Appendix R - Correlation, Energy Spectral Density and Power Spectral Density

R.1 Introduction

In signal and system analysis the characteristics of individual signals are, of course, important, but often the relationships between signals are just as important. Relationships between signals often indicate whether the physical phenomena that caused the signals are related or whether one signal is a modified version of the other. The relationship between two signals in a system can be used to measure system characteristics. For example, in a flowing liquid stream a heater can be placed upstream of a temperature sensor (Figure R-1).

![Figure R-1](image)

Figure R-1 Measurement of flow by examining the relationship between two signals

Then the heater power can be modulated with a known signal shape. By knowing the spacing $d$ between the heater and the downstream temperature sensor and by observing the signal coming from the temperature sensor and watching for that same shape (or a similar shape) to arrive at a later time, the flow rate can be determined from the spacing $d$ and the time delay between the signals. This is a very simple system whose excitation-response relationship is like a filter with a time delay and some frequency-dependent attenuation and the relationship between the two signals indicates what the attenuation and time delay are. The relation between signals often indicates whether one depends on the other, or both depend on some common phenomenon.

In this appendix we will explore mathematical techniques for comparing two signals. These comparison methods can be applied to all kinds of signals, continuous-time and discrete-time, deterministic and random. An exploration of the properties of random signals is beyond the scope of this appendix but the basic ideas of how to compare signals will be presented here with examples using both random and non-random signals but with exercises using only non-random signals.
R.2 Correlation and the Correlogram

The word "correlation" comes from the basic word "relation", which has to do with how one thing is related to another. How do we determine whether two signals are correlated? The natural answer is to simply look at them and try to detect any similarity between them. Human beings are very good at seeing similarities between images, especially faces. It is an important evolutionary survival skill. We can each recognize a large number of people as distinct individuals. We can read text printed in a multitude of different fonts, upper or lower case, or even handwritten. But we need a mathematical method to precisely and quantitatively indicate the correlation between signals.

In Figure R-2 through Figure R-5 below are illustrated pairs of signals. Each pair of signals is graphed versus time and then the two signals are graphed, not versus time, but rather versus each other. This third graph is called a correlogram and it can help in determining whether or not two signals are correlated.

![Figure R-2 A pair of discrete-time signals and their correlogram](image)

It might not have been obvious at first glance that the two discrete-time signals in Figure R-2 are very similar but the correlogram illustrates this relationship very clearly. When the second signal is graphed versus the first, the correlogram follows a straight line through the origin with a negative slope. The correlogram is simply indicating that when $x_1[n]$ goes positive from zero, $x_2[n]$ always goes negative from zero by a proportional amount, and vice versa. In this example, the slope of the correlogram line is minus one. That means that when $x_1[n]$ deviates positively from zero, $x_2[n]$ deviates negatively from zero by the same amount. This is strong evidence that there is a simple mathematical relation between the two signals,

$$x_2[n] = -x_1[n].$$
If a correlogram tends to form a straight line, the two signals used to form the correlogram are said to be highly correlated. The closer the correlogram is to a straight line, the more correlated the signals are. If the line has a positive slope the signals are positively correlated and if the line has a negative slope the signals are negatively correlated.

The two continuous-time signals in Figure R-3 have similar characteristics. They deviate about the same amount from zero, their average values both appear to be about zero and they tend to vary as a function of time at the same general rate. But are they correlated? There is no obvious similarity from just examining them and the correlogram confirms that there is no general trend of one signal varying in the same direction as the other or in the opposite direction. Since there is no apparent linearity in the correlogram we would conclude, based on this evidence, that these two signals are uncorrelated.

As in the previous case, the two discrete-time signals in Figure R-4 also have similar characteristics. If we visually examine them we will probably notice some similarity, but they are certainly not identical. The correlogram confirms that there is
some similarity because it tends to stay fairly close to a straight line with a positive slope. That is, it spends much more time in the first and third quadrants than in the second and fourth quadrants. The correlogram is telling us that these two signals are not completely correlated but they are not completely uncorrelated either. There is some relationship between them but it is not a simple proportionality as for the signals in Figure R-2. A typical situation that would cause this kind of relationship would be that $x_2[n]$ is some constant $K$ times $x_1[n]$ plus a third signal, usually random noise $n[n]$. The relationship could be described mathematically by

$$x_2[n] = K x_1[n] + n[n].$$

Figure R-5 A pair of continuous-time signals and their correlogram

Figure R-5 is different because even though we can see by looking at the two continuous-time signals that their shapes are very similar, the correlogram tells us that they are not exactly proportional to each other because it does not form a straight line (although it tends to lie in the first and third quadrants more than in the second and fourth). But it does have an interesting shape. It tends to form pseudo-elliptical shapes centered on a positive-slope line. What is this shape telling us? If you look closely at the two time plots you will notice a very small time shift between them (Figure R-6).
The second signal is a time-shifted version of the first. In this case, the second signal moves in the same direction as the first but earlier in time. So there is a relation between them, but with time shift. That kind of relationship can be described mathematically by

\[ x_2(t) = K x_1(t - \tau), \]

where, in this case, \( K = 1 \) and \( \tau < 0 \). If we were to shift the second signal a little later in time, we would get a straight-line correlogram with a positive slope indicating a strong positive correlation.

One way to see why the correlogram has this distinctive shape when there is time delay between the signals is to plot a correlogram for two very simple discrete-time signals, two sinusoids of the same frequency with a 45 degree phase shift (a one-eighth fundamental period time delay) between them (Figure R-7).
If the phase shift is changed to 90° we get a correlogram like Figure R-8.

![Figure R-8 A correlogram for two continuous-time sinusoids with a 90° phase difference](image)

Two terms commonly used in descriptions of relations between signals are **correlation** and **independence**. We have begun, at least qualitatively, to define correlation. Positive correlation simply means the tendency of two signals to move in the same direction at the same time and negative correlation means the tendency of two signals to move in opposite directions at the same time. The commonly-accepted definition of the term independence is that if two signals are independent that means that there is no commonality between them. That is, there is no mathematical relationship between the generation of one and the generation of the other.

Since independence and correlation seem to be opposite concepts, it is tempting at this point to think that if two signals are not independent they are correlated, but that is not always true. This last correlogram (Figure R-8) is a good illustration of the difference between correlation and dependence. The two continuous-time signals are obviously not independent since they are sinusoids of the same frequency and there is a simple mathematical relationship between them. Knowing one of them and the phase difference would allow calculation of the other. But they are uncorrelated. The lack of any discernible linearity in the correlogram is what indicates they are uncorrelated. As we will soon see, that can be shown mathematically from the definition of correlation.
Figure R-9 presents another interesting type of correlogram. The two discrete-time signals look quite different and the correlogram certainly does not tend to form a straight line, yet, looking at the correlogram, the feeling that there is some mathematical relationship between the two signals is irresistible. Even though the two signals are not linearly related it is apparent from the correlogram that they are non-linearly related. According to the usual definition of correlation, these signals are not highly correlated but they are obviously closely related because the correlogram, although not linear, forms a very definite single smooth curve. In this case, the actual mathematical relationship between the two signals is $x_2[n] = x_1^2[n]$. Correlation is ordinarily defined based on a linear relationship between signals. In this case, we could show that relationship by plotting the square of $x_1[n]$ versus $x_2[n]$. Then we would get a straight line and we would say that $x_1^2[n]$ and $x_2[n]$ are highly correlated.

Plotting correlograms in MATLAB is quite simple. For discrete-time signals we graph one signal against the other, plotting only dots. For example,

```matlab
% Assign values of one discrete-time signal to x1 and the other discrete-time signal to x2
plot(x1,x2,'k.');
```

For continuous-time signals we must first sample the signals well above the higher of the two Nyquist rates, then graph one signal against the other, plotting lines between points. For example,
Assign samples of one continuous-time signal to x1 and samples from the other continuous-time signal to x2.

plot(x1,x2,'k')
.
.

R.3 The Correlation Function

Conceptual Basis

The correlogram is useful as a visualization tool but it would be nice to have a precise mathematical way of expressing the relationship between two signals. Correlation is the mathematical technique that indicates in a precise quantitative way whether two signals are related and how related they are.

The mathematical calculation of correlation is based on an analysis of whether two signals tend to move in the same direction at the same time or tend to move in opposite directions at the same time. If one signal moves in a positive direction and the other signal also moves in a positive direction, at the same time, they are correlated, at least for that time. The same is true if they both move in a negative direction together. If, over a long period of time, the signals tend to move in the same direction at the same time, they are said to be positively correlated. If, over a long period of time, two signals tend to move in opposite directions at the same time they are also correlated, but in a negative sense. If, over a long period of time, the two signals tend to move in the same direction about half the time and in opposite directions the other half of the time, they are said to be uncorrelated. (This is true of the two sinusoids above that were 90° out of phase.) These statements are not mathematically precise but they describe in a conceptual way what correlation is.

The mathematical definition of correlation must somehow embody these ideas about how signals move relative to each other. The way it does this is by looking at the average value of the product of the functions. Consider first two signals, each of which has an average value of zero (Figure R-10 and Figure R-11). If they tend to move together in the same direction, their product tends to be positive. If they are both positive the product is positive and if they are both negative their product is still positive. Similarly, if they move in opposite directions most of the time, their product will tend to be negative most of the time. Therefore the average of their product over a long period of time is a good measure of how correlated they are and in which sense.
If the average values of the signals are both non-zero then a bias (a constant) will be added to the product. If the average values of the signals are large compared with their variations around the average values, the product of the average values will dominate the average of the product. In the common usage of the word correlation it is usually intended to indicate how the signals vary with time in relation to each other. The key word in the last sentence is “vary”. A constant does not vary. So to evaluate correlation in this sense we should observe not just the average of the product but the average of the product of the signals compared with the product of the average values of the signals. The variation around the bias will indicate whether the signal variations are moving in the same or opposite directions (Figure R-12 and Figure R-13).
If the average of the product of the signals is greater than the product of the average values of the two individual signals the variations of the signals are positively correlated. If the average of the product is less than the product of the averages, the variations of the signals are negatively correlated. If the average of the product equals the product of the averages the variations of the signals are uncorrelated. A method that is often used in signal analysis is to subtract the average value of each signal from the signal and then find the average of the product. Then the conclusions in the previous examples apply.

A careful look at the uncorrelated case in Figure R-13 will reveal that the average of the product and the product of the averages are not exactly the same, although they are very close. This occurs because the average is taken over a finite time. As the time over which the average is taken tends to infinity, these two values approach the same limit.

Energy Signals

The mathematical definition of correlation depends on the type of signals being analyzed. There are two commonly-accepted definitions, one for energy signals and one
for power signals. For two continuous-time energy signals $x(t)$ and $y(t)$ correlation is defined by $\int_{-\infty}^{\infty} x(t)y^*(t)dt$. For two discrete-time energy signals $x[n]$ and $y[n]$ correlation is defined by $\sum_{n=-\infty}^{\infty} x[n]y^*[n]$. For the common case in which both signals are real, these definitions simplify to $\int_{-\infty}^{\infty} x(t)y(t)dt$ and $\sum_{n=-\infty}^{\infty} x[n]y[n]$.

It is much more common in signal and system analysis to refer to the correlation function instead of just “the correlation”. The correlation function is a mathematical expression of how correlated two signals are as a function of how much one of them is shifted in time. The correlation between two functions is a single number. The correlation function between two functions is itself a function, a function of the shift amount. The mathematical definition of the correlation function $R_{xy}$ between two continuous-time energy signals $x(t)$ and $y(t)$ is\footnote{Unfortunately different authors use different definitions of the correlation function. The differences occur in the specification of which signal is to be shifted, which direction it is shifted and in the symbol used for the shift variable. Typical definitions for continuous-time signals are $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t + \tau)y(t)dt$ and $R_{yx}(\tau) = \int_{-\infty}^{\infty} x(t)y(t - \tau)dt$.}

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t + \tau)dt = \int_{-\infty}^{\infty} x(t - \tau)y^*(t)dt,$$

or, in the very common case in which both signals $x(t)$ and $y(t)$ are real,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t + \tau)dt = \int_{-\infty}^{\infty} x(t - \tau)y(t)dt.$$

For discrete-time energy signals,

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]y^*[n + m] = \sum_{n=-\infty}^{\infty} x[n - m]y^*[n],$$

It would be nice, of course, if everyone could agree on a common definition. But all that is really important is that a definition is established and used consistently. The fundamental characteristics of the correlation function and the implications for signal and system analysis are the same regardless of which definition is used.
or, if both signals \( x[n] \) and \( y[n] \) are real,

\[
R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]y[n + m] = \sum_{n=-\infty}^{\infty} x[n - m]y[n].
\]

Notice the similarity between the correlation function for two energy signals and the aperiodic convolution of two signals. The aperiodic convolution of two signals \( x \) and \( y \) is

\[
x(t) * y(t) = \int x(t - \tau) y(\tau) d\tau \quad \text{or} \quad x[n] * y[n] = \sum_{m=-\infty}^{\infty} x[n - m]y[m].
\]

The only difference is that in convolution one of the signals is time-inverted before the shifting process occurs and in correlation the time inversion is omitted. For energy signals, there is a simple mathematical relationship between correlation and convolution,

\[
R_{xy}(\tau) = x(-\tau) * y(\tau) \quad \text{or} \quad R_{xy}[m] = x[-m] * y[m].
\]

Since there is such a close relationship between convolution and correlation for energy signals, we can use the multiplication-convolution duality of the Fourier transform to help in calculating correlations. Convolution in the time domain corresponds to multiplication in the frequency domain. Therefore, using

\[
x(-t) \leftrightarrow X^*(f) \quad \text{and} \quad x[-n] \leftrightarrow X^*(F),
\]

the correlation function for energy signals can be expressed as

\[
R_{xy}(\tau) \leftrightarrow X^*(f)Y(f)
\]

or

\[
R_{xy}[m] \leftrightarrow X^*(F)Y(F).
\]

We can use the results summarized in Chapter 10 to find approximations to these correlation functions using the DFT.

\[
\left[ x(t) * h(t) \right]_{n=nT_s} \equiv T_s \times DFT^{-1}\left( DFT \left( x(nT_s) \right) \times DFT \left( h(nT_s) \right) \right)
\]

\[
R_{xy}(nT_s) = \left[ x(-\tau) * y(\tau) \right]_{\tau=nT_s} \equiv T_s \times DFT^{-1}\left( \left[ DFT \left( x(nT_s) \right) \right]^* \times DFT \left( y(nT_s) \right) \right)
\]

\[
x[n] * h[n] \equiv DFT^{-1}\left( DFT \left( x[n] \right) \times DFT \left( h[n] \right) \right)
\]

\[
R_{xy}[m] = x[-m] * y[m] \equiv DFT^{-1}\left( \left[ DFT \left( x[m] \right) \right]^* \times DFT \left( y[m] \right) \right)
\]
Power Signals

The correlation function between two continuous-time power signals \( x(t) \) and \( y(t) \) is mathematically defined by

\[
R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{T} x(t) y^*(t+\tau) dt = \lim_{T \to \infty} \frac{1}{T} \int_{T} x(t-\tau) y^*(t) dt .
\]

If \( x(t) \) and \( y(t) \) are both real,

\[
R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{T} x(t)y(t+\tau) dt = \lim_{T \to \infty} \frac{1}{T} \int_{T} x(t-\tau)y(t) dt .
\]

The correlation function between two discrete-time power signals \( x[n] \) and \( y[n] \) is mathematically defined by

\[
R_{xy}[m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y^*[n+m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n-m] y^*[n] .
\]

If \( x[n] \) and \( y[n] \) are both real,

\[
R_{xy}[m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y[n+m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n-m] y[n] .
\]

An important special case of correlation of power signals is the correlation between two periodic signals whose fundamental periods are such that the product of the two signals is also periodic. This will happen anytime the fundamental periods of the two periodic signals have a finite least common multiple (LCM).

For two periodic functions whose product has a period \( T \) or \( N \) the general form of the correlation function (for real power functions),

\[
R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{T} x(t)y(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y[n+m]
\]

can be simplified to

\[
R_{xy}(\tau) = \frac{1}{T} \int_{T} x(t) y(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y[n+m] \quad \text{(R.1)}
\]

because the integral over one period of the product, divided by the period, (which is the average of the integrand over one period) is the same as the average over any integer number of periods, including infinitely many periods. The right-hand sides of the two
equations in (R.1) are very similar to periodic convolutions. In fact we can express the correlation between two periodic signals over any period they have in common as a periodic convolution,

\[ R_{xy}(\tau) = \frac{x(-\tau) \otimes y(\tau)}{T} \quad \text{and} \quad R_{xy}[m] = \frac{x[-m] \otimes y[m]}{N} \]

over that common period or, using the CTFS or DFT, and their multiplication-convolution duality properties,

\[ x(t) \otimes y(t) \xrightarrow{T} \mathcal{F} \rightarrow T \cdot c_x[k] c_y[k] \]
and

\[ x[n] \otimes y[n] \xrightarrow{N} \mathcal{F} \rightarrow Y[k] X[k] \]

\[ R_{xy}(\tau) \xrightarrow{T} c_x[k] c_y[k] \quad \text{and} \quad R_{xy}[m] \xrightarrow{N} (1/N) X^*[k] Y[k] \]

where, in each case, the Fourier-series representation is taken over a time \( T \) or \( N \), which is any period that is common to both functions.

We can use the results summarized in Chapter 10 to find approximations to the continuous-time correlation functions using the DFT.

\[ \left[ x(t) \otimes h(t) \right]_{t \rightarrow nT_s} \equiv T_s \times \mathcal{DFT}^{-1} \left( \mathcal{DFT} \left( x(nT_s) \right) \times \mathcal{DFT} \left( h(nT_s) \right) \right) \]

\[ R_{xy}(nT_s) = \left[ \frac{x(-\tau) \otimes h(\tau)}{T} \right]_{\tau \rightarrow nT_s} \equiv (T_s / T) \times \mathcal{DFT}^{-1} \left( \left[ \mathcal{DFT} \left( x(nT_s) \right) \right]^* \times \mathcal{DFT} \left( y(nT_s) \right) \right) \]

The reason we have two definitions of the correlation function is that if we applied the definition for energy signals,

\[ R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n] y[n+m] \]

to two power signals, the result would be infinite and if we applied the definition for power signals,

\[ R_{xy}(\tau) = \lim_{T \to \infty} \left( 1/T \right) \int_{-\infty}^{\infty} x(t)y(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \lim_{N \to \infty} \left( 1/N \right) \sum_{n=-N}^{N} x[n] y[n+m] \]

to two energy signals the result would be zero. It is natural to ask at this point what to use if one signal is an energy signal and the other signal is a power signal. The answer is to use the energy signal definition,
\[
R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)\,dt \quad \text{or} \quad R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]y[n+m]
\]

The finite energy of the energy signal will keep the integral of the product from being infinite.

As stated previously, the correlation function is more general than just “the correlation” because it is a function of how much the second function is shifted. Some functions are uncorrelated at one shift and highly correlated at another shift, for example a continuous-time sine and a cosine of the same frequency. If neither one is shifted, they are uncorrelated. If one is shifted by 90° they are highly correlated, either positively or negatively (Figure R-14).

![Graphical illustration of the correlation between a cosine and a sine at different shifts](image)

Figure R-14  Graphical illustration of the correlation between a cosine and a sine at different shifts

Example R-1  Correlation function for two continuous-time energy signals

Find the correlation function for the energy signals in Figure R-15.

![Two energy signals](image)

Figure R-15  Two energy signals
Method 1:

\[ x_1(t) = 4 \text{rect}(t/4) \xrightarrow{\mathcal{F}} X_1(f) = 16 \text{sinc}(4f) \]

\[ x_2(t) = \text{rect}\left(\frac{t+1}{2}\right) - \text{rect}\left(\frac{t-1}{2}\right) \xrightarrow{\mathcal{F}} X_2(f) = 2 \text{sinc}(2f)\left(e^{j2\pi f} - e^{-j2\pi f}\right) \]

\[ R_{12}(\tau) \xrightarrow{\mathcal{F}} X_1^*(f)X_2(f) = 32 \text{sinc}(4f)\text{sinc}(2f)\left(e^{j2\pi f} - e^{-j2\pi f}\right) \]

Using

\[
\frac{a+b}{2} \text{tri}\left(\frac{2t}{a+b}\right) - \frac{a-b}{2} \text{tri}\left(\frac{2t}{a-b}\right) \xrightarrow{\mathcal{F}} \mid ab \mid \text{sinc}(af)\text{sinc}(bf), a > b > 0
\]

\[ 12 \text{tri}\left(\frac{2t}{3}\right) - 4 \text{tri}(2t) \xrightarrow{\mathcal{F}} 32 \text{sinc}(4f)\text{sinc}(2f), a > b > 0. \]

Then using the time-shifting property of the CTFT,

\[
4 \left\{ 3 \text{tri}\left(\frac{2(t+1)}{3}\right) - \text{tri}(2(t+1)) \right\} \xrightarrow{\mathcal{F}} 32 \text{sinc}(4f)\text{sinc}(2f)\left(e^{j2\pi f} - e^{-j2\pi f}\right).
\]

Therefore

\[
R_{12}(\tau) = 4 \left\{ 3 \text{tri}\left(\frac{2(\tau+1)}{3}\right) - \text{tri}(2(\tau+1)) - 3 \text{tri}\left(\frac{2(\tau-1)}{3}\right) + \text{tri}(2(\tau-1)) \right\}.
\]

The function, \(3 \text{tri}\left(2(t+1)/3\right) - \text{tri}(2(t+1))\), is a trapezoid of height two whose lower base extends from -4 to 2 and whose upper base extends from -2 to zero. It is therefore centered at -1. The function \(3 \text{tri}\left(2(t-1)/3\right) - \text{tri}(2(t-1))\) is identical except shifted to the right by 2 to be centered at +1. When we subtract the second function from the first and multiply by four we get the function in Figure R-16.

Figure R-16 Correlation function
Method 2:

The definition of the correlation function for energy signals is

\[ R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)\,dt \]

or, in this case,

\[ R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t+\tau)\,dt . \]

The integral depends on the shift amount, \( \tau \), as illustrated in Figure R-17.

Case 1: \( \tau > 4 \)

In this case the non-zero parts of the signals do not overlap and the correlation function is zero.

Case 2: \( 2 < \tau < 4 \)

For this case the correlation function is

\[ R_{12}(\tau) = \int_{-2}^{-\tau+2} 4 \times (-1)\,dt = 4(\tau - 4) . \]

Case 3: \( 0 < \tau < 2 \)

For this case the correlation function is

\[ R_{12}(\tau) = \int_{-2}^{-\tau} 4 \times (+1)\,dt + \int_{-\tau}^{-\tau+2} 4 \times (-1)\,dt = 4 \left( \int_{-2}^{-\tau} dt - \int_{-\tau}^{-\tau+2} dt \right) = -4\tau . \]
Case 4: $-2 < \tau < 0$

Because of the even symmetry of the first signal and the odd symmetry of the second signal, the result is the same as Case 3, $R_{12}(\tau) = -4\tau$.

Case 5: $-4 < \tau < 2$

Again, from symmetry considerations,

$$ R_{12}(\tau) = 4(\tau + 4) $$

Case 6: $\tau < -4$

In this case the non-zero parts of the signals do not overlap and the correlation function is again zero.

When this result is graphed it is exactly the same as the previous result in Figure R-16.

Example R-2  Correlation function for two discrete-time power signals

Find the correlation between the discrete-time power signals,

$$ x[n] = 5\cos\left(\frac{2\pi n}{5}\right) \quad \text{and} \quad y[n] = 2\cos\left(\frac{2\pi n}{7}\right) $$

Method 1:

Use the relation

$$ R_{xy}[m] \xleftarrow{\text{DFT}} (1 / N) X^*[k] Y[k]. $$

Before we can use this result we must find a common period for the two signals. The two individual periods are 5 and 7. The least common multiple of those two periods is 35. The two DFT harmonic functions are

$$ X[k] = \frac{165}{2}(\delta_{35}[k - 7] + \delta_{35}[k + 7]) $$

and

$$ Y[k] = 35(\delta_{35}[k - 5] + \delta_{35}[k + 5]). $$

Therefore

$$ R_{xy}[m] \xleftarrow{\text{DFT}} (1 / 35)(165 / 2)(35)(\delta_{35}[k - 7] + \delta_{35}[k + 7])(\delta_{35}[k - 5] + \delta_{35}[k + 5]). $$
\[ R_{xy}[m] = \frac{1}{35} (165/2) (\delta_{35}[k-7] + \delta_{35}[k+7]) (\delta_{35}[k-5] + \delta_{35}[k+5]) \]

This is the product of two periodic sequences of discrete-time impulses. Therefore the product is zero except where \( X[k] \) and \( Y[k] \) both have a non-zero impulse that occurs at the same value of \( k \). But the impulses in \( X[k] \) and \( Y[k] \) never occur at the same value of \( k \). Therefore the correlation function is zero,

\[ R_{xy}[m] = 0. \]

Method 2:

The general expression for the correlation function for two discrete-time power signals with a common period is

\[ R_{xy}[m] = \frac{1}{N} \sum_{n=(N)} x[n]y[n+m] \]

Applying that to \( x[n] \) and \( y[n] \) we get

\[ R_{xy}[m] = \frac{1}{35} \sum_{n=(35)} 5 \cos \left( \frac{2\pi n}{5} \right) \cos \left( \frac{2\pi (n+m)}{7} \right) \]

Using

\[ \cos(x)\cos(y) = \frac{1}{2} \left[ \cos(x-y) + \cos(x+y) \right] \]

we get

\[ R_{xy}[m] = \frac{5}{35} \sum_{n=(35)} \left[ \cos \left( \frac{2\pi n}{5} - \frac{2\pi (n+m)}{7} \right) + \cos \left( \frac{2\pi n}{5} + \frac{2\pi (n+m)}{7} \right) \right] \]

\[ R_{xy}[m] = \frac{1}{7} \sum_{n=(35)} \left[ \cos \left( \frac{4\pi n}{35} - \frac{2\pi m}{7} \right) + \cos \left( \frac{24\pi n}{35} + \frac{2\pi m}{7} \right) \right] \]

Then, using

\[ \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \]

\[ R_{xy}[m] = \frac{1}{7} \left[ \cos \left( -\frac{2\pi m}{7} \right) \sum_{n=(35)} \cos \left( \frac{4\pi n}{35} \right) - \sin \left( -\frac{2\pi m}{7} \right) \sum_{n=(35)} \sin \left( \frac{4\pi n}{35} \right) \right. \]

\[ + \cos \left( \frac{2\pi m}{7} \right) \sum_{n=(35)} \cos \left( \frac{24\pi n}{35} \right) - \sin \left( \frac{2\pi m}{7} \right) \sum_{n=(35)} \sin \left( \frac{24\pi n}{35} \right) \left. \right] \]
Each summation in this equation is of a sinusoid over exactly one period. So each summation is zero and the correlation function is also zero.

\[ R_{xy}[m] = 0. \]

These two discrete-time power signals are completely uncorrelated. The lack of correlation is a consequence of the fact that the two signals have different frequencies and the correlation of a power signal is computed over a common period. A comparison between the first and second methods illustrates in a convincing way the power of transform methods in signal and system analysis.

The result of Example R-2 leads to an important general conclusion. The correlation between two sinusoids of different frequencies is zero. Let \( x_1(t) = A_1 \cos(2\pi f_{01}t + \theta_1) \) and \( x_2(t) = A_2 \cos(2\pi f_{02}t + \theta_2) \). Their CTFS harmonic functions consist of impulses at different locations and the product is zero. Therefore the correlation function is also zero. We can also show that the correlation is zero directly from the definition. The correlation function is

\[
R_{12}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A_1 \cos(2\pi f_{01}t + \theta_1) A_2 \cos(2\pi f_{02}(t + \tau) + \theta_2) dt.
\]

We can use the trigonometric identity,

\[
\cos(x)\cos(y) = (1/2)[\cos(x - y) + \cos(x + y)],
\]

to write

\[
R_{12}(\tau) = \lim_{T \to \infty} \frac{A_1 A_2}{2T} \int_{-T/2}^{T/2} \left[ \cos(2\pi(f_{01} - f_{02})t - 2\pi f_{02}\tau + \theta_1 - \theta_2) + \cos(2\pi(f_{01} + f_{02})t + 2\pi f_{02}\tau + \theta_1 + \theta_2) \right] dt.
\]

If \( f_{01} \neq f_{02} \), then

\[
R_{12}(\tau) = \lim_{T \to \infty} \frac{A_1 A_2}{2T} \left[ \frac{\sin(2\pi(f_{01} - f_{02})t - 2\pi f_{02}\tau + \theta_1 - \theta_2)}{2\pi(f_{01} - f_{02})} \right]_{-T/2}^{T/2} \times \frac{\sin(2\pi(f_{01} + f_{02})t + 2\pi f_{02}\tau + \theta_1 + \theta_2)}{2\pi(f_{01} + f_{02})} \left[ \frac{\sin(2\pi(f_{01} + f_{02})t + 2\pi f_{02}\tau + \theta_1 + \theta_2)}{2\pi(f_{01} + f_{02})} \right]_{-T/2}^{T/2}.
\]

In the limit as \( T \) approaches infinity, the division by \( T \) of a bounded quantity is zero. Therefore if \( f_{01} \neq f_{02} \), \( R_{12}(\tau) = 0 \).
Example R-3  Correlation function for two offset continuous-time cosines

Find the correlation function between

\[ x(t) = 10 + \cos(2\pi t) \quad \text{and} \quad y(t) = 10 - \cos(2\pi t). \]

\[ R_{xy}(\tau) \leftarrow \frac{\mathcal{F}}{\mathcal{T}} c_x[k] c_y[k] \]

These signals share a common period, \( T = 1 \). Therefore

\[ R_{xy}(\tau) \leftarrow \frac{\mathcal{F}}{\mathcal{T}} \left[ 10\delta[k] + \left( \frac{1}{2} \right)(\delta[k-1] + \delta[k+1]) \right]^* \left[ 10\delta[k] - \left( \frac{1}{2} \right)(\delta[k-1] + \delta[k+1]) \right] \]

\[ R_{xy}(\tau) \leftarrow \frac{\mathcal{F}}{\mathcal{T}} 100\delta[k] - \left( \frac{1}{4} \right)(\delta[k-1] + \delta[k+1]) \]

\[ R_{xy}(\tau) = 100 - \frac{1}{4}\cos(2\pi \tau) \]

This correlation function is dominated by the constant 100 and has a small variation about that constant.

It is important in light of this result to comment on conventional terminology relative to the word, “correlated” and to the significance of the correlation function as defined here. In the theories of random variables and stochastic processes the conventional meaning of the word “correlated” is that if two signals have variations about their average values that tend to move in the same direction at the same time they are positively correlated. If the variations tend to move in the opposite directions at the same time they are negatively correlated. As illustrated in this example, the correlation function actually indicates something different. This correlation function is positive for all \( \tau \) because of the dominance of the large positive average value. But the variations about the average value are actually negatively correlated.

This is a conflict between the most natural interpretation of the terms, “correlated” and “correlation function”. Shouldn’t the correlation function indicate whether or not two signals are correlated? If we remove the average value from signals before calculating the correlation function, there is no conflict of meaning. There is another function commonly defined in random variable theory, the \textbf{covariance function}, that does exactly that. So we could say that the covariance function is the real indicator of whether or not two signals are correlated.
R.4 Autocorrelation

Relation to Signal Energy and Signal Power

A very important special case of the correlation function is the correlation of a function with itself. This type of correlation function is called **autocorrelation**. If \( x(t) \) is an energy signal its autocorrelation is

\[
R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) x(t + \tau) \, dt \quad \text{or} \quad R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n] x[n + m]
\]

At a shift of zero that becomes

\[
R_{xx}(0) = \int_{-\infty}^{\infty} x^2(t) \, dt \quad \text{or} \quad R_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n]
\]

which is the total signal energy of the signal.

If \( x(t) \) or \( x[n] \) is a power signal the autocorrelation at zero shift is

\[
R_{xx}(0) = \lim_{T \to \infty} \left( \frac{1}{T} \right) \int_{-T/2}^{T/2} x^2(t) \, dt \quad \text{or} \quad R_{xx}[0] = \lim_{N \to \infty} \left( \frac{1}{N} \right) \sum_{n=-N}^{N} x^2[n]
\]

which is the average signal power of the signal. Correlation functions defined in this way mesh nicely into previous theory about signal energy and power.

Properties of Autocorrelation

The autocorrelation depends on the choice of the shift amount so we cannot say what the autocorrelation function looks like until we know what the function is. But we can say that the value of the autocorrelation can never be bigger than it is at zero shift. That is,

\[
R_{xx}(0) \geq R_{xx}(\tau) \quad \text{or} \quad R_{xx}[0] \geq R_{xx}[m]
\]

because at a shift of zero, the correlation with itself is obviously as large as it can get since the shifted and unshifted versions coincide. Also,

\[
R_{xx}(-\tau) = \int_{-\infty}^{\infty} x(t) x(t - \tau) \, dt \quad \text{or} \quad R_{xx}(-\tau) = \lim_{T \to \infty} \left( \frac{1}{T} \right) \int_{-T/2}^{T/2} x(t) x(t - \tau) \, dt
\]
Then if we make the change of variable,
\[ t' = t - \tau \quad \text{and} \quad dt' = dt \]
we can show that
\[ R_{xx}(\tau) = R_{xx}(-\tau) . \]

It can be shown by a similar technique that
\[ R_{xx}[m] = R_{xx}[-m] \]
or, in other words, all autocorrelation functions are even functions (but not all correlation functions).

Another characteristic of the autocorrelation function is that a time shift of a signal does not change its autocorrelation. Let \( R_{xx}[m] \) be the autocorrelation function of a discrete-time energy signal \( x[n] \). Then
\[ R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m] . \]
Now let \( y[n] = x[n-n_0] \). Then
\[ R_{yy}[m] = \sum_{n=-\infty}^{\infty} y[n]y[n+m] = \sum_{n=-\infty}^{\infty} x[n-n_0]x[n-n_0+m] . \]
We can make a change of variable \( q = n - n_0 \). Then
\[ R_{yy}[m] = \sum_{q=-\infty}^{\infty} x[q]x[q+m] = R_{xx}[m] , \]
proving that the autocorrelation functions of \( x[n] \) and \( y[n] \) are the same, regardless of what the value of \( n_0 \) is. The same rule holds for continuous-time energy signals and continuous-time and discrete-time power signals.

The autocorrelation of a sum of sinusoids of different frequencies is the sum of the autocorrelations of the individual sinusoids. To demonstrate this idea let a continuous-time power signal \( x(t) \) be a sum of two sinusoids \( x_1(t) \) and \( x_2(t) \) where
\[ x_1(t) = A_1 \cos(2\pi f_0_1 t + \theta_1) \quad \text{and} \quad x_2(t) = A_2 \cos(2\pi f_0_2 t + \theta_2) , \quad f_0_1 \neq f_0_2 . \]
The autocorrelation of this signal is
\[ R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)d\tau \]

R-23
The autocorrelation function is describing an important feature of these signals. Each of them is random fluctuation around zero for even very small non-zero values of the shift \( m \). The autocorrelation functions have a sharp peak at \( m = 0 \) and then very quickly go to a small random fluctuation around zero for even very small non-zero values of the shift \( m \). The autocorrelation function is describing an important feature of these signals. Each of them

\[
R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[ x_1(t)x_1(t+\tau) + x_1(t)x_2(t+\tau) + x_2(t)x_1(t+\tau) + x_2(t)x_2(t+\tau) \right] d\tau
\]

The correlations \( R_{12}(\tau) \) and \( R_{21}(\tau) \) are both zero because they are the correlations between sinusoids of different frequencies. Therefore

\[ R_x(\tau) = R_1(\tau) + R_2(\tau). \]

**Autocorrelation Examples**

In Figure R-18 and Figure R-19 are some graphical examples of some energy signals and their autocorrelation functions. Figure R-18 is an illustration of the autocorrelation functions for three random discrete-time energy signals.

Since they are random these signals are all different but they do have some similar properties. One of the similarities is seen in their autocorrelation functions. All three autocorrelation functions have a sharp peak at \( m = 0 \) and then very quickly go to a small random fluctuation around zero for even very small non-zero values of the shift \( m \). The autocorrelation function is describing an important feature of these signals. Each of them
changes very rapidly with time to new values that have practically no correlation with past or future values, even at a very short time in the past or future.

Figure R-19 is an illustration of the autocorrelation functions for two continuous-time sinusoidal bursts. These waveforms are typical of some types of communication signals that encode binary data for transmission.

![Cosine and sine bursts and their autocorrelation functions](image)

Figure R-19 Cosine and sine bursts and their autocorrelation functions

Notice that, even though one is a cosine burst and the other is a sine burst, their autocorrelation functions are almost identical. Also notice that, even though the sine function is odd, its autocorrelation function is even. The autocorrelation function is even because it indicates how a function relates to itself when shifted, not how the function itself varies with time. For these two signals, the relation of each to a shifted version of itself is almost exactly the same. (When we get to power signals, we will see that a cosine and sine of the same frequency and amplitude have exactly the same autocorrelation function.) The fact that these two autocorrelation functions are so close comes from the same basic reason that when a function is time shifted its autocorrelation function does not change. In the case of the two sinusoidal bursts, the sine burst is not simply a time-shifted version of the cosine burst, but it almost is. That is why the two autocorrelation functions are almost the same.

Try to visualize the shifting process inherent in the autocorrelation (Figure R-20).
At zero shift either sinusoidal burst and its shifted version coincide and the area under the product is a maximum. That is why the autocorrelation functions have a maximum value at $\tau = 0$. Then as we shift one version of the signal, at a shift of half a fundamental period of the underlying sinusoid the positive and negative peaks line up and we get a large negative area under the product. Then as we shift another half fundamental period, the positive peaks line up again but now, because of the shift, the peaks on opposite ends of the two versions are multiplied by zero. Therefore, although the area under the product reaches a positive peak, it is a smaller positive peak than the one for zero shift. As the shift proceeds further, the peaks, both positive and negative, gradually fall to zero because of the diminishing overlap between the non-zero portions of the signals.

Figure R-21 illustrates the autocorrelation functions for three random continuous-time power signals.

Notice how the rate of variation of the autocorrelation functions with time indicates generally how fast the signals themselves change with time. In other words, the autocorrelation function tells us something about the frequency content of the signal. We will soon make that relationship concrete when we define energy spectral density and power spectral density.
Figure R-22 illustrates the autocorrelation functions for a cosine and a sine. As indicated earlier, since a sine of the same frequency and amplitude as a cosine is just a time-shifted cosine, the two autocorrelation functions must be the same.

![Figure R-22](image)

Figure R-22 A cosine and a sine of the same amplitude and frequency and their identical autocorrelation functions

Example R-4  Autocorrelation function for a continuous-time rectangular wave

Find the autocorrelation of the continuous-time power signal in Figure R-23.

![Figure R-23](image)

Figure R-23 A square-wave signal

This is a power signal so the autocorrelation is

$$R_x(\tau) \leftarrow \frac{\delta x}{T} \rightarrow c_x[k] c_x[k] = |c_x[k]|^2.$$

The signal is described by

$$x(t) = A \text{rect}(2t / T_0) * \delta_{T_0}(t)$$

and its CTFS harmonic function (with $T = T_0$) is

$$c_x[k] = (A / 2) \text{sinc}(k / 2).$$

Therefore the autocorrelation is
\[ R_x(\tau) \xrightarrow{\mathcal{F}} (A/2) \text{sinc}(k/2)^2 = (A/2)^2 \text{sinc}^2(k/2) \]

and, using

\[ \text{tri}(t/w) \ast \delta_{T_0}(t) \xrightarrow{\mathcal{F}} (w/T_0) \text{sinc}^2(wk/T_0), \]

\[ R_x(\tau) = \left( A^2/2 \right) \text{tri}(2\tau/T_0) \ast \delta_{T_0}(\tau). \]

(Figure R-24).

![Figure R-24 Autocorrelation of the square wave signal](image)

Example R-5  Autocorrelation function for a discrete-time cosine modulated by a sinc

Find the autocorrelation of the discrete-time energy signal,

\[ x[n] = \cos(\pi n) \text{sinc}(n/2) \]

(Figure R-25).

![Figure R-25 A discrete-time signal](image)

We can use,

\[ R_x[m] \xrightarrow{\mathcal{F}} X^*(F)X(F), \]

to help find this autocorrelation. The DTFT of \( x[n] \) is

\[ X(F) = \left( \frac{1}{2} \right) \left[ \delta_i(F-1/2) + \delta_i(F+1/2) \right] \otimes \left( 2\text{rect}(2F) \ast \delta_i(F) \right) \]

or

\[ X(F) = \left[ \delta(F-1/2) + \delta(F+1/2) \right] \ast \left( 2\text{rect}(2F) \ast \delta_i(F) \right) \]

or

\[ X(F) = \text{rect}(2(F-1/2)) \ast \delta_i(F) + \text{rect}(2(F+1/2)) \ast \delta_i(F). \]
This is the sum of two periodic rectangular functions, that, because of the two discrete-
time frequency shifts, \( F - 1/2 \) and \( F + 1/2 \), exactly coincide. So the sum is just twice
either of the two periodic rectangular functions (Figure R-26).

![Figure R-26 Magnitude of the DTFT of \( x[n] = \cos(\pi n) \text{sinc}(n/2) \)]

Since \( X(F) \) is purely real, \( X(F) = X^*(F) \). \( X(F) \) is a periodically-repeated rectangle of
height, 2, and width, 1/2, with fundamental period, one,

\[
X(F) = 2 \text{rect}(2(F - 1/2)) \ast \delta_1(F).
\]

Therefore

\[
R_x[m] \leftarrow \mathcal{F}^{-1} \left( 2 \text{rect}(2(F - 1/2)) \ast \delta_1(F) \right)^2 = 4 \left( \text{rect}^2(2(F - 1/2)) \ast \delta_1(F) \right)
\]

The CTFT or inverse CTFT of a unit rectangle and the square of a unit rectangle are the
same because their values only differ at two isolated points. The inverse Fourier
transform of the unit-height rectangle-squared convolved with \( \delta_1(F) \) is the same as the
inverse Fourier transform of the unit-height rectangle convolved with \( \delta_1(F) \),

\[
\text{sinc}(n/w) \leftarrow \mathcal{F}^{-1} w \text{rect}^2(wF) \ast \delta_1(F),
\]

and, using the frequency-shifting property of the DTFT,

\[
e^{j2\pi F_0 n} x[n] \leftarrow \mathcal{F} \left( X(F - F_0) \right),
\]

we get

\[
R_x[m] = 2 \text{sinc}(n/2) e^{jn} = 2 \text{sinc}(n/2) \left( \cos(\pi n) + j \sin(\pi n) \right)
\]

or

\[
R_x[m] = 2 \cos(\pi n) \text{sinc}(n/2).
\]

So we get the surprising result that the autocorrelation function for \( \cos(\pi n) \text{sinc}(n/2) \) is
2\( \cos(\pi n) \text{sinc}(n/2) \). Except for a factor of two, it is its own autocorrelation!
The most important use of autocorrelation is in the analysis of the effect of LTI systems on random signals. Consider the following qualitative argument to see how autocorrelation describes a random signal. Let a signal $x(t)$ be a linear combination of sinusoids of different frequencies

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_0 t + \theta_k).$$

Then, since the sinusoids are all of different frequencies,

$$R_x(\tau) = \sum_{k=1}^{N} R_k(\tau),$$

where $R_k(\tau)$ is the autocorrelation of $A_k \cos(2\pi f_0 t + \theta_k)$. Also $R_k(\tau)$ is independent of the choice of $\theta_k$. Now imagine that we were to form several versions of $x(t)$ by using randomly-chosen phase shifts $\theta_k$ but the same amplitudes (Figure R-27 and Figure R-28).

![Figure R-27](image-url) Two illustrations of groups of four random signals with identical autocorrelation functions
Two more illustrations of groups of four random signals with identical autocorrelation functions

In each group of four signals in Figure R-27 and Figure R-28 the signals are all different but they have identical autocorrelation functions. Looking at the signals in each group of four it is apparent that are similar but not exactly the same. Their common characteristics (the amplitudes and frequencies of the sinusoids that form them) are described by the autocorrelation function. The autocorrelation function describes a signal generally, but not exactly. It is the best description of a random signal, short of an exact description which, as a practical matter, is often either not available or not needed.

Correlation is the basis of a widely-used technique in communication systems called **matched filtering**. In digital communication systems the only important thing is that the 1’s and 0’s in the data stream be distinguishable from each other so the receiver can reproduce the bit pattern that was transmitted. A 1 is sent as a signal of some shape and a 0 is sent as a signal of some different shape, ideally a shape that is easily distinguished from the shape that represents a 1. They could be sent as different voltage-level pulses, or as sinusoidal bursts with different phase or different frequency, or in a variety of other ways. The object of the receiver is to recognize the bits. The designer of the communication system knows the shapes of the signals representing the bits so the receiver is designed to optimally detect those shapes in the presence of noise, which is always present in any system at some level.

It has been shown that in the presence of the most common type of additive random noise, the best way to detect a signal of a certain shape is to use a filter that is matched to that shape. Let the signal representing a 1 be \( x_1(t) \) and let the signal representing a 0 be \( x_0(t) \). A matched filter is an LTI system whose impulse response, \( h(t) \), is a scaled, and perhaps shifted, version of the time inverse of the signal to be detected. Typical shapes for 1’s and 0’s and the corresponding matched filter impulse responses are illustrated in Figure R-29.
Some signals representing 1’s and 0’s and the impulse responses of matched filters designed to optimally detect them in the presence of noise

Suppose we are designing the part of the system that detects 1’s. Let a transmitted 1 be \( x_{1T}(t) = x_1(t) \) and let a received 1 be \( x_{1R}(t) = Ax_1(t-t_D) \) where \( A \) is some constant representing the attenuation in transmission and \( t_D \) is a constant representing the propagation delay in transmission. The general form of the impulse response of that system would be \( h_1(t) = Bx_1(-t+t_0) \) where \( B \) is an arbitrary constant. The response, \( y_1(t) \), of the receiver is the convolution of the excitation (the received signal) with the impulse response,

\[
y_1(t) = x_{1R}(t) * h_1(t) = Ax_1(t-t_D) * Bx_1(-t+t_0)
\]

or

\[
y_1(t) = AB \int_{-\infty}^{\infty} x_1(\tau-t_D)x_1(-(t-\tau)+t_0)d\tau
\]

Making the change of variable, \( \tau-t_D = \lambda \),

\[
y_1(t) = AB \int_{-\infty}^{\infty} x_1(\lambda)x_1(\lambda-(t-\tau-t_0))d\lambda.
\]
Applying the definition of autocorrelation and the fact that it is an even function,

\[ R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt, \]

for continuous-time energy signals, we get

\[ y_1(t) = AB \times R_{x_1}(t-t_D-t_0). \]

The matched filter response for 1’s is a scaled version of the autocorrelation function of the signal representing 1’s, delayed in time by the propagation delay \( t_D \) and the filter delay \( t_0 \). For this reason another common name for the matched filter is the correlation filter. Figure R-30 is an illustration of a signal, without noise and with noise, and the response of a matched filter for each case.

![Figure R-30](image)

Figure R-30  A 1 followed by a 0, an filter impulse response matched to the 1 and the response of the filter with and without noise

An autocorrelation is a maximum when its argument is zero, so the response of the matched filter is a maximum when \( t = t_D + t_0 \) and if a 1 is present in the signal the matched filter will be a maximum at that time. If the signal representing a 0 is the negative of the signal representing a 1 the same filter can be used to detect both. If, at the end of a bit time, the signal from the matched filter is positive then the bit is probably a 1 and if it is negative the bit is probably a 0.

### R.5 Crosscorrelation

**Properties of Crosscorrelation**

A common term for the correlation function between two different signals is cross correlation to distinguish it from autocorrelation. Autocorrelation is simply a special case
of the cross correlation function. Cross correlation is more general than autocorrelation so the properties are not as numerous but there is one property that is sometimes useful,

\[ R_{xy}(\tau) = R_{yx}(-\tau) \quad \text{or} \quad R_{xy}[m] = R_{yx}[-m] \]

Notice that when \( y(t) = x(t) \) or \( y[n] = x[n] \) this property reduces to the property of autocorrelation functions that they are even functions of shift.

**Examples of Crosscorrelation**

As an illustration of crosscorrelation, suppose that the two continuous-time power signals \( x(t) \) and \( y(t) \) are the ones illustrated in Figure R-31. (These illustrations show the signals for a finite time. They are assumed to be power signals that are similar for other time ranges.)

![Figure R-31](image)

Their crosscorrelation function is illustrated in Figure R-32.

![Figure R-32](image)

It might not have been obvious at first glance that the two waveforms were highly correlated but one look at the largest peak in the cross correlation function indicates that they are. The peak in the cross correlation function occurs at a shift that is exactly the
amount of shift between \( x(t) \) and \( y(t) \) at which they “line up”. That is, if the \( y(t) \) signal is shifted to the left by that amount all the peaks of \( x(t) \) and \( y(t) \) coincide in time and there is maximum similarity or correlation between the two waveforms at that shift amount. Figure R-33 is a graph of the two signals with \( y(t) \) shifted to emphasize the point.

![Figure R-33](image)

Figure R-33  Original functions with \( y(t) \) shifted to show correlation

In signal analysis it is sometimes very important whether two signals are correlated or not. When two signals are added, the signal power in the sum depends strongly on whether the two signals are correlated. Consider three discrete-time signals, \( x[n] \), \( y[n] \) and \( z[n] \) (Figure R-34). All are sinusoidal with the same amplitude and frequency.

![Figure R-34](image)

Figure R-34  Three discrete-time sinusoidal signals

Now consider the signals \( x[n] + y[n] \) and \( x[n] + z[n] \) (Figure R-35).
The signal $x[n] + y[n]$ definitely has a greater amplitude than the signal $x[n] + z[n]$ and, therefore, has a greater average signal power. This occurs basically because $x[n]$ and $y[n]$ are positively correlated (they are equal) and $x[n]$ and $z[n]$ are uncorrelated (they are $90^\circ$ out of phase).

Next consider three continuous-time random signals $x(t)$, $y(t)$ and $z(t)$ (Figure R-36), graphed on the same scale, all of which have an average value of zero and the same signal power.

Below are plots of $x(t) + y(t)$ and $x(t) + z(t)$ on the same scale (Figure R-37).
It should be apparent that there is a qualitative difference between $x(t) + y(t)$ and $x(t) + z(t)$. That is, $x(t) + z(t)$ generally deviates farther from zero than $x(t) + y(t)$ does.

The power in a signal is proportional to its square. When $x(t) + y(t)$ and $x(t) + z(t)$ are squared the difference becomes more apparent. Again they are graphed on the same scale (Figure R-38).

The average power of a signal is proportional to the mean of its square. The mean of the square of $x(t) + z(t)$ is bigger than the mean of the square of $x(t) + y(t)$ by a factor of two. A graph of the crosscorrelation functions between the signals, on the same scale, reveals why (Figure R-39).
It is now apparent that \( x(t) \) and \( z(t) \) are highly correlated at a shift of zero. (They are in fact identical.) But \( x(t) \) and \( y(t) \) are not well correlated at all, at any shift. When \( x(t) \) and \( z(t) \) are added the result is simply \( 2x(t) \). Everywhere \( x(t) \) is positive, so is \( z(t) \) and everywhere \( x(t) \) is negative so is \( z(t) \). The squaring operation makes both positive and negative sums positive.

What would happen to the average power of \( x(t) + z(t) \) if \( x(t) \) were equal to the negative of \( z(t) \)? Obviously then \( x(t) + z(t) \) would be zero everywhere and the average power would also be zero. The cross correlation between \( x(t) \) and \( z(t) \) would then have the form illustrated in Figure R-40.

![Figure R-40 Cross correlation between two signals with perfect negative correlation](image)

What if \( z(t) \) were time-shifted a little before adding it to \( x(t) \)? Notice that the correlation between \( x(t) \) and \( z(t) \) is very high at zero shift but quickly goes to a very low value for even a small shift. When \( z(t) \) is shifted even a small amount, the power in \( x(t) + z(t) \) immediately goes to the same as in \( x(t) + y(t) \), because the correlation goes to approximately zero.

In Figure R-41 and Figure R-42 are some examples of pairs of signals and their cross correlations.

![Figure R-41 Cross correlations between periodic, non-sinusoidal continuous-time signals](image)
Recall the formulas for the trigonometric CTFS harmonic function of a periodic signal over exactly one fundamental period,

\[ a_x[k] = \frac{2}{T_0} \int_{T_0} x(t) \cos\left(\frac{2\pi kt}{T_0}\right) dt \quad , \quad k = 1, 2, 3, \ldots \]

and

\[ b_x[k] = \frac{2}{T_0} \int_{T_0} x(t) \sin\left(\frac{2\pi kt}{T_0}\right) dt \quad , \quad k = 1, 2, 3, \ldots \]

Each value of \( a_x[k] \) or \( b_x[k] \) is simply twice the cross correlation, at zero shift, between the function \( x(t) \) and sines and cosines of different fundamental periods. That is,

\[ a_x[k] = 2R_{xc}(0) \quad , \quad b_x[k] = 2R_{xs}(0) \]

where

\[ c(t) = \cos\left(\frac{2\pi kt}{T_0}\right) \quad \text{and} \quad s(t) = \sin\left(\frac{2\pi kt}{T_0}\right). \]

Similarly,

\[ c_x[k] = \frac{1}{T_0} \int_{T_0} x(t) e^{-j2\pi kt/T_0} dt = R_{xz}(0) \]

where

\[ z(t) = e^{+j2\pi kt/T_0}. \]

(Notice that in the equation for \( c_x[k] \), the general form of cross correlation for complex functions must be used. That is what makes the sign in the exponent of \( e \) in the last equation + instead of -.)

Now the representation of a signal by a Fourier series, which is a process of decomposing a signal into a linear combination of sinusoidal functions, can be seen as a
process of correlating the signal with the sinusoids to find out whether any particular sinusoid or complex exponential is present in the signal and, if so, how much of it is there.

R.7 Energy Spectral Density (ESD)

In the remaining sections of this chapter there will be a discussion of energy spectral density (ESD) and then power spectral density (PSD) and their relation to autocorrelation. During the development of these concepts it is natural to wonder why autocorrelation, ESD and PSD are necessary and useful since what happens to a signal seems to be completely and directly determined by the use of linear system concepts and the Fourier transform without appealing to the concepts of autocorrelation, ESD and PSD. But this is only true if one has an exact description of the signal. As mentioned earlier, most real signals in real systems do not have an exact description but the autocorrelation and power spectral density can still be found (or at least estimated). This type of signal is called a random signal. The best way to analyze random signals as they progress through systems is through their autocorrelation, ESD and/or PSD. Since random variables are not covered in this appendix we will apply the ideas of ESD, PSD and autocorrelation to deterministic signals to illustrate the principles involved.

Definition and Derivation of Energy Spectral Density

Parseval’s theorem relates the total signal energy in a signal \( x(t) \) or \( x[n] \) to its Fourier transform \( X(f) \) or \( X(F) \) through

\[
E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad \text{or} \quad E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{f} |X(F)|^2 dF
\]

The quantity \( |X(f)|^2 \) or \( |X(F)|^2 \) is called the energy spectral density and is given the symbol \( \Psi \). That is,

\[
\Psi_x(f) = |X(f)|^2 \quad \text{or} \quad \Psi_x(F) = |X(F)|^2
\]

It is called energy spectral density because it describes mathematically the variation of the signal energy with frequency. If \( x(t) \) or \( x[n] \) is a real function, \( \Psi_x(f) \) or \( \Psi_x(F) \) is even, non-negative and real. Therefore the signal energy can be written as

\[
E_x = 2 \int_{0}^{\infty} \Psi_x(f) df \quad \text{or} \quad E_x = 2 \int_{0}^{1/2} \Psi_x(F) dF .
\]

Effects of Systems on ESD

The usefulness of the concept of energy spectral density can be seen if we analyze
the effect of bandpass filtering an excitation continuous-time signal $x(t)$ to create a response signal $y(t)$. If the filter is ideal, with unity gain and linear phase in its passband, that part of the signal within the passband will be unaffected (except possibly for a time shift) and that part of the signal outside the passband will be eliminated. Signal energy of a continuous-time signal is found by integrating ESD over all frequencies. If the signal has no ESD over some range of frequencies, the range of the integral need only cover the range over which the signal is non-zero. Then the total signal energy of $y(t)$ can be found by integrating its ESD,

$$E_y = 2\int_0^\infty \Psi_y(f)\,df = 2\int_0^\infty |Y(f)|^2\,df = 2\int_0^\infty |H(f)X(f)|^2\,df$$

$$E_y = 2\int_0^{f_u} |H(f)|^2 \Psi_x(f)\,df = 2\int_{f_L}^{f_u} \Psi_x(f)\,df$$

This integral can also be thought of as that part of the signal energy of $x(t)$ that lies within the filter’s passband. In general, the ESD of the response of a linear continuous-time system is related to the ESD of the excitation by

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f)H^*(f)\Psi_x(f)$$

and the ESD of the response of a linear discrete-time system is related to the ESD of the excitation by

$$\Psi_y(F) = |H(F)|^2 \Psi_x(F) = H(F)H^*(F)\Psi_x(F)$$

The units of ESD depend on the units of the signal to which it applies and whether the signal is continuous-time or discrete-time. For example, if the signal unit is the volt (V), and it is a continuous-time signal, its Fourier transform has units of V/Hz and its ESD has units of (V/Hz)$^2$ or (V·s)$^2$. These units can be rearranged into a more meaningful form as V$^2$·s/Hz, which expresses the ESD as a signal energy in V$^2$·s per unit frequency in Hz. For discrete-time signals, the unit of ESD is simply the square of the signal unit, whatever that may be.

The ESD Concept

The ESD of a signal is a description of the distribution of the signal energy of the signal versus frequency. In the signal processing discipline there are two ways of conceiving ESD, double-sided and single-sided. Mathematically, double-sided ESD’s are more convenient in complicated system analysis but, because of the conceptual difficulty
of imagining a negative frequency, ESD’s are often discussed as though all the signal energy resides in positive frequency space. Since the double-sided ESD (the one derived above) is an even function, the relation between double- and single-sided ESD’s is simple. The single-sided ESD of a signal is twice the double-sided ESD of the same signal for positive frequencies and zero for negative frequencies. Defined this way the total energy in a signal is the integral over all frequency space of either ESD.

The name, energy spectral density, comes from the fact that ESD is a mathematical functional description of how the signal energy of a signal is distributed in frequency. Figure R-43 is a conceptual block diagram of how the single-sided ESD of a continuous-time signal could be measured using an array of filters, squarers, integrators and dividers to estimate the ESD versus frequency. It is rarely, if ever, actually measured this way but the diagram does aid understanding of what ESD really is.

![Conceptual block diagram illustrating the concept of energy spectral density for a continuous-time signal](image)

**Relation of ESD to Autocorrelation**

For energy signals, the time-domain counterpart to ESD is autocorrelation. The autocorrelation of an energy signal \( x(t) \) or \( x[n] \) is the inverse Fourier transform of its ESD

\[ R_x(t) \xrightarrow{\mathcal{F}} \Psi_x(f) \quad \text{or} \quad R_x[n] \xrightarrow{\mathcal{F}} \Psi_x(F) \]
This can be proven by the following logic. From the definition of ESD,

\[ \Psi_s(f) = \left| X(f) \right|^2 \quad \text{or} \quad \Psi_s(F) = \left| X(F) \right|^2, \]

we can write

\[ R_s(t) \xlongleftarrow{\mathcal{F}} X^*(f) X(f) \quad \text{or} \quad R_s[n] \xlongleftarrow{\mathcal{F}} X^*(F) X(F) \]

Translating multiplication in the frequency domain into convolution in the time domain, and using the Fourier transform property relating to complex conjugates,

\[ R_s(t) = x(-t) \ast x(t) = \int_{-\infty}^{\infty} x(-\tau)x(t-\tau)d\tau \]

or

\[ R_s[n] = x[-n] \ast x[n] = \sum_{m=-\infty}^{\infty} x[-m]x[n-m] \]

and these can be simplified to

\[ R_s(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau+t)d\tau \]

or

\[ R_s[n] = \sum_{m=-\infty}^{\infty} x[m]x[m+n] \]

which are exactly the definitions of autocorrelation for continuous-time and discrete-time energy signals. (The two symbols \( t \) and \( \tau \) or \( n \) and \( m \) have exchanged places but that does not invalidate the result.)

**R.8 Power Spectral Density (PSD)**

**Definition and Derivation of Power Spectral Density**

PSD has the same relation to power signals as ESD has to energy signals. Many signals in systems are thought of and analyzed as if they were power signals even though they are not, since no real signal can endure an infinite time. But they are often steady-state signals that have been active a long time and are expected to continue for a long time.

Since the total signal energy of a power signal cannot be found, let us first find the ESD of a truncated version of a continuous-time signal \( x_T(t) \).
\[ x_T(t) = \begin{cases} x(t) & , \ |t| < T/2 \\ 0 & , \ \text{otherwise} \end{cases} = \text{rect}(t/T)x(t), \]

and

\[ \Psi_{x_T}(f) = |X_T(f)|^2 \]

where

\[ X_T(f) = \int_{-\infty}^{\infty} x_T(t)e^{-j2\pi ft}dt = \int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt. \]

The average signal power of the signal \( x_T(t) \) in this time interval is the signal energy of the signal in this time interval divided by the length of the time interval. Therefore it is analogous and logical to define the PSD of the truncated signal as its ESD divided by the time,

\[ G_{x_T}(f) = \frac{\Psi_{x_T}(f)}{T} = \frac{1}{T}|X_T(f)|^2. \]

As the time interval \( T \) becomes larger, the PSD of this truncated signal approaches that of the original signal. Therefore,

\[ G_x(f) = \lim_{T \to \infty} G_{x_T}(f) = \lim_{T \to \infty} \left( \frac{1}{T} \right)|X_T(f)|^2. \]

In a manner analogous to that derived for ESD, the power of a finite-signal-power signal in a bandwidth \( f_L \) to \( f_H \) is given by

\[ \text{Power} = 2 \int_{f_L}^{f_H} G(f)df. \]

The equivalent result for the PSD of a discrete-time signal is

\[ G_x(F) = \lim_{N \to \infty} G_{x_N}(F) = \lim_{N \to \infty} \left( \frac{1}{N} \right)|X_N(F)|^2. \]

**Effects of Systems on PSD**

The relationship between the PSD of an excitation and the PSD of the response of a linear system is similar to the relation between the ESD of an excitation and the ESD of a response of a linear system. The PSD of the response of a linear system is related to the PSD of the excitation by

\[ G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f). \]

or

R-44
\[ G_x(F) = |H(F)|^2 G_x(F) = H(F)H^*(F)G_x(F). \]

This is a very important result and is the starting point for most analyses of how noise propagates through an LTI system.

The units of PSD again depend on the units of the underlying signal to which it applies and whether it is continuous-time or discrete-time. If the signal unit of a continuous-time signal is the amp (A), the units of PSD are A^2/Hz. If the signal unit is the volt (V) the units of PSD are V^2/Hz. Since signal power is the integral of PSD over a range of frequency the ”/Hz” is integrated out. Therefore the signal power of a current signal has units of A^2 and the signal power of a voltage signal has units of V^2. For discrete-time signals, the unit is simply the square of the signal unit. For convenience, in many analyses in which the signal units are consistent throughout a system, analysis is done without using units. But in any analysis in which the final result must be related back to a physical quantity, the units must ultimately be considered and shown to be consistent.

The PSD Concept

One way to visualize the concept of PSD is to imagine that a continuous-time signal is processed by a system as illustrated in Figure R-44.

![Block diagram illustrating the concept of power spectral density](image)

**Figure R-44** Block diagram illustrating the concept of power spectral density

The signal power of the continuous-time signal \( x(t) \) is first split into small
frequency ranges by ideal bandpass filters, each with bandwidth $\Delta f$. Each signal so formed is then squared (to form instantaneous signal power) and time-averaged (to form average signal power), then divided by $\Delta f$ to form time-average signal power per unit frequency. Then the output signals $G_x(f_k)$ are estimates of the PSD at discrete frequencies. If $N$ goes to infinity, the output signals $G_x(f_k)$ cover all frequency space. The exact single-sided PSD of $x(t)$ is simply the limit of this process as $\Delta f$ approaches zero and $N$ approaches infinity so that the coverage is uniform and continuous over all frequency space.

As an example of what the signals in the system above might look like, let the input signal be $x(t)$ and let $N = 4$. Some of the signals are illustrated in Figure R-45.

![Typical signals in the conceptual system of Figure R-44 with N = 4](image)

Figure R-45 Typical signals in the conceptual system of Figure R-44 with $N = 4$

Relation of PSD to Autocorrelation

For power signals, the time-domain counterpart to PSD is autocorrelation. The autocorrelation of a power signal is the inverse Fourier transform of the PSD. That is,

$$R(t) \overset{\mathcal{F}}{\leftrightarrow} G(f) \quad \text{or} \quad R[n] \overset{\mathcal{F}}{\leftrightarrow} G(F)$$

The proof is similar to the one presented above for energy signals.
Example R-6  Power spectral density of a continuous-time rectangular wave

Find the PSD of the power signal in Figure R-46.

\[ x(t) \]

\[ ... \]

\[ A \]

\[ T_0 \]

\[ ... \]

\[ t \]

Figure R-46  A square-wave signal

We have already found the autocorrelation function (Figure R-47) for this signal in a previous example,

\[ R_x(t) = \left( \frac{A^2}{2} \right) \text{tri}(2t/T_0) * \delta_{T_0}(t). \]

This function can be compactly described by

\[ R_x(t) = \left( \frac{A^2}{2} \right) \text{tri}(2t/T_0) * \delta_{T_0}(t). \]

Now the PSD can be found by Fourier transforming the autocorrelation.

\[ G_x(f) = \left( \frac{A^2}{4} \right) \text{sinc}^2(T_0f/2) \delta_{1/T_0}(f) \]

or

\[ G_x(f) = \left( \frac{A^2}{4} \right) \sum_{n=-\infty}^{\infty} \text{sinc}^2(n/2) \delta(f - nf_0) = \left( \frac{A^2}{4} \right) \sum_{n=-\infty}^{\infty} \text{sinc}^2(n/2) \delta(f - nf_0) \]

(Figure R-48).
The PSD indicates that the signal has significant power at frequencies of zero and the fundamental frequency of the signal $f_0 = 1/T_0$.

Suppose we decide to take the other approach to finding the PSD, the direct approach using the definition

$$G_x(f) = \lim_{T \to \infty} (1/T)|X_T(f)|^2.$$ 

The time signal $x(t)$ is

$$x(t) = A \text{rect}(2t/T_0) \ast \delta_{T_0}(t).$$

The truncated time signal is

$$x_T(t) = A \{\text{rect}(2t/T_0) \ast \delta_{T_0}(t)\} \text{rect}(t/T).$$

Its Fourier transform $X_T(f)$ is

$$X_T(f) = (A/2) \text{sinc}(T_0 f/2) \delta_{f_0}(f) \ast T \text{sinc}(Tf).$$

Then

$$G_x(f) = \lim_{T \to \infty} (1/T)\left|(A/2) \text{sinc}(T_0 f/2) \delta_{f_0}(f) \ast T \text{sinc}(Tf)\right|^2$$

$$G_x(f) = \lim_{T \to \infty} (1/T)\left|\left(A/2\right) \sum_{n = -\infty}^{\infty} \text{sinc}(n/2) \delta(f - nf_0) \ast T \text{sinc}(Tf)\right|^2$$

$$G_x(f) = \lim_{T \to \infty} (1/T)\left|\left(A/2\right) \sum_{n = -\infty}^{\infty} \text{sinc}(n/2)T \text{sinc}\left[T(f - nf_0)\right]\right|^2.$$ 

As $T$ gets larger, the sinc functions in the summation become thinner and overlap less and in the limit do not overlap at all. In that limit, the square of the summation equals the summation of the squares of the individual functions since they do not overlap. Therefore,

$$G_x(f) = \lim_{T \to \infty} (1/T) \sum_{n = -\infty}^{\infty} \left(A^2/4\right) \text{sinc}^2(n/2)T^2 \text{sinc}^2\left[T(f - nf_0)\right].$$

R-48
Taking the limit inside,

\[ G_x(f) = \sum_{n=-\infty}^{\infty} \left( \frac{A^2}{4} \right) \text{sinc}^2 \left( \frac{n}{2} \right) \lim_{T \to \infty} \left\{ T \text{sinc}^2 \left[ T \left( f - n f_0 \right) \right] \right\}. \]

To properly interpret the limiting process on the sinc\(^2\) function consider the following:

\[ \mathcal{F}^{-1} \left[ a \text{sinc}^2 (af) \right] = \text{tri}(t/a) \]

Then, using the fact that the area under a frequency-domain function is its inverse transform evaluated at \( t = 0 \), the area under the sinc\(^2\) function is one. If \( a \) is allowed to approach infinity, the area under the sinc\(^2\) function stays constant at one because the triangle function is just getting wider and its value at \( t = 0 \) stays the same. At the same time the width of the sinc\(^2\) function is decreasing toward zero. A function whose area is constant while its width approaches zero is an impulse (in the limit). Therefore

\[ G_x(f) = \left( \frac{A^2}{4} \right) \sum_{n=-\infty}^{\infty} \text{sinc}^2 \left( \frac{n}{2} \right) \delta(f - n f_0) \]

which agrees with the previous result after considerably more mathematical and conceptual effort.

Example R-7  Power spectral density of a discrete-time periodic impulse

Find the PSD of the discrete-time signal, 

\[ x[n] = \delta_{N_0}[n]. \]

First find the autocorrelation function of this periodic signal using

\[ \delta_{N_0}[n] \xrightarrow{\mathcal{F}} 1. \]  \hspace{1cm} (R.2)

\[ R_X[m] \xrightarrow{\mathcal{F}} (1 / N_0) X^*[k] X[k] = 1 / N_0 \]

Then, again using (R.2)

\[ R_x[m] = (1 / N_0) \delta_{N_0}[m]. \]

The PSD is the DTFT of the autocorrelation function, which is
\[ G_x(F) = \left(\frac{1}{N_0^2}\right)\delta_{1/N_0}(F). \]

We can check the reasonableness of this result by finding the average signal power from the PSD. It is

\[ P_x = \int G_x(F) dF = \left(\frac{1}{N_0^2}\right)\int \delta_{1/N_0}(F) dF = \left(\frac{1}{N_0^2}\right)\int \sum_{m=-\infty}^{\infty} \delta(F - N/N_0) dF \]

We can choose any discrete-time interval of length one for the integration. No matter which one we choose there are exactly \( N_0 \) impulses in it so the integral evaluates to \( N_0 \). Then the average power is

\[ P_x = 1 / N_0. \]

This result implies that as the fundamental period increases the average power decreases. Since each impulse in the periodic impulse function has the same energy, this is reasonable. When the impulses occur less frequently, the average power decreases. Also this agrees with the autocorrelation function evaluated at \( m = 0 \). Therefore the answer seems reasonable.

---

**R.9 Summary of Important Points**

1. Relationships between signals are often as important as the signals themselves.

2. The correlogram is a good way to illustrate whether and how much two signals are correlated.

3. Correlation and independence are not exactly opposite concepts although for many signals they seem to be.

4. The correlation function indicates how correlated two signals are as a function of how much one of them is shifted in time.

5. There are two definitions of the correlation function, one for energy signals and one for power signals.

6. Correlation and convolution are closely related mathematical processes.

7. The correlation of a signal with a shifted version of itself is called autocorrelation.

8. Autocorrelation is closely related to signal energy or power and contains important information about how rapidly a signal varies in time.

9. The Fourier series harmonic function can be viewed as the correlation of a signal with a succession of sinusoids, complex or real.
10. Energy spectral density and power spectral density are the frequency-domain counterparts of autocorrelation, related through the Fourier transform.

11. Energy spectral density and power spectral density indicate how the energy or power of a signal varies with frequency.

**Exercises With Answers**

1. Plot correlograms of the following pairs of continuous-time and discrete-time signals.

   (a) \( x_1(t) = \cos(2\pi t) \) and \( x_2(t) = 2\cos(4\pi t) \)

   (b) \( x_1[n] = \sin(2\pi n / 16) \) and \( x_2[n] = 2\cos(2\pi n / 8) \)

   (c) \( x_1(t) = e^{-t}u(t) \) and \( x_2(t) = e^{-2t}u(t) \)

   (d) \( x_1[n] = e^{-n/10} \cos(2\pi n / 10)u[n] \) and \( x_1[n] = e^{-n/10} \sin(2\pi n / 10)u[n] \)

   Answers:

   ![Diagram](image1)

2. Plot correlograms of the following pairs of continuous-time and discrete-time signals.

   (a) \( x_1(t) = \cos(2\pi t) \) and \( x_2(t) = \cos^2(2\pi t) \)

   (b) \( x_1[n] = n \) and \( x_2[n] = n^3 \), \( -10 < n < 10 \)

   (c) \( x_1(t) = t \) and \( x_2(t) = 2 - t^2 \), \( -4 < t < 4 \)

R-51
3. In MATLAB generate two vectors, $x_1$ and $x_2$, representing discrete-time signals using the following code fragment,

$$x_1 = \text{randn}(100,1) ; \quad x_2 = \text{randn}(100,1) ; \quad x_3 = \text{randn}(100,1) ;$$

Then graph correlograms of the following pairs of discrete-time signals.

(a) $x_1$ and $x_2$
(b) $x_1$ and $x_1 + x_2$
(c) $x_1 + x_2$ and $x_1 + x_3$
(d) $x_1 + x_2/10$ and $-x_1 + x_3/10$

Answers:

4. Plot the correlation function for each of the following pairs of energy signals.

(a) $x_1(t) = 4 \text{rect}(t)$ and $x_2(t) = -3 \text{rect}(2t)$
(b) $x_1[n] = 2(u[n + 3] - u[n - 4])$

and $x_2[n] = 5(u[n + 4] - u[n - 5])$

(c) $x_1(t) = 4e^{-t}u(t)$ and $x_2(t) = 4e^{-t}u(t)$

(d) $x_1[n] = 2e^{-n/16}\sin\left(\frac{2\pi n}{8}\right)u[n]$

and $x_2[n] = -3e^{-n/16}\sin\left(\frac{2\pi n}{8} - \frac{\pi}{4}\right)u[n]$

Answers:
5. Plot the correlation function for each of the following pairs of power signals.

(a) \( x_1(t) = 6 \sin(12\pi t) \) and \( x_2(t) = 5 \cos(12\pi t) \)

(b) \( x_1[n] = 6 \sin(2\pi n / 12) \) and \( x_2[n] = 5 \sin(2\pi n / 12) \)

(c) \( x_1(t) = 6 \sin(12\pi t) \) and \( x_2(t) = 5 \sin(12\pi t - \pi / 4) \)

![Correlation Function Graphs]

Answers:

6. Find the autocorrelations of the following continuous-time and discrete-time energy and power signals and show that, at zero shift, the value of the autocorrelation is the signal energy or power and that all the properties of autocorrelation functions are satisfied.

(a) \( x(t) = e^{-3t} u(t) \)

(b) \( x[n] = (u[n + 5] - u[n - 6]) \)

(c) \( x(t) = \text{rect}(2(t - 1/4)) - \text{rect}(2(t - 3/4)) \)

Answers: \( (1/6)e^{-3t}, 11 \text{tri}(m/11), \text{tri}(2t) - (1/2)[\text{tri}(2(t - 1/2)) + \text{tri}(2(t + 1/2))] \)

7. Find the autocorrelation functions of the following power signals.

(a) \( x(t) = 5 \sin(24\pi t) - 2 \cos(18\pi t) \)

(b) \( x[n] = -4 \sin(2\pi n / 36) - 2 \cos(2\pi n / 40) \)

Answers: \( (25/2) \cos(24\pi t) + 2 \cos(18\pi t), 8 \cos(2\pi n / 36) + 2 \cos(2\pi n / 40) \)

8. A signal is sent from a transmitter to a receiver and is corrupted by noise along the way. The signal shape is of the functional form,

\[
x(t) = A \sin(2\pi f_0 t) \text{rect}\left((f_0 / 4)(t - 1/2f_0)\right)
\]
What is the frequency response of a matched filter for this signal?

Answer: \[ H(f) = \frac{2K}{f_0} e^{-j2\pi f (f_0 - f_0^2)} \left[ \text{sinc} \left( \frac{4(f + f_0)}{f_0} \right) - \text{sinc} \left( \frac{4(f - f_0)}{f_0} \right) \right] \]

9. Find the signal power of the following sums or differences of signals and compare it to the power in the individual signals. How does the comparison relate to the correlation between the two signals that are summed or differenced?

(a) \( x(t) = \sin(2\pi t) + \cos(2\pi t) \)  \quad (b) \( x(t) = \sin(2\pi t) + \cos(2\pi t - \pi/4) \)

(c) \( x[n] = (u[n+2] - u[n-3]) * \delta_{10}[n] - \text{tri}(n/2) * \delta_{10}[n] \)

(d) \( x[n] = (u[n+2] - u[n-3]) * \delta_{10}[n] + \text{tri} \left( \frac{n-5}{2} \right) * \delta_{10}[n] \)

Answers: \( 0.65 = 0.65 \), \( 0.5 = 0.5 \), \( 0.25 < 0.65 \), \( 1.707 > 1 \)

10. Find the crosscorrelation functions of the following pairs of periodic signals.

(a) \( x_1(t) = 24 \text{rect}(t/6) * \delta_{24}(t) \) and \( x_2(t) = 24 \text{rect} \left( \frac{t-3}{6} \right) * \delta_{24}(t) \)

(b) \( x_1[n] = \sin^2(2\pi n/8) \) and \( x_2[n] = \sin^2(2\pi n/10) \)

(c) \( x_1(t) = e^{-j10\pi t} \) and \( x_2(t) = \cos(10\pi t) \)

Answers: \( (1/2)e^{-j10\pi t} \), \( 1/4 \), \( 144 \text{tri} \left( \frac{t-3}{6} \right) * \delta_{24}(t) \)

11. Find the ESD’s of the following energy signals.

(a) \( x[n] = A\delta[n-n_0] \) \quad (b) \( x(t) = e^{-100t} u(t) \)

(c) \( x[n] = 10(7/8)^n \sin(2\pi n/12) u[n] \) \quad (d) \( x(t) = A\text{tri} \left( \frac{t-t_0}{w} \right) \)
Answers: $A^2$, $(Aw)^2 \text{sinc}^4(wf)$, $\frac{1}{10^4 + \omega^2}$, $100 \left[ 1 - 1.515 \cos(\Omega) + 0.7656 \cos(2\Omega) \right]^2 + \left[ 1.515 \sin(\Omega) - 0.7656 \sin(2\Omega) \right]^2$

12. Find the ESD of the response $y(t)$ or $y[n]$ of each system with impulse response $h(t)$ or $h[n]$ to the excitation $x(t)$ or $x[n]$.

(a) $x[n] = \delta[n]$, $h[n] = (-9/10)^n u[n]$

(b) $x(t) = e^{-100t} u(t)$, $h(t) = e^{-100t} u(t)$


(d) $x(t) = 4e^{-t} \cos(2\pi t) u(t)$, $h(t) = \text{rect}(t-1/2)$

Answers: $\frac{1 + \omega^2}{[(2\pi)^2 + 1 - \omega^2]^2 + 4\omega^2} \text{sinc}^2\left(\frac{\omega}{2\pi}\right)$, $\frac{\sin^2(7\pi F) \sin^2(5\pi F)}{\sin^2(\pi F) \sin^2(\pi F)}$, $\left(\frac{1}{10^4 + \omega^2}\right)^2 \frac{100}{181 + 180 \cos(\Omega)}$

13. Find the PSD's of these signals.

(a) $x(t) = A \cos(2\pi f_0 t + \theta)$

(b) $x(t) = 3 \text{rect}(100t) \ast \text{comb}(25t)$

(c) $x[n] = 8 \sin(2\pi n / 12)$

(d) $x[n] = 3(u[n+4] - u[n-5]) \ast \delta[n]$

Answers: $9 \times 10^{-4} \text{sinc}^2\left(\frac{f}{100}\right) \delta_{25}(f)$, $\frac{9}{400} \frac{\sin^2(9\pi F)}{\sin^2(\pi F)} \delta_{1/20}(F)$, $16[\delta_{1}(F - 1/12) + \delta_{1}(F + 1/12)]$, $(A^2 / 4)[\delta(f - f_0) + \delta(f + f_0)]$

14. Find the PSD of the response $y(t)$ or $y[n]$ of each system with impulse response $h(t)$ or $h[n]$ to the excitation $x(t)$ or $x[n]$. 

R-55
Exercises Without Answers

15. Plot correlograms of the following pairs of continuous-time and discrete-time signals.

(a) \[ x(t) = 4 \cos(32\pi t - \pi / 4), \quad h(t) = e^{-t/10} u(t) \]
and \[ x_2(t) = \sin(2\pi t) \]

(b) \[ x(t) = 2\delta_{1/2}(t), \quad h(t) = \text{rect}(t - 1) \]

(c) \[ x[n] = 2\delta_8[n], \quad h[n] = (11/12)^n u[n - 1] \]

(d) \[ x[n] = (-0.9)^n u[n], \quad h[n] = (0.5)^n u[n] \]

Answers: \[ 0, \quad 400 \frac{\delta(f - 16) + \delta(f + 16)}{1 + (320\pi)^2}, \quad 16\delta_2(f) \text{sinc}^2(f), \quad \frac{0.0525\delta_{1/8}(F)}{1 - 1.8333\cos(2\pi F) + 0.8403} \]

16. Plot a correlogram for the following sets of samples from two signals \( x \) and \( y \). In each case, from the nature of the correlogram what relationship, if any, exists between the two sets of data?

(a) \[ x = \{6,5,8,-2,3,-10,9,-2,-4,3,-2,6,0,-5,-7,1,9,9,4,-6\} \]
\[ y = \{-1,-10,-4,4,5,-2,3,-5,-9,2,6,-5,-1,-10,-9,0,4,-10,9,-1\} \]
21. Find all cross correlation and autocorrelation functions for these three signals:

   \[ x_1(t) = \cos(2\pi t), \quad x_2(t) = \sin(2\pi t), \quad x_3(t) = \cos(4\pi t) \]
Check your autocorrelation answers by finding the average power of each signal.

22. Find and sketch the crosscorrelation between a unit-amplitude, one Hz cosine and a 50% duty-cycle square wave that has a peak-to-peak amplitude of two, a fundamental period of one, an average value of zero and is an even function.

23. Find and sketch the ESD of each of these signals:

(a) \( x(t) = A \text{rect}(t/w) \)  
(b) \( x(t) = A \text{rect}(\frac{t+1}{w}) \)  
(c) \( x(t) = A \text{sinc}(t/w) \)  
(d) \( x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \)

24. Find the PSD’s of

(a) \( x(t) = A \)  
(b) \( x(t) = A \cos(2\pi f_0 t) \)  
(c) \( x(t) = A \sin(2\pi f_0 t) \)

25. Which of the following functions could not be the autocorrelation function of a real signal and why?

(a) \( R(\tau) = \text{tri}(\tau) \)  
(b) \( R(\tau) = A \sin(2\pi f_0 \tau) \)  
(c) \( R(\tau) = \text{rect}(\tau) \)  
(d) \( R(\tau) = A \text{sinc}(B\tau) \)