorded worldwide in 2005 with magnitude at least 7.0.
(A) How much energy is released by a magnitude 7.0 earthquake?
(B) The total average daily consumption of energy for the entire United States in 2006 was 2.88×10^{14} joules. How many days could the energy released by a magnitude 7.0 earthquake power the United States? 2.76 days

104. EARTHQUAKES On December 26, 2004, a magnitude 9.0 earthquake struck in the Indian Ocean, causing a massive tsunami that resulted in over 230,000 deaths.
(A) How much energy was released by this earthquake?
(B) The total average daily consumption of energy for the entire United States in 2006 was 2.88×10^{14} joules. How many days could the energy released by a magnitude 9.0 earthquake power the United States? 2,760 days

105. EARTHQUAKES There were 10 earthquakes worldwide in 2005 with magnitudes between 7.0 and 7.9. Assume that these earthquakes had an average magnitude of 7.5. How long could the total energy released by these ten earthquakes power the United States, which had a total energy consumption of 1.05×10^{17} joules in 2006? 0.426 years, or about 155 days

106. EARTHQUAKES There were 144 earthquakes worldwide in 2005 with magnitudes between 6.0 and 6.9. Assume that these earthquakes had an average magnitude of 6.5. How long could the total energy released by these 144 earthquakes power the United States, which had a total energy consumption of 1.05×10^{17} joules in 2006? 0.193 years, or about 70.4 days

CHAPTER 4 Review

4-1 Exponential Functions

The equation $f(x) = b^x$, $b > 0$, $b \neq 1$, defines an **exponential function** with **base *b***. The domain of f is $(-\infty, \infty)$ and the range is $(0, \infty)$. The graph of f is a continuous curve that has no sharp corners; passes through (0, 1); lies above the x axis, which is a horizontal asymptote; increases as x increases if $b > 1$; decreases as x increases if $b < 1$; and intersects any horizontal line at most once. The function f is one-to-one and has an inverse. The following **exponential function properties** are useful in working with these functions.

1. $a^x a^y = a^{x+y}$ $\quad (a^x)^y = a^{xy}$ $\quad (ab)^x = a^x b^x$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

2. $a^x = a^y$ if and only if $x = y$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$.

As x approaches ∞, the expression $[1 + (1/x)]^x$ approaches the irrational number $e \approx 2.718\,281\,828\,459$. The function $f(x) = e^x$ is called the **exponential function with base *e*.** The growth of money in an account paying **compound interest** is described by $A = P(1 + r/m)^n$, where P is the **principal,** r is the annual **rate,** m is the number of compounding periods in 1 year, and A is the **amount** in the account after n compounding periods.

If the account pays **interest compounded continuously,** the amount A in the account after t years is given by $A = Pe^{rt}$.

4-2 Exponential Models

Exponential functions are used to model various types of growth:

1. *Population growth* can be modeled by using the **doubling time growth model** $A = A_0 2^{t/d}$, where A is the population at time t, A_0 is the population at time $t = 0$, and d is the **doubling time**—the time it takes for the population to double. Another model of population growth, $A = A_0 e^{kt}$, where A_0 is the population at time zero and k is a positive constant called the **relative growth rate**, uses the exponential function with base e. This model is used for many other types of quantities that exhibit exponential growth as well.

2. *Radioactive decay* can be modeled by using the **half-life decay model** $A = A_0(\frac{1}{2})^{t/h} = A_0 2^{-t/h}$, where A is the amount at time t, A_0 is the amount at time $t = 0$, and h is the **half-life**—the time it takes for half the material to decay. Another model of radioactive decay, $A = A_0 e^{-kt}$, where A_0 is the amount at time zero and k is a positive constant, uses the exponential function with base e. This model can be used for other types of quantities that exhibit negative exponential growth as well.

3. *Limited growth*—the growth of a company or proficiency at learning a skill, for example—can often be modeled by the equation $y = A(1 - e^{-kt})$, where A and k are positive constants.

Logistic growth is another limited growth model that is useful for modeling phenomena like the spread of an epidemic, or sales of a new product. The logistic model is $y = M/(1 + ce^{-kt})$, where c, k, and M are positive constants. A good comparison of these different exponential models can be found in Table 3 at the end of Section 4-2.

Exponential regression can be used to fit a function of the form $y = ab^x$ to a set of data points. Logistic regression can be use to find a function of the form $y = c/(1 + ae^{-bx})$.

4-3 Logarithmic Functions

The **logarithmic function with base *b*** is defined to be the inverse of the exponential function with base b and is denoted by $y = \log_b x$. Thus, $y = \log_b x$ if and only if $x = b^y$, $b > 0$, $b \neq 1$. This relationship can be used to convert an expression from logarithmic to exponential form, and vice versa.

The domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of a logarithmic function is a continuous curve that always passes through the point (1, 0) and has the y axis as a vertical asymptote. The following **properties of logarithmic functions** are useful in working with these functions:

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^x = x$

4. $b^{\log_b x} = x, x > 0$

5. $\log_b M = \log_b N$ if and only if $M = N$

6. $\log_b MN = \log_b M + \log_b N$

7. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

8. $\log_b M^p = p \log_b M$

Logarithms to the base 10 are called **common logarithms** and are denoted by $\log x$. Logarithms to the base e are called **natural**

logarithms and are denoted by $\ln x$. Thus, $\log x = y$ is equivalent to $x = 10^y$, and $\ln x = y$ is equivalent to $x = e^y$.

The **change-of-base formula,** $\log_b N = (\log_a N)/(\log_a b)$, relates logarithms to two different bases and can be used, along with a calculator, to evaluate logarithms to bases other than e or 10.

4-4 Logarithmic Models

Logarithmic functions increase very slowly as the input gets very large, so they can be used to scale down quantities that involve very large numbers, like the intensity of sound waves and the energy released by earthquakes.

The following applications involve logarithmic functions:

1. The **decibel** is defined by $D = 10 \log (I/I_0)$, where D is the **decibel level** of a sound, I is the **intensity** of the sound, and $I_0 = 10^{-12}$ watts per square meter is a standardized sound level.
2. The **magnitude** M of an earthquake on the **Richter scale** is given by $M = \frac{2}{3} \log (E/E_0)$, where E is the energy released by the earthquake and $E_0 = 10^{4.40}$ joules is a standardized energy level.
3. The **velocity** v of a rocket at burnout is given by the **rocket equation** $v = c \ln(W_t/W_b)$, where c is the exhaust velocity, W_t is the takeoff weight, and W_b is the burnout weight.

Logarithmic regression is used to fit a function of the form $y = a + b \ln x$ to a set of data points.

4-5 Exponential and Logarithmic Equations

Exponential equations are equations in which the variable appears in an exponent. If the exponential expression is isolated, applying a logarithmic function to both sides and using the property $\log_b N^p = p \log_b N$ will enable you to remove the variable from the exponent. If the exponential expression is not isolated, we can use previously developed techniques to first solve for the exponential, then solve as above.

Logarithmic equations are equations in which the variable appears inside a logarithmic function. In most cases, the key to solving them is to change the equation to the equivalent exponential expression. For equations with multiple log expressions, properties of logarithms can be used to combine the expressions before solving.

CHAPTER 4 Review Exercises

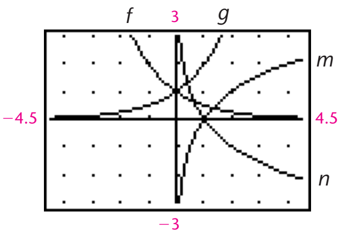
Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

A

1. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = \log_2 x$ m (B) $y = 0.5^x$ f

(C) $y = \log_{0.5} x$ n (D) $y = 2^x$ g *(4–1, 4–3)*



2. Write in logarithmic form using base 10: $m = 10^n$. $\log m = n$ *(4–3)*

3. Write in logarithmic form using base e: $x = e^y$. $\ln x = y$ *(4–3)*

Write Problems 4 and 5 in exponential form.

4. $\log x = y$ $x = 10^y$ *(4–3)* **5.** $\ln y = x$ $y = e^x$ *(4–3)*

6. (A) Plot at least five points, then draw a hand sketch of the graph of $y = (\frac{4}{3})^x$.

(B) Use your result from part A to sketch the graph of $y = \log_{4/3} x$.

In Problems 7 and 8, simplify using properties of exponents.

7. $\dfrac{7^{x+2}}{7^{2-x}}$ 7^{2x} *(4–1)* **8.** $\left(\dfrac{e^x}{e^{-x}}\right)^x$ e^{2x^2} *(4–1)*

Solve Problems 9–11 for x exactly. Do not use a calculator or table.

9. $\log_2 x = 3$ $x = 8$ *(4–3)* **10.** $\log_x 25 = 2$ $x = 5$ *(4–3)* **11.** $\log_3 27 = x$ $x = 3$ *(4–3)*

Solve Problems 12–15 for x to three significant digits.

12. $10^x = 17.5$ $x = 1.24$ *(4–5)*

13. $e^x = 143{,}000$ $x = 11.9$ *(4–5)*

14. $\ln x = -0.015\ 73$ $x = 0.984$ *(4–3)*

15. $\log x = 2.013$ $x = 103$ *(4–3)*

Evaluate Problems 16–19 to four significant digits using a calculator.

16. $\ln \pi$ 1.145 *(4–3)*

17. $\log(-e)$ Not defined *(4–3)*

18. $\pi^{\ln 2}$ 2.211 *(4–3)*

19. $\dfrac{e^{\pi} + e^{-\pi}}{2}$ 11.59 *(4–1)*

B **20.** Write as a single log: $\log x + 3 \log y - \frac{1}{2} \log z$ $\log \dfrac{xy^3}{\sqrt{z}}$ *(4–3)*

21. Write in terms of $\ln x$ and $\ln y$: $\ln \frac{x^3}{y}$ $3 \ln x - \ln y$ *(4–3)*

Solve Problems 22–34 for x exactly. Do not use a calculator or table.

22. $\ln(2x - 1) = \ln(x + 3)$ $x = 4$ *(4–5)*

23. $\log(x^2 - 3) = 2 \log(x - 1)$ $x = 2$ *(4–5)*

24. $e^{x^2 - 3} = e^{2x}$ $x = 3, -1$ *(4–5)*

25. $4^{x-1} = 2^{1-x}$ $x = 1$ *(4–5)*

26. $4 - 3^x = 2$ $x = \ln 2/\ln 3$ *(4–5)*

27. $5 + \frac{1}{2}e^x = \frac{17}{2}$ $x = \ln 7$ *(4–5)*

28. $2x^2e^{-x} = 18e^{-x}$ $x = 3, -3$ *(4–5)*

29. $\log_{1/4} 16 = x$ $x = -2$ *(4–5)*

30. $\log_x 9 = -2$ $x = \dfrac{1}{3}$ *(4–5)*

31. $\log_{16} x = \frac{3}{2}$ $x = 64$ *(4–5)*

32. $\log_x e^5 = 5$ $x = e$ *(4–5)*

33. $10^{\log_{10} x} = 33$ $x = 33$ *(4–5)*

34. $\ln x = 0$ $x = 1$ *(4–5)*

Solve Problems 35–44 for x to three significant digits.

35. $x = 2(10^{1.32})$ $x = 41.8$ *(4–1)*

36. $x = \log_5 23$ $x = 1.95$ *(4–3)*

37. $\ln x = -3.218$ $x = 0.0400$ *(4–3)*

38. $x = \log(2.156 \times 10^{-7})$ $x = -6.67$ *(4–3)*

39. $x = \dfrac{\ln 4}{\ln 2.31}$ $x = 1.66$ *(4–3)*

40. $25 = 5(2^x)$ $x = 2.32$ *(4–5)*

41. $4{,}000 = 2{,}500(e^{0.12x})$ $x = 3.92$ *(4–5)*

42. $0.01 = e^{-0.05x}$ $x = 92.1$ *(4–5)*

43. $5^{2x-3} = 7.08$ $x = 2.11$ *(4–5)*

44. $\dfrac{e^x - e^{-x}}{2} = 1$ $x = 0.881$ *(4–5)*

Solve Problems 45–50 for x exactly. Do not use a calculator or table.

45. $\log 3x^2 - \log 9x = 2$ $x = 300$ *(4–5)*

46. $\log x - \log 3 = \log 4 - \log(x + 4)$ $x = 2$ *(4–5)*

47. $\ln(x + 3) - \ln x = 2 \ln 2$ $x = 1$ *(4–5)*

48. $\ln(2x + 1) - \ln(x - 1) = \ln x$ $x = \dfrac{3 + \sqrt{13}}{2}$ *(4–5)*

49. $(\log x)^3 = \log x^9$ $x = 1, 10^3, 10^{-3}$ *(4–5)*

50. $\ln(\log x) = 1$ $x = 10^e$ *(4–5)*

In Problems 51 and 52, simplify.

51. $(e^x + 1)(e^{-x} - 1) - e^x(e^{-x} - 1)$ $e^{-x} - 1$ *(4–1)*

52. $(e^x + e^{-x})(e^x - e^{-x}) - (e^x - e^{-x})^2$ $2 - 2e^{-2x}$ *(4–1)*

In Problems 53–56, use a graphing calculator to help you draw the graph of each function. Then find the domain and range, intercepts, and asymptotes. Round all approximate values to two decimal places.

53. $y = 2^{x-1}$

54. $f(t) = 10e^{-0.08t}$

55. $y = \ln(x - 1)$

56. $N = \dfrac{100}{1 + 3e^{-t}}$

57. If the graph of $y = e^x$ is reflected through the line $y = x$, the graph of what function is obtained? Discuss the functions that are obtained by reflecting the graph of $y = e^x$ through the x axis and the y axis.

58. Approximate all real zeros of $f(x) = 4 - x^2 + \ln x$ to three decimal places. 0.018, 2.187 *(4–5)*

59. Find the coordinates of the points of intersection of $f(x) = 10^{x-3}$ and $g(x) = 8 \log x$ to three decimal places. (1.003, 0.010), (3.653, 4.502) *(4–5)*

C *Solve Problems 60–63 for the indicated variable in terms of the remaining symbols.*

60. $D = 10 \log \dfrac{I}{I_0}$ for I (sound intensity) $I = I_0(10^{D/10})$ *(4–5)*

61. $y = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for x (probability) $x = \pm\sqrt{-2 \ln(\sqrt{2\pi}y)}$ *(4–5)*

62. $x = -\dfrac{1}{k} \ln \dfrac{I}{I_0}$ for I (x-ray intensity) $I = I_0(e^{-kx})$ *(4–5)*

63. $r = P \dfrac{i}{1 - (1 - i)^{-n}}$ for n (finance)

64. (A) Explain why the equation $e^{-x/3} = 4 \ln(x + 1)$ has exactly one solution.

(B) Find the solution of the equation to three decimal places. 0.258 *(4–5)*

65. Write $\ln y = -5t + \ln c$ in an exponential form free of logarithms; then solve for y in terms of the remaining symbols. $y = ce^{-5t}$ *(4–5)*

66. For $f(x) = \log_2 x$, graph f and f^{-1} on the same coordinate system. What are the domains and ranges for f and f^{-1}?

67. Explain why 1 cannot be used as a logarithmic base.

68. Prove that for any positive M, N, and b ($b \neq 1$), $\log_b MN = \log_b M + \log_b N$. (*Hint:* Start by writing $u = \log_b M$ and $v = \log_b N$ and changing each to exponential form.)

APPLICATIONS

Solve these application problems algebraically or graphically, whichever seems more appropriate.

69. POPULATION GROWTH Many countries have a population growth rate of 3% (or more) per year. At this rate, how many years will it take a population to double? Use the annual compounding growth model $P = P_0(1 + r)^t$. Compute the answer to three significant digits. 23.4 years *(4–2)*

70. POPULATION GROWTH Repeat Problem 69 using the continuous compounding growth model $P = P_0e^{rt}$. 23.1 years *(4–2)*

71. CARBON 14-DATING How many years will it take for carbon-14 to diminish to 1% of the original amount after the death of a plant or animal? Use the formula $A = A_0e^{-0.000124t}$. Compute the answer to three significant digits. 37,100 years *(4–2)*

72. MEDICINE One leukemic cell injected into a healthy mouse will divide into two cells in about $\frac{1}{2}$ day. At the end of the day these two cells will divide into four. This doubling continues until 1 billion cells are formed; then the animal dies with leukemic cells in every part of the body.

(A) Write an equation that will give the number N of leukemic cells at the end of t days. $N = 2^{2t}$ (or $N = 4^t$)

(B) When, to the nearest day, will the mouse die? 15 days *(4–2)*

73. MONEY GROWTH Assume \$1 had been invested at an annual rate of 3% compounded continuously in the year A.D. 1. What would be the value of the account in the year 2011? Compute the answer to two significant digits. 1.5×10^{26} dollars *(4–2)*

74. PRESENT VALUE Solving $A = Pe^{rt}$ for P, we obtain $P = Ae^{-rt}$, which is the **present value** of the amount A due in t years if money is invested at a rate r compounded continuously.

(A) Graph $P = 1{,}000(e^{-0.08t})$, $0 \le t \le 30$.

(B) What does it appear that P tends to as t tends to infinity? [*Conclusion:* The longer the time until the amount A is due, the smaller its present value, as we would expect.] 0

75. EARTHQUAKES The 1971 San Fernando, California, earthquake released 1.99×10^{14} joules of energy. Compute its magnitude on the Richter scale using the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules. Compute the answer to one decimal place. 6.6 *(4–4)*

76. EARTHQUAKES Refer to Problem 75. If the 1906 San Francisco earthquake had a magnitude of 8.3 on the Richter scale, how much energy was released? Compute the answer to three significant digits. $10^{16.85}$ or 7.08×10^{16} joules *(4–4)*

***77.** SOUND If the intensity of a sound from one source is 100,000 times that of another, how much more is the decibel level of the louder sound than the softer one? Use the formula $D = 10\log(I/I_0)$.

The level of the louder sound is 50 decibels more. *(4–4)*

78. MARINE BIOLOGY The intensity of light entering water is reduced according to the exponential function

$$I = I_0e^{-kd}$$

where I is the intensity d feet below the surface, I_0 is the intensity at the surface, and k is the coefficient of extinction. Measurements in the Sargasso Sea in the West Indies have indicated that half the surface light reaches a depth of 73.6 feet. Find k, and find the depth at which 1% of the surface light remains. Compute answers to three significant digits.
$k = 0.00942$, $d = 489$ feet *(4–4)*

79. WILDLIFE MANAGEMENT A lake formed by a newly constructed dam is stocked with 1,000 fish. Their population is expected to increase according to the logistic curve

$$N = \frac{30}{1 + 29e^{-1.35t}}$$

where N is the number of fish, in thousands, expected after t years. The lake will be open to fishing when the number of fish reaches 20,000. How many years, to the nearest year, will this take? 3 years *(4–2)*

MODELING AND DATA ANALYSIS

80. MEDICARE The annual expenditures for Medicare (in billions of dollars) by the U.S. government for selected years since 1980 are shown in Table 1. Let x represent years since 1980.

(A) Find an exponential regression model of the form $y = ab^x$ for these data. Round to three significant digits. Estimate (to the nearest billion) the total expenditures in 2010 and in 2020.

(B) When (to the nearest year) will the total expenditures reach \$900 billion? 2015 *(4–2)*

Table 1 Medicare Expenditures

Year	Billion \$
1980	37
1985	72
1990	111
1995	181
2000	225
2005	342

Source: U.S. Bureau of the Census

81. AGRICULTURE The total U.S. corn consumption (in millions of bushels) is shown in Table 2 for selected years since 1975. Let x represent years since 1900.

(A) Find a logarithmic regression model of the form $y = a + b \ln x$ for these data. Round to four significant digits. Estimate (to the nearest million) the total consumption in 1996 and in 2010.

(B) The actual consumption in 1996 was 1,583 million bushels. How does this compare with the estimated consumption in part A? What effect will this additional 1996 information have on the estimate for 2010? Explain.

Table 2 Corn Consumption

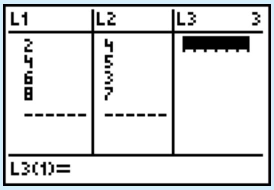
Year	Total consumption (million bushels)
1975	522
1980	659
1985	1,152
1990	1,373
1995	1,690

Source: U.S. Department of Agriculture

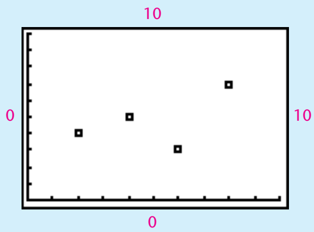
CHAPTER 4

››› GROUP ACTIVITY Comparing Regression Models

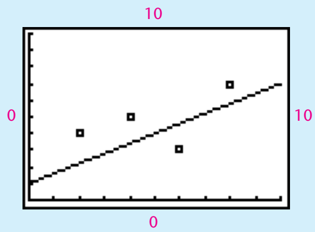
We have used polynomial, exponential, and logarithmic regression models to fit curves to data sets. And there are other equations that can be used for curve fitting (the TI-84 graphing calculator has 12 different equations on its STAT-CALC menu). How can we determine which equation provides the best fit for a given set of data? There are two principal ways to select models. The first is to use information about the type of data to help make a choice. For example, we expect the weight of a fish to be related to the cube of its length. And we expect most populations to grow exponentially, at least over the short term. The second method for choosing among equations involves developing a measure of how closely an equation fits a given data set. This is best introduced through an example. Consider the data set in Figure 1, where L1 represents the x coordinates and L2 represents the y coordinates. The graph of this data set is shown in Figure 2. Suppose we arbitrarily choose the equation $y_1 = 0.6x + 1$ to model these data (Fig. 3).



› **Figure 1**



› **Figure 2**



› **Figure 3** $y_1 = 0.6x + 1$.

To measure how well the graph of y_1 fits these data, we examine the difference between the y coordinates in the data set and the corresponding y coordinates on the graph of y_1 (L3 in Figs. 4 and 5). Each of these differences is called a **residual.** The most commonly accepted measure of the fit provided by a given model is the **sum of the squares of the residuals (SSR).** Computing this quantity is a simple matter on a graphing calculator (Fig. 6).

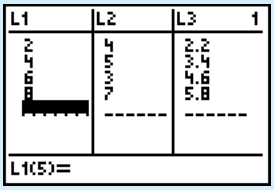


Figure 4

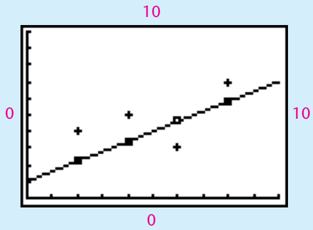


Figure 5 Here + is L2 and □ is L3.

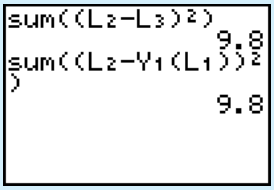


Figure 6 Two ways to calculate SSR.

(A) Find the linear regression model for the data in Figure 1, compute the SSR for this equation, and compare it with the one we computed for y_1.

It turns out that among all possible linear polynomials, **the linear regression model minimizes the sum of the squares of the residuals.** For this reason, the linear regression model is often called the **least-squares line.** A similar statement can be made for polynomials of any fixed degree. That is, the quadratic regression model minimizes the SSR over all quadratic polynomials, the cubic regression model minimizes the SSR over all cubic polynomials, and so on. The same statement cannot be made for exponential or logarithmic regression models. Nevertheless, the SSR can still be used to compare exponential, logarithmic, and polynomial models.

(B) Find the exponential and logarithmic regression models for the data in Figure 1, compute their SSRs, and compare with the linear model.

(C) National annual advertising expenditures for selected years since 1950 are shown in Table 1 where x is years since 1950 and y is total expenditures in billions of dollars. Which regression model would fit this data best: a quadratic model, a cubic model, or an exponential model? Use the SSRs to support your choice.

Table 1 Annual Advertising Expenditures, 1950–2005

x (years)	0	5	10	15	20	25	30	35	40	45	50	55
y (billion \$)	5.7	9.2	12.0	15.3	19.6	27.9	53.6	94.8	128.6	160.9	243.3	271.1

Source: U.S. Bureau of the Census

CHAPTERS 3–4 Cumulative Review

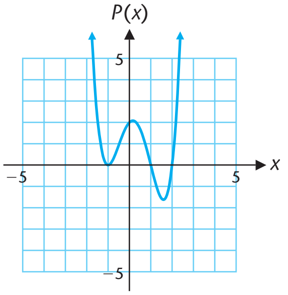
Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

A

1. Let $P(x)$ be the polynomial whose graph is shown in the figure on the next page.

(A) Assuming that $P(x)$ has integer zeros and leading coefficient 1, find the lowest-degree equation that could produce this graph.

(B) Describe the left and right behavior of $P(x)$.

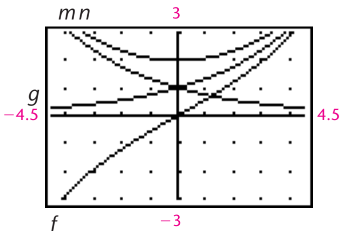


2. Draw the graph of a polynomial with lowest possible degree that has zeros −5, 1, and 6, and has a negative leading coefficient.

3. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = (\frac{3}{4})^x$ *m* (B) $y = (\frac{4}{3})^x$ *g*

(C) $y = (\frac{3}{4})^x + (\frac{4}{3})^x$ *n* (D) $y = (\frac{4}{3})^x - (\frac{3}{4})^x$ *f* *(4–1)*



4. For $P(x) = 3x^3 + 5x^2 - 18x - 3$ and $D(x) = x + 3$, use synthetic division to divide $P(x)$ by $D(x)$, and write the answer in the form $P(x) = D(x)Q(x) + R$.

5. Let $P(x) = 2(x + 2)(x - 3)(x - 5)$. What are the zeros of $P(x)$? −2, 3, 5 *(3–1)*

6. Let $P(x) = 4x^3 - 5x^2 - 3x - 1$. How do you know that $P(x)$ has at least one real zero between 1 and 2?

7. Let $P(x) = x^3 + x^2 - 10x + 8$. Find all rational zeros for $P(x)$. 1, 2, −4 *(3–4)*

8. Solve for x.

(A) $y = 10^x$ $x = \log y$ (B) $y = \ln x$ $x = e^y$ *(4–3)*

9. Simplify using properties of exponents.

(A) $(2e^x)^3$ $8e^{3x}$ (B) $\dfrac{e^{3x}}{e^{-2x}}$ e^{5x} *(4–1)*

10. Solve for x exactly. Do not use a calculator or a table.

(A) $\log_3 x = 2$ 9 (B) $\log_3 81 = x$ 4 (C) $\log_x 4 = -2$ $\frac{1}{2}$ *(4–3)*

11. Solve for x to three significant digits.

(A) $10^x = 2.35$ 0.371 (B) $e^x = 87{,}500$ 11.4

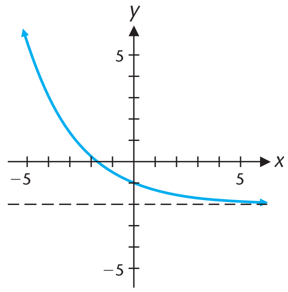
(C) $\log x = -1.25$ 0.0562 (D) $\ln x = 2.75$ 15.6 *(4–3)*

In Problems 12 and 13, translate each statement into an equation using k as the constant of proportionality.

12. E varies directly as p and inversely as the cube of x.

13. F is jointly proportional to q_1 and q_2 and inversely proportional to the square of r.

B 14. Explain why the graph in the figure is not the graph of a polynomial function.



15. Explain why the graph in the figure is not the graph of a rational function.

16. The function f subtracts the square root of the domain element from three times the natural log of the domain element. Write an algebraic definition of f. $f(x) = 3 \ln x - \sqrt{x}$ *(4–3)*

17. Write a verbal description of the function

$$f(x) = 100e^{0.5x} - 50.$$

18. Let $f(x) = \dfrac{2x + 8}{x + 2}$.

(A) Find the domain and the intercepts for f.

(B) Find the vertical and horizontal asymptotes for f.

(C) Sketch the graph of f. Draw vertical and horizontal asymptotes with dashed lines.

19. Find all zeros of $P(x) = (x^3 + 4x)(x + 4)$, and specify those zeros that are x intercepts.

20. Solve $(x^3 + 4x)(x + 4) \le 0$. $[-4, 0]$ *(3–3)*

21. If $P(x) = 2x^3 - 5x^2 + 3x + 2$, find $P(\frac{1}{2})$ using the remainder theorem and synthetic division.

22. One of the zeros of $P(x) = 3x^3 - 7x^2 - 18x - 8$ is $x = -1$. Find all others. −2/3 and 4 *(3–2)*

23. Which of the following is a factor of

$$P(x) = x^{25} - x^{20} + x^{15} + x^{10} - x^5 + 1$$

(A) $x - 1$ (B) $x + 1$ $x + 1$ (3–2)

24. Let $P(x) = x^4 - 8x^2 + 3$.

(A) Graph $P(x)$ and describe the graph verbally, including the number of x intercepts, the number of turning points, and the left and right behavior.

(B) Approximate the largest x intercept to two decimal places. 2.76

25. Let $P(x) = x^5 - 8x^4 + 17x^3 + 2x^2 - 20x - 8$.

(A) Approximate the zeros of $P(x)$ to two decimal places and state the multiplicity of each zero.

(B) Can any of these zeros be approximated with the bisection method? The MAXIMUM command? The MINIMUM command? Explain.

26. Let $P(x) = x^4 + 2x^3 - 20x^2 - 30$.

(A) Use the upper and lower bound theorem to find the smallest positive and largest negative integers that are upper and lower bounds, respectively, for the real zeros of $P(x)$. Upper bound: 4; lower bound: −6

(B) If $(k, k + 1)$, k an integer, is the interval containing the largest real zero of $P(x)$, determine how many additional intervals are required in the bisection method to approximate this zero to one decimal place. Four intervals

(C) Approximate the real zeros of $P(x)$ to two decimal places. −5.68, 3.80 (3–3)

27. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = 4x^3 - 20x^2 + 29x - 15$.

28. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = x^4 + 5x^3 + x^2 - 15x - 12$, and factor $P(x)$ into linear factors.

Solve Problems 29–39 for x exactly. Do not use a calculator or a table.

29. $2^{x^2} = 4^{x+4}$ $x = 4, -2$ (4–5)

30. $\frac{13}{2} - 3^x = \frac{1}{2}$ $x = \ln 6/\ln 3$ (4–5)

31. $2x^2e^{-x} + xe^{-x} = e^{-x}$ $x = \frac{1}{2}, -1$ (4–5)

32. $e^{\ln x} = 2.5$ $x = 2.5$ (4–5)

33. $\log_x 10^4 = 4$ $x = 10$ (4–5)

34. $\log_9 x = -\frac{3}{2}$ $x = \frac{1}{27}$ (4–5)

35. $\ln(x + 4) - \ln(x - 4) = 2\ln 3$ $x = 5$ (4–5)

36. $\ln(2x^2 + 2) = 2\ln(2x - 4)$ $x = 7$ (4–5)

37. $\log x + \log(x + 15) = 2$ $x = 5$ (4–5)

38. $\log(\ln x) = -1$ $x = e^{1/10}$ (4–5)

39. $4(\ln x)^2 = \ln x^2$ $x = 1, e^{1/2}$ (4–5)

Solve Problems 40–44 for x to three significant digits.

40. $x = \log_3 41$ $x = 3.38$ (4–5)

41. $\ln x = 1.45$ $x = 4.26$ (4–5)

42. $4(2^x) = 20$ $x = 2.32$ (4–5)

43. $10e^{-0.5x} = 1.6$ $x = 3.67$ (4–5)

44. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{2}$ $x = 0.549$ (4–5)

In Problems 45–49, use a graphing calculator to draw the graph of each function. The find the domain and range, intercepts, and asymptotes. Round all approximate values to two decimal places.

45. $f(x) = 3^{1-x}$

46. $g(x) = \ln(2 - x)$

47. $A(t) = 100e^{-0.3t}$

48. $h(x) = -2e^{-x} + 3$

49. $N(t) = \dfrac{6}{2 + e^{-0.1t}}$

50. If the graph of $y = \ln x$ is reflected through the line $y = x$, the graph of what function is obtained? Discuss the functions that are obtained by reflecting the graph of $y = \ln x$ in the x axis and in the y axis.

51. (A) Explain why the equation $e^{-x} = \ln x$ has exactly one solution.

(B) Approximate the solution of the equation to two decimal places. 1.31 (4–3)

In Problems 52 and 53, factor each polynomial in two ways:
(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)
(B) As a product of linear factors with complex coefficients

52. $P(x) = x^4 + 9x^2 + 18$

53. $P(x) = x^4 - 23x^2 - 50$

54. G is directly proportional to the square of x. If $G = 10$ when $x = 5$, find G when $x = 7$. $G = 19.6$ (3–6)

55. H varies inversely as the cube of r. If $H = 162$ when $r = 2$, find H when $r = 3$. $H = 48$ (3–6)

56. Graph f and indicate any horizontal, vertical, or slant asymptotes with dashed lines:

$$f(x) = \frac{x^2 + 4x + 8}{x + 2}$$

57. Solve $\dfrac{x^3 - x}{x^3 - 8} \geq 0$. $(-\infty, -1] \cup [0, 1] \cup (2, \infty)$ (3–5)

C **58.** Let $P(x) = x^4 - 28x^3 + 262x^2 - 922x + 1{,}083$. Approximate (to two decimal places) the x intercepts and the local extrema.

59. Find a polynomial of lowest degree with leading coefficient 1 that has zeros −1 (multiplicity 2), 0 (multiplicity 3), and $3 - 5i$. Leave the answer in factored form. What is the degree of the polynomial?

60. If $P(x)$ is a fourth-degree polynomial with integer coefficients and if i is a zero of $P(x)$, can $P(x)$ have any irrational zeros? Explain.

61. Let $P(x) = x^4 + 9x^3 - 500x^2 + 20{,}000$.

(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that are upper and lower bounds, respectively, for the real zeros of $P(x)$. Upper bound: 20; lower bound: -30

(B) Approximate the real zeros of $P(x)$ to two decimal places. $-26.98, -6.22, 7.23, 16.67$ *(3–3)*

62. Find all zeros (rational, irrational, and imaginary) exactly for

$$P(x) = x^5 - 4x^4 + 3x^3 + 10x^2 - 10x - 12$$

and factor $P(x)$ into linear factors.

63. Find rational roots exactly and irrational roots to two decimal places for

$$P(x) = x^5 + 4x^4 + x^3 - 11x^2 - 8x + 4$$

-2 (double), $-1.88, 0.35, 1.53$ *(3–4)*

64. Give an example of a rational function $f(x)$ that satisfies the following conditions: the real zeros of f are 5 and 8; $x = 1$ is the only vertical asymptote; and the line $y = 3$ is a horizontal asymptote.

65. Use natural logarithms to solve for n.

$$A = P\frac{(1 + i)^n - 1}{i}$$

66. Solve $\ln y = 5x + \ln A$ for y. Express the answer in a form that is free of logarithms. $y = Ae^{5x}$ *(4–5)*

67. Solve for x.

$$y = \frac{e^x - 2e^{-x}}{2}$$

68. Solve (to three decimal places)

$$\frac{4x}{x^2 - 1} < 3$$

$(-\infty, -1) \cup (-0.535, 1) \cup (1.869, \infty)$ *(3–5)*

APPLICATIONS

69. PROFIT ANALYSIS The daily profit in dollars made by the snack bar at a small college can be modeled by the function

$$P(x) = -4.8x^3 + 47x^2 - 35x - 40 \quad (0 \le x \le 12)$$

where x is the number of hours the snack bar is open per day.

(A) How many hours should the snack bar be open to maximize its profit? 6.1 hours

(B) How long will the snack bar need to stay open to make a profit of \$300? 4.3 or 7.6 hours

(C) For what range of hours will the snack bar at least break even? 1.5 to 8.9 hours *(3–1, 3–3)*

70. EFFICIENCY After learning how to solve a Rubik's Cube puzzle, a student practices for 2 hours each week, trying to decrease her best time to solve the puzzle. Suppose that the function

$$T(w) = 540 - \frac{450w}{w + 2}$$

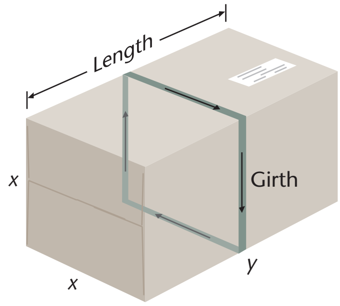
describes her best time in seconds after w weeks of practice.

(A) What was her best time after one week of practice? 390 seconds

(B) Find the horizontal asymptote of this rational function. What does it tell you about this student's performance?

(C) Explain why the vertical asymptote is not relevant to this problem.

71. SHIPPING A mailing service provides customers with rectangular shipping containers. The length plus the girth of one of these containers is 10 feet (see the figure). If the end of the container is square and the volume is 8 cubic feet, find the dimensions. Find rational solutions exactly and irrational solutions to two decimal places.



72. GEOMETRY The diagonal of a rectangle is 2 feet longer than one of the sides, and the area of the rectangle is 6 square feet. Find the dimensions of the rectangle. Find rational solutions exactly and irrational solutions to two decimal places. 1.79 feet by 3.35 feet *(3–4)*

73. ASTRONOMY The square of the time t required for a planet to make one orbit around the sun varies directly as the cube of its mean (average) distance d from the sun. Write the equation of variation, using k as the constant of variation. $t^2 = kd^3$ *(3–6)*

74. PHYSICS Atoms and molecules that make up the air constantly fly about like microscopic missiles. The velocity v of a particle at a fixed temperature varies inversely as the square root of its molecular weight w. If an oxygen molecule in air at room temperature has an average velocity of 0.3 mile/second, what will be the average velocity of a hydrogen molecule, given that the hydrogen molecule is one-sixteenth as heavy as the oxygen molecule? 1.2 miles per second *(3–6)*

75. POPULATION GROWTH If the Democratic Republic of the Congo has a population of about 60 million people and a doubling time of 23 years, find the population in *(3–6)*

(A) 5 years 69.8 million (B) 30 years 148 million

Compute answers to three significant digits.

76. COMPOUND INTEREST How long will it take money invested in an account earning 7% compounded annually to double? Use the annual compounding growth model $P = P_0(1 + r)^t$, and compute the answer to three significant digits. 10.2 years *(4–1)*

77. COMPOUND INTEREST Repeat Problem 76 using the continuous compound interest model $P = P_0 e^{rt}$. 9.90 years *(4–1)*

78. EARTHQUAKES If the 1906 and 1989 San Francisco earthquakes registered 8.3 and 7.1, respectively, on the Richter scale, how many times more powerful was the 1906 earthquake than the 1989 earthquake? Use the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules, and compute the answer to one decimal place. 63.1 times as powerful *(4–4)*

79. SOUND If the decibel level at a rock concert is 88, find the intensity of the sound at the concert. Use the formula $D = 10\log(I/I_0)$, where $I_0 = 10^{-12}$ watts per square meter, and compute the answer to two significant digits. 6.31×10^{-4} w/m^2 *(4–4)*

MODELING AND DATA ANALYSIS

80. Table 1 shows the life expectancy (in years) at birth for residents of the United States from 1970 to 1995. Let x represent years since 1970. Use the indicated regression model to estimate the life expectancy (to the nearest tenth of a year) for a U.S. resident born in 2010.

(A) Linear regression 78.9 years

(B) Quadratic regression 78.0 years

(C) Cubic regression 79.1 years

(D) Exponential regression 79.1 years *(3–1, 4–2)*

Table 1

Year	Life expectancy
1970	70.8
1975	72.6
1980	73.7
1985	74.7
1990	75.4
1995	75.9
2000	77.0
2005	77.7

Source: U.S. Census Bureau

81. Refer to Problem 80. The Census Bureau projected the life expectancy for a U.S. resident born in 2010 to be 77.9 years. Which of the models in Problem 80 is closest to the Census Bureau projection? Quadratic *(3–1, 4–2)*