Chapter 2

1. The quarterly interest rate is

$$j = \frac{i^{(4)}}{4} = \frac{.06}{4} = .015$$

and all time periods are measured in quarters. Using the end of the third year as the comparison date

$$3000(1+j)^{12} + X = 2000v^4 + 5000v^{28}$$
$$X = 2000(.94218) + 5000(.65910) - 3000(1.19562)$$
$$= \$1593.00.$$

2. The monthly interest rate is

$$j = \frac{i^{(12)}}{12} = \frac{.18}{12} = .015.$$

Using the end of the third month as the comparison date

$$X = 1000(1+j)^3 - 200(1+j)^2 - 300(1+j)$$
$$= 1000(1.04568) - 200(1.03023) - 300(1.015)$$
$$= \$535.13.$$

3. We have

$$200v^5 + 500v^{10} = 400.94v^5$$
$$v^{10} = .40188v^5$$
$$v^5 = .40188 \quad \text{or} \quad (1+i)^5 = 2.4883.$$

Now using time $t = 10$ as the comparison date

$$P = 100(1+i)^{10} + 120(1+i)^5$$
$$= 100(2.4883)^2 + 120(2.4883) = \$917.76.$$

4. The quarterly discount rate is 1/41 and the quarterly discount factor is $1 - 1/41 = 40/41$. The three deposits accumulate for 24, 16, and 8 quarters, respectively. Thus,

$$A(28) = 100\left[(1.025)\left(\frac{40}{41}\right)^{-24} + (1.025)^3\left(\frac{40}{41}\right)^{-16} + (1.025)^5\left(\frac{40}{41}\right)^{-8}\right].$$

However,

$$\left(\frac{40}{41}\right)^{-1} = 1.025$$

so that

$$A(28) = 100\left[(1.025)^{25} + (1.025)^{19} + (1.025)^{13}\right] = \$483.11.$$

5. (*a*) At time $t = 10$, we have

$$X = 100(1+10i) + 100(1+5i) \quad \text{with} \quad i = .05$$
$$= 200 + 1500(.05) = \$275.$$

(*b*) At time $t = 15$, we have

$$X(1+5i) = 100(1+15i) + 100(1+10i) \quad \text{with} \quad i = .05$$
$$X(1.25) = 200 + 2500(.05) = 325$$

and

$$X = \frac{325}{1.25} = \$260.$$

6. The given equation of value is

$$1000(1.06)^n = 2000(1.04)^n$$

so that

$$\left(\frac{1.06}{1.04}\right)^n = 2$$
$$n[\ln 1.06 - \ln 1.04] = \ln 2$$

and

$$n = \frac{.693147}{.058269 - .039221} = 36.4 \text{ years}.$$

7. The given equation of value is

$$3000 + 2000v^2 = 5000v^n + 5000v^{n+5}$$
$$3000 + 2000(1-d)^2 = 5000(1-d)^n\left[1 + (1-d)^5\right]$$
$$\text{and} \quad 3000 + 2000(.94)^2 = 5000(.94)^n\left[1 + (.94)^5\right]$$

since $d = .06$. Simplifying, we have

$$4767.20 = 8669.52(.94)^n$$
$$(.94)^n = \frac{4767.20}{8669.52} = .54988$$
$$n\ln(.94) = \ln(.54988)$$
$$\text{and} \quad n = \frac{\ln(.54988)}{\ln(.94)} = 9.66 \quad \text{years.}$$

8. The given equation of value is

$$100 = 100v^n + 100v^{2n}$$

which is a quadratic in v^n. Solving

$$v^{2n} + v^n - 1 = 0$$

$$v^n = \frac{-1 \pm \sqrt{1-(4)(1)(-1)}}{2} = \frac{-1+\sqrt{5}}{2}$$

$= .618034$ rejecting the negative root.

We are given $i = .08$, so that

$$(1.08)^n = 1/.61803 = 1.618034$$

and $n = \dfrac{\ln 1.618034}{\ln 1.08} = 6.25$ years.

9. Applying formula (2.2)

$$\bar{t} = \frac{n^2 + (2n)^2 + \cdots + (n^2)^2}{n + 2n + \cdots + n^2} = \frac{n^2(1+2^2+\cdots+n^2)}{n(1+2+\cdots+n)}.$$

We now apply the formulas for the sum of the first n positive integers and their squares (see Appendix C) to obtain

$$\frac{n^2\left(\frac{1}{6}\right)(n)(n+1)(2n+1)}{n\left(\frac{1}{2}\right)(n)(n+1)} = \frac{1}{3}(n)(2n+1) = \frac{2n^2+n}{3}.$$

10. We parallel the derivation of formula (2.4)

$$(1+i)^n = 3 \quad \text{or} \quad n = \frac{\ln 3}{\ln(1+i)}$$

and approximating i by .08, we obtain

$$n \approx \frac{\ln 3}{i} \cdot \frac{.08}{\ln(1.08)} = \frac{1.098612}{i} \cdot \frac{.08}{.076961}$$

$= \dfrac{1.14}{i}$ or a rule of 114, i.e. $n = 114$.

11. Use time $t = 10$ as the comparison date

$$\text{A: } 10[1+(10)(.11)] + 30[1+(5)(.11)] = 67.5$$

$$\text{B: } 10(1.0915)^{10-n} + 30(1.0915)^{10-2n} = 67.5$$

$$10v^n + 30v^{2n} = 67.5(1.0915)^{-10} = 28.12331$$

which gives the quadratic

$$v^{2n} + .33333v^n - .93744 = 0.$$

Solving

$$v^n = \frac{-.33333 \pm \sqrt{(.33333)^2 - (4)(1)(-.93744)}}{2} = .81579$$

and

$$n = \frac{\ln(.81579)}{-\ln(1.0915)} = 2.33 \text{ years.}$$

12. Let t measure time in years. Then

$$a^A(t) = (1.01)^{12t} \quad \text{and}$$

$$a^B(t) = e^{\int_0^t r/6dr} = e^{t^2/12}.$$

Equate the two expressions and solve for t

$$(1.01)^{12t} = e^{t^2/12} \quad \text{or} \quad (1.01)^{144t} = e^{t^2}$$

$$144t \ln(1.01) = t^2$$

$$\text{and} \quad t = 144\ln(1.01) = 1.43 \text{ years.}$$

13. Let j be the semiannual interest rate. We have

$$1000(1+j)^{30} = 3000$$

$$\text{and} \quad j = 3^{1/30} - 1 = .0373.$$

The answer is

$$i^{(2)} = 2j = 2(.0373) = .0746, \quad \text{or} \quad 7.46\%.$$

14. The given equation of value is

$$300(1+i)^2 + 200(1+i) + 100 = 700.$$

Simplifying, we get a quadratic

$$3(1+2i+i^2) + 2(1+i) - 6 = 0$$

$$3i^2 + 8i - 1 = 0.$$

Solving the quadratic

$$i = \frac{-8 \pm \sqrt{8^2 - (4)(3)(-1)}}{(2)(3)} = \frac{-8 \pm \sqrt{76}}{6}$$

$$= \frac{-8 + 2\sqrt{19}}{6} = \frac{\sqrt{19} - 4}{3} \quad \text{rejecting the negative root.}$$

15. The given equation of value is

$$100 + 200v^n + 300v^{2n} = 600v^{10}.$$

Substituting the given value of v^n

$$100 + 200(.75941) + 300(.75941)^2 = 600v^{10}$$
$$v^{10} = .708155 \quad \text{or} \quad (1+i)^{10} = 1.41212$$
$$\text{and} \quad i = (1.41212)^{.1} - 1 = .0351, \quad \text{or} \quad 3.51\%.$$

16. The total amount of interest equals
$$1000i(1+2+\cdots+10) = 55,000i.$$
Thus, we have
$$1000 + 55,000i = 1825$$
$$\text{and} \quad i = \frac{1825}{55,000} = .015, \quad \text{or} \quad 1.5\%.$$

17. We have
$$a(10) = e^{\int_0^{10} \delta_t dt} = e^{\int_0^{10} ktdt} = e^{50k} = 2$$
so that
$$50k = \ln 2 \quad \text{and} \quad k = \frac{\ln 2}{50}.$$

18. We will use i to represent both the interest rate and the discount rate, which are not equivalent. We have
$$(1+i)^3 + (1-i)^3 = 2.0096$$
$$(1+3i+3i^2+i^3) + (1-3i+3i^2-i^3) = 2.0096$$
$$2 + 6i^2 = 2.0096 \quad \text{or} \quad 6i^2 = .0096$$
$$i^2 = .0016 \quad \text{and} \quad i = .04, \quad \text{or} \quad 4\%.$$

19. (*a*) Using Appendix A December 7 is Day 341
August 8 is Day 220.

We then have

1941:	24	$= 365 - 341$
1942:	365	
1943:	365	
1944:	366	(leap year)
1945:	220	
Total =	1340	days.

(*b*) Applying formula (2.5)
$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$$
$$= 360(1945 - 1941) + 30(8 - 12) + (8 - 7) = 1321 \text{ days.}$$

20. (*a*) $I = (10,000)(.06)\left(\frac{62}{365}\right) = \$101.92.$

(*b*) $I = (10,000)(.06)\left(\frac{60}{360}\right) = \$100.00.$

(*c*) $I = (10,000)(.06)\left(\frac{62}{360}\right) = \$103.33.$

21. (*a*) Bankers Rule: $I = Pr\left(\frac{n}{360}\right)$

Exact simple interest: $I = Pr\left(\frac{n}{365}\right)$

where n is the exact number of days in both. Clearly, the Banker's Rule always gives a larger answer since it has the smaller denominator and thus is more favorable to the lender.

(*b*) Ordinary simple interest: $I = Pr\left(\frac{n^*}{360}\right)$

where n^* uses 30-day months. Usually, $n \geq n^*$ giving a larger answer which is more favorable to the lender.

(*c*) Invest for the month of February.

22. (*a*) The quarterly discount rate is

$$(100-96)/100 = .04. \quad \text{Thus,}$$
$$d^{(4)} = 4(.04) = .16, \quad \text{or } 16\%.$$

(*b*) With an effective rate of interest

$$96(1+i)^{.25} = 100$$

$$\text{and} \quad i = \left(\frac{100}{96}\right)^4 - 1 = .1774, \quad \text{or } 17.74\%.$$

23. (*a*) Option A - 7% for six months:

$$(1.07)^{.5} = 1.03441.$$

Option B - 9% for three months:

$$(1.09)^{.25} = 1.02178.$$

The ratio is

$$\frac{1.03441}{1.02178} = 1.0124.$$

(*b*) Option A - 7% for 18 months:

$$(1.07)^{1.5} = 1.10682.$$

Option B - 9% for 15 months:

$$(1.07)^{1.25} = 1.11374.$$

The ratio is

$$\frac{1.10682}{1.11374} = .9938.$$

24. The monthly interest rates are:

$$y_1 = \frac{.054}{12} = .0045 \quad \text{and} \quad y_2 = \frac{.054 - .018}{12} = .003.$$

The 24-month CD is redeemed four months early, so the student will earn 16 months at .0045 and 4 months at .003. The answer is

$$5000(1.0045)^{16}(1.003)^4 = \$5437.17.$$

25. The APR = 5.1% compounded daily. The APY is obtained from

$$1 + i = \left(1 + \frac{.051}{365}\right)^{365} = 1.05232$$

or APY = .05232. The ratio is

$$\frac{\text{APY}}{\text{APR}} = \frac{.05232}{.051} = 1.0259.$$

Note that the term "APR" is used for convenience, but in practice this term is typically used only with consumer loans.

26. (*a*) No bonus is paid, so $i = .0700$, or 7.00%.

(*b*) The accumulated value is $(1.07)^3(1.02) = 1.24954$, so the yield rate is given by

$$(1+i)^3 = 1.24954 \quad \text{or} \quad i = (1.24954)^{1/3} - 1 = .0771, \quad \text{or} \quad 7.71\%.$$

(*c*) The accumulated value is

$$(1.07)^3(1.02)(1.07) = (1.07)^4(1.02) = 1.33701,$$

so the yield rate is given by

$$(1+i)^3 = 1.33701 \quad \text{or} \quad i = (1.33701)^{1/3} - 1 = .0753, \quad \text{or} \quad 7.53\%.$$

27. This exercise is asking for the combination of CD durations that will maximize the accumulated value over six years. All interest rates are convertible semiannually. Various combinations are analyzed below:

4-year/2-year: $1000(1.04)^8(1.03)^4 = 1540.34.$

3-year/3-year: $1000(1.035)^{12} = 1511.08.$

All other accumulations involving shorter-term CD's are obviously inferior. The maximum value is \$1540.34.

28. Let the purchase price be *R*. The customer has two options:

One: Pay .9*R* in two months.

Two: Pay $(1-.01X)R$ immediately.

The customer will be indifferent if these two present values are equal. We have

$$(1-.01X)R = .9R(1.08)^{-1/6}$$
$$1-.01X = .9(1.08)^{-1/6} = .88853$$

and

$$X = 100(1-.88853) = 11.15\%.$$

29. Let the retail price be *R*. The retailer has two options:

One: Pay .70*R* immediately.

Two: Pay .75*R* in six months.

The retailer will be indifferent if these two present values are equal. We have

$$.70R = .75R(1+i)^{-.5}$$
$$.70(1+i)^{.5} = .75$$

and

$$i = \left(\frac{.75}{.70}\right)^2 - 1 = .1480, \quad \text{or} \quad 14.80\%.$$

30. At time 5 years

$$1000(1+i/2)^{10} = X.$$

At time 10.5 years:

$$1000(1+i/2)^{14}(1+2i/4)^{14} = 1980.$$

We then have

$$(1+i/2)^{28} = 1.98$$
$$(1+i/2)^{10} = (1.98)^{10/28} = 1.276$$

and the answer is

$$1000(1.276) = \$1276.$$

31. We are given

$$A(1.06)^{20} + B(1.08)^{20} = 2000$$
$$2A(1.06)^{10} = B(1.08)^{10}$$

which is two linear equations in two unknowns. Solving these simultaneous equations gives:

$$A = 182.82 \quad \text{and} \quad B = 303.30.$$

The answer then is

$$A(1.06)^5 + B(1.08)^5 = (182.82)(1.06)^5 + (303.30)(1.08)^5$$
$$= \$690.30.$$

32. We are given that

$$10,000(1+i)(1+i-.05) = 12,093.75.$$

Solving the quadratic

$$1 + i - .05 + i + i^2 - .05i = 1.209375$$
$$i^2 + 1.95i - .259375 = 0$$
$$i = \frac{-1.95 \pm \sqrt{(1.95)^2 - (4)(1)(-.259375)}}{2}$$
$$= .125 \quad \text{rejecting the negative root.}$$

We then have

$$10,000(1 + .125 + .09)^3 = 10,000(1.215)^3$$
$$= \$17,936.$$

33. The annual discount rate is

$$d = \frac{1000 - 920}{1000} = \frac{80}{1000} = .08.$$

The early payment reduces the face amount by *X*. We then have

$$X\left[1 - \tfrac{1}{2}(.08)\right] = 288,$$

so that

$$X = \frac{288}{.96} = 300$$

and the face amount has been reduced to

$$1000 - 300 = \$700.$$