Chapter 4

1. The nominal rate of interest convertible once every two years is *j*, so that

$$1+j=\left(1+\frac{.07}{2}\right)^4 \quad \text{and} \quad j=(1.035)^4-1=.14752.$$

The accumulated value is taken 4 years after the last payment is made, so that

$$\begin{aligned}2000s_{\overline{8}|j}(1+j)^2 &= 2000(13.60268)(1.31680)\\ &= \$35{,}824 \quad \text{to the nearest dollar.}\end{aligned}$$

2. The quarterly rate of interest j is obtained from

$$(1+j)^4=1.12 \quad \text{so that} \quad j=.02874.$$

The present value is given by

$$\begin{aligned}&600\ddot{a}_{\overline{40}|j}-200\ddot{a}_{\overline{20}|j}\\ &=600(24.27195)-200(15.48522)\\ &=\$11{,}466 \quad \text{to the nearest dollar.}\end{aligned}$$

3. The equation of value at time $t=8$ is

$$100\left[(1+8i)+(1+6i)+(1+4i)+(1+2i)\right]=520$$

so that

$$4+20i=5.2, \text{ or } 20i=1.2, \quad \text{and} \quad i=.06, \text{ or } 6\%.$$

4. Let the quarterly rate of interest be *j*. We have

$$400a_{\overline{40}|j}=10{,}000 \quad \text{or} \quad a_{\overline{40}|j}=25.$$

Using the financial calculator to find an unknown *j*, set $\mathrm{N}=40 \quad \mathrm{PV}=25 \quad \mathrm{PMT}=-1$ and $\mathrm{CPT\ I}$ to obtain $j=.02524$, or 2.524%. Then

$$\left(1+\frac{i^{(12)}}{12}\right)^{12}=(1.02524)^4 \quad \text{and} \quad i^{(12)}=.100, \text{ or } 10.0\%.$$

5. Adapting formula (4.2) we have

$$\begin{aligned}&2000\frac{s_{\overline{32}|.035}}{s_{\overline{4}|.035}}(1.035)^8\\ &=(2000)\left(\frac{57.33450}{4.21494}\right)(1.31681)=\$35{,}824 \text{ to the nearest dollar.}\end{aligned}$$

6. (*a*) We use the technique developed in Section 3.4 that puts in imaginary payments and then subtracts them out, together with adapting formula (4.1), to obtain

$$\frac{200}{s_{\overline{4}|}}\left(a_{\overline{176}|}-a_{\overline{32}|}\right).$$

Note that the number of payments is $\frac{176-32}{4}=36,$ which checks.

(*b*) Similar to part (*a*), but adapting formula (4.3) rather than (4.1), we obtain

$$\frac{200}{a_{\overline{4}|}}\left(a_{\overline{180}|}-a_{\overline{36}|}\right).$$

Again we have the check that

$$\frac{180-36}{4}=36.$$

7. The monthly rate of discount is $d_j=\frac{d^{(12)}}{12}=\frac{.09}{12}=.0075$ and the monthly discount factor is $v_j=1-d_j=.9925.$ From first principles, the present value is

$$300\left[1+(.9925)^6+(.9925)^{12}+\cdots+(.9925)^{114}\right]=300\frac{1-(.9925)^{120}}{1-(.9925)^6}$$

upon summing the geometric progression.

8. Using first principles and summing an infinite geometric progression, we have

$$v^3+v^6+v^9+\ldots=\frac{v^3}{1-v^3}=\frac{1}{(1+i)^3-1}=\frac{125}{91}$$

and

$$(1+i)^3-1=\frac{91}{125}\quad\text{or}\quad(1+i)^3=\frac{216}{125}$$

and $1+i=\left(\frac{216}{125}\right)^{1/3}=\frac{6}{5}=1.2$ which gives $i=.20,$ or $20\%.$

9. Using first principles with formula (1.31), we have the present value

$$100\left[1+e^{-.02}+e^{-.04}+\cdots+e^{-.38}\right]$$

and summing the geometric progression

$$100\frac{1-e^{-.4}}{1-e^{-.02}}.$$

10. This is an unusual situation in which each payment does not contain an integral number of interest conversion periods. However, we again use first principles measuring time in 3-month periods to obtain $1+v^{4/3}+v^{8/3}+\cdots+v^{140/3}$ and summing the geometric progression, we have

$$\frac{1-v^{48}}{1-v^{4/3}}.$$

11. Adapting formula (4.9) we have

$$2400\ddot{a}^{(4)}_{\overline{10}|.12}-800\ddot{a}^{(4)}_{\overline{5}|.12}.$$

Note that the proper coefficient is the "annual rent" of the annuity, not the amount of each installment. The nominal rate of discount $d^{(4)}$ is obtained from

$$\left(1-\frac{d^{(4)}}{4}\right)^{-4}=1+i=1.12 \quad \text{and} \quad d^{(4)}=4\left[1-(1.12)^{-1/4}\right]=.11174.$$

The answer is

$$2400\cdot\frac{1-(1.12)^{-10}}{.11174}-800\frac{1-(1.12)^{-5}}{.11174}=\$11,466 \text{ to the nearest dollar.}$$

12. (*a*) $$\frac{1}{m}\sum_{t=1}^{m}v^{t/m}\ddot{a}_{\overline{n}|}=\ddot{a}_{\overline{n}|}\sum_{t=1}^{m}\frac{1}{m}v^{1/m}=\ddot{a}_{\overline{n}|}a^{(m)}_{\overline{1}|}=\frac{1-v^n}{d}\cdot\frac{1-v}{i^{(m)}}=\frac{1-v^n}{i^{(m)}}=a^{(m)}_{\overline{n}|}.$$

(*b*) The first term in the summation is the present value of the payments at times $1/m, 1+1/m, \ldots, n-1+1/m$. The second term is the present value of the payments at times $2/m, 1+2/m, \ldots, n-1+2/m$. This continues until the last term is the present value of the payments at times $1, 2, \ldots, n$. The sum of all these payments is $a^{(m)}_{\overline{n}|}$.

13. The equation of value is

$$1000\,{}_{n|}\ddot{a}^{(2)}_{\overline{\infty}|}=10,000 \quad \text{or} \quad {}_{n|}\ddot{a}^{(2)}_{\overline{\infty}|}=10,$$

where *n* is the deferred period. We then have

$${}_{n|}\ddot{a}^{(2)}_{\overline{\infty}|}=v^n\ddot{a}^{(2)}_{\overline{\infty}|}=\frac{v^n}{d^{(2)}}=10 \quad \text{or} \quad v^n=10d^{(2)}.$$

Now expressing the interest functions in terms of *d*, we see that

$$v=1-d \quad \text{and} \quad d^{(2)}=2\left[1-(1-d)^{1/2}\right].$$

We now have

$$(1-d)^n = 20\left[1-(1-d)^{.5}\right]$$
$$n\ln(1-d) = \ln 20\left[1-(1-d)^{.5}\right]$$
$$\text{and} \quad n = \frac{\ln 20\left[1-(1-d)^{.5}\right]}{\ln(1-d)}.$$

14. We have

$$3a_{\overline{n}|}^{(2)} = 2a_{\overline{2n}|}^{(2)} = 45s_{\overline{1}|}^{(2)}$$
$$\text{or} \quad 3\left(\frac{1-v^n}{i^{(2)}}\right) = 2\left(\frac{1-v^{2n}}{i^{(2)}}\right) = 45\frac{i}{i^{(2)}}.$$

Using the first two, we have the quadratic

$$3(1-v^n) = 2(1-v^{2n}) \quad \text{or} \quad 2v^{2n} - 3v^n + 1 = 0$$

which can be factored $(2v^n - 1)(v^n - 1) = 0$ or $v^n = 1/2$, rejecting the root $v = 1$. Now using the first and third, we have

$$3(1-v^n) = 45i \quad \text{or} \quad i = \frac{3\left(1-1/2\right)}{45} = \frac{1}{30}.$$

15. Using a similar approach to Exercise 10, we have

$$1 + v^{3/4} + v^{6/4} + \cdots + v^{141/4} = \frac{1-v^{36}}{1-v^{3/4}}.$$

16. Each of the five annuities can be expressed as $1-v^n$ divided by $i, i^{(m)}, \delta, d^{(m)}$, and d, respectively. Using the result obtained in Exercise 32 in Chapter 1 immediately establishes the result to be shown. All five annuities pay the same total amount. The closer the payments are to time $t = 0$, the larger the present value.

17. The equation of value is

$$2400\bar{a}_{\overline{n}|} = 40,000 \quad \text{or} \quad \bar{a}_{\overline{n}|} = 50/3.$$

Thus

$$\bar{a}_{\overline{n}|} = \frac{1-e^{-.04n}}{.04} = \frac{50}{3}$$

or

$$1-e^{-.04n}=\frac{2}{3} \qquad e^{-.04n}=\frac{1}{3}$$

and

$$.04n=\ln 3=1.0986, \quad \text{so that} \quad n=27.47.$$

18. We have

$$\bar{a}_{\overline{n}|}=\frac{1-v^n}{\delta}=4 \quad \text{or} \quad v^n=1-4\delta$$

and

$$\bar{s}_{\overline{n}|}=\frac{(1+i)^n-1}{\delta}=12 \quad \text{or} \quad (1+i)^n=1+12\delta.$$

Thus, $1+12\delta=\dfrac{1}{1-4\delta}$ leading to the quadratic $1+8\delta-48\delta^2=1,$ so that $\delta=\dfrac{8}{48}=\dfrac{1}{6}.$

19. Using formula (4.13) in combination with formula (1.27), we have

$$\bar{a}_{\overline{n}|}=\int_0^n v^t dt=\int_0^n e^{-\int_0^t \delta_r dr}=\int_0^n e^{-\int_0^t (1+r)^{-1} dr} dt$$

Now

$$e^{-\int_0^t (1+r)^{-1} dr}=e^{-\ln(1+t)}=(1+t)^{-1}.$$

Thus,

$$\bar{a}_{\overline{n}|}=\int_0^n (1+t)^{-1} dt=\ln(1+t)\Big]_0^n=\ln(n+1).$$

20. Find t such that $v^t=\bar{a}_{\overline{1}|}=\dfrac{1-v}{\delta}=\dfrac{iv}{\delta}.$ Thus, $t\ln v=\ln v+\ln\dfrac{i}{\delta}$ and $t=1-\dfrac{1}{\delta}\ln\dfrac{i}{\delta}.$

21. Algebraically, apply formulas (4.23) and (4.25) so that $(Ia)_{\overline{n}|}=\dfrac{\ddot{a}_{\overline{n}|}-nv^n}{i}$ and $(Da)_{\overline{n}|}=\dfrac{n-a_{\overline{n}|}}{i}.$ Thus,

$$(Ia)_{\overline{n}|}+(Da)_{\overline{n}|}=\frac{1}{i}\left(\ddot{a}_{\overline{n}|}-nv^n+n-a_{\overline{n}|}\right)$$

$$=\frac{1}{i}\left(a_{\overline{n}|}+1-v^n-nv^n+n-a_{\overline{n}|}\right)=(n+1)\left(\frac{1-v^n}{i}\right)=(n+1)a_{\overline{n}|}.$$

Diagrammatically,

Time:	0	1	2	3	…	$n-1$	n
$(Ia)_{\overline{n}\rvert}$:		1	2	3	…	$n-1$	n
$(Da)_{\overline{n}\rvert}$:		n	$n-1$	$n-2$	…	2	1
Total:		$n+1$	$n+1$	$n+1$	…	$n+1$	$n+1$

22. Applying formula (4.21) directly with $P=6$, $Q=1$, and $n=20$

$$Pa_{\overline{n}\rvert}+Q\frac{a_{\overline{n}\rvert}-nv^n}{i}=6a_{\overline{20}\rvert}+\frac{a_{\overline{20}\rvert}-20v^{20}}{i}.$$

23. The present value is

$$\begin{aligned}v^4(Da)_{\overline{10}\rvert}&=\frac{v^4}{i}\left(10-a_{\overline{10}\rvert}\right)=\frac{1}{i}\left(10v^4-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}\right)\\&=\frac{1}{i}\left[10\left(1-ia_{\overline{4}\rvert}\right)-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}\right]\\&=\frac{1}{i}\left[10-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}(1-10i)\right].\end{aligned}$$

24. Method 1:

$$\begin{aligned}\text{PV}&=(Ia)_{\overline{n}\rvert}+v^n na_{\overline{\infty}\rvert}=\frac{1}{i}\left(\ddot{a}_{\overline{n}\rvert}-nv^n+nv^n\right)\\&=\frac{\ddot{a}_{\overline{n}\rvert}}{i}=\frac{(1+i)a_{\overline{n}\rvert}}{i}=\frac{a_{\overline{n}\rvert}}{d}.\end{aligned}$$

Method 2:

$$\begin{aligned}\text{PV}&=(Ia)_{\overline{\infty}\rvert}-v^n(Ia)_{\overline{\infty}\rvert}=(1-v^n)\left(\frac{1}{i}+\frac{1}{i^2}\right)\\&=\left(\frac{1-v^n}{i}\right)\left(1+\frac{1}{i}\right)=\frac{1-v^n}{id}=\frac{a_{\overline{n}\rvert}}{d}.\end{aligned}$$

25. We are given that $11v^6=13v^7$ from which we can determine the rate of interest. We have $11(1+i)=13,$ so that $i=2/11.$ Next, apply formula (4.27) to obtain

$$\frac{P}{i}+\frac{Q}{i^2}=\frac{1}{i}+\frac{2}{i^2}=\frac{11}{2}+2\left(\frac{11}{2}\right)^2=66.$$

26. We are given:

$$X=va_{\overline{\infty}\rvert}=\frac{v}{i}\quad\text{and}\quad 20X=v^2(Ia)_{\overline{\infty}\rvert}=\frac{v^2}{id}.$$

Therefore,

$$X = \frac{v}{i} = \frac{v^2}{20id} \quad \text{or} \quad 20d = v = 1-d \quad \text{and} \quad d = 1/21.$$

27. The semiannual rate of interest $j = .16/2 = .08$ and the present value can be expressed as

$$300a_{\overline{10}|.08} + 50(Da)_{\overline{10}|.08} = 300a_{\overline{10}|.08} + 50\left(\frac{10-a_{\overline{10}|}}{.08}\right)$$
$$= 300A + 50\left(\frac{10-A}{.08}\right) = 6250 - 325A.$$

28. We can apply formula (4.30) to obtain

$$\text{PV} = 600\left[1 + \frac{1.05}{1.1025} + \left(\frac{1.05}{1.1025}\right)^2 + \cdots + \left(\frac{1.05}{1.1025}\right)^{19}\right]$$
$$= 600\left[\frac{1-(1.05/1.1025)^{20}}{1-(1.05/1.1025)}\right] = \$7851 \text{ to the nearest dollar.}$$

29. We can apply formula (4.31)

$$i' = \frac{i-k}{1+k} = \frac{.1025-.05}{1+.05} = .05, \quad \text{or} \quad 5\%,$$

which is the answer.

Note that we could have applied formula (4.32) to obtain $\text{PV} = 600\ddot{a}_{\overline{20}|.05} = \$7851$ as an alternative approach to solve Exercise 28.

30. The accumulated value of the first 5 deposits at time $t = 10$ is

$$1000\ddot{s}_{\overline{5}|.08}(1.08)^5 = (1000)(6.33593)(1.46933) = 9309.57.$$

The accumulated value of the second 5 deposits at time $t = 10$ is

$$1000\left[(1.05)(1.08)^5 + (1.05)^2(1.08)^4 + \cdots + (1.05)^5(1.08)\right]$$
$$= 1000(1.05)(1.08)^5\left[\frac{1-(1.05/1.08)^5}{1-1.05/1.08}\right] = 7297.16.$$

The total accumulated value is $9309.57 + 7297.16 = \$16,607$ to the nearest dollar.

31. We have the equation of value

$$4096 = 1000\left[\frac{1}{(1.25)^5} + \frac{1+.01k}{(1.25)^6} + \frac{(1+.01k)^2}{(1.25)^7} + \cdots\right]$$

or

$$4.096 = \frac{1/(1.25)^5}{1-(1+.01k)/1.25} = \frac{1}{(1.25)^4(.25-.01k)}$$

upon summing the infinite geometric progression. Finally, solving for k

$$10 = \frac{1}{.25-.01k} \quad \text{and} \quad k = 15\%.$$

32. The first contribution is $(40,000)(.04) = 1600$. These contributions increase by 3% each year thereafter. The accumulated value of all contributions 25 years later can be obtained similarly to the approach used above in Exercise 30. Alternatively, formula (4.34) can be adapted to an annuity-due which gives

$$1600\frac{(1.05)^{25}-(1.03)^{25}}{.05-.03}(1.05) = \$108,576 \text{ to the nearest dollar.}$$

33. Applying formula (4.30), the present value of the first 10 payments is

$$100\left[\frac{1-(1.05/1.07)^{10}}{.07-.05}\right](1.07) = 919.95.$$

The 11th payment is $100(1.05)^9(.95) = 147.38$. Then the present value of the second 10 payments is $147.38\left[\frac{1-(.95/1.07)^{10}}{.07-.05}\right](1.07)(1.07)^{-10} = 464.71$. The present value of all the payments is $919.95 + 464.71 = \$1385$ to the nearest dollar.

34. We have

$$\text{PV} = \frac{1}{m^2}\left[v^{\frac{1}{m}} + 2v^{\frac{2}{m}} + \cdots + nmv^{\frac{nm}{m}}\right]$$

$$\text{PV}(1+i)^{\frac{1}{m}} = \frac{1}{m^2}\left[1 + 2v^{\frac{1}{m}} + \cdots + nmv^{n-\frac{1}{m}}\right]$$

$$\text{PV}\left[(1+i)^{\frac{1}{m}} - 1\right] = \frac{1}{m^2}\left[1 + v^{\frac{1}{m}} + \cdots + v^{n-\frac{1}{m}} - nmv^{\frac{nm}{m}}\right]$$

$$= \frac{1}{m}\left[\ddot{a}_{\overline{n}|}^{(m)} - nv^n\right].$$

Therefore

$$\text{PV} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{m\left[(1+i)^{\frac{1}{m}} - 1\right]} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}}.$$

35. (*a*) $\frac{1}{12}\left[(\overbrace{1+1+\cdots+1}^{(12\text{ terms})}) + (\overbrace{2+2+\cdots+2}^{(12\text{ terms})})\right] = \frac{1}{12}(12+24) = 3.$

(*b*) $\frac{1}{144}\left[(1+2+\cdots+12)+(13+14+\cdots+24)\right] = \frac{(24)(25)}{(2)(144)} = \frac{25}{12}.$

36. We have

$$\begin{aligned}
\text{PV} &= \left[v^5 + v^6 + 2v^7 + 2v^8 + 3v^9 + 3v^{10} + \cdots\right] \\
&= \left(v^5 + v^7 + v^9 + \cdots\right)\ddot{a}_{\overline{\infty}|} = \frac{v^5}{1-v^2}\cdot\frac{1}{d} \\
&= \frac{v^5}{1-v^2}\cdot\frac{1}{iv} = \frac{v^4}{i-vd}.
\end{aligned}$$

37. The payments are $1, 6, 11, 16, \ldots$. This can be decomposed into a level perpetuity of 1 starting at time $t = 4$ and on increasing perpetuity of $1, 2, 3, \ldots$ starting at time $t = 8$. Let i_4 and d_4 be effective rates of interest and discount over a 4-year period. The present value of the annuity is

$$\frac{1}{i_4} + 5(1+i_4)^{-1}\left(\frac{1}{i_4 \cdot d_4}\right) \quad \text{where} \quad i_4 = (1+i)^4 - 1.$$

We know that

$$(1+i)^4 = (.75)^{-1} = 4/3, \text{ or } i_4 = \frac{4}{3} - 1 = \frac{1}{3} \quad \text{and} \quad d_4 = \frac{1/3}{1+1/3} = \frac{1}{4}.$$

Thus, the present value becomes

$$3 + (5)\left(\frac{3}{4}\right)\left(\frac{1}{\frac{1}{3}\cdot\frac{1}{4}}\right) = 3 + 45 = 48.$$

38. Let j be the semiannual rate of interest. We know that $(1+j)^2 = 1.08$, so that $j = .03923$. The present value of the annuity is

$$1 + \frac{1.03}{1.03923} + \left(\frac{1.03}{1.03923}\right)^2 + \cdots = \frac{1}{1 - 1.03/1.03923} = 112.59$$

upon summing the infinite geometric progression.

39. The ratio is

$$\frac{\int_5^{10} t\,dt}{\int_0^5 t\,dt} = \frac{\frac{1}{2}t^2\Big]_5^{10}}{\frac{1}{2}t^2\Big]_0^5} = \frac{75/2}{25/2} = 3.$$

40. Taking the limit of formula (4.42) as $n \to \infty$, we have

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{\bar{a}_{\overline{\infty}|}}{\delta} = \frac{1}{\delta^2} = \frac{1}{(.08)^2} = 156.25.$$

41. Applying formula (4.43) we have the present value equal to

$$\int_0^\infty f(t)v^t dt = \int_0^\infty \left(\frac{1+k}{1+i}\right)^t dt = \left.\frac{\left(\frac{1+k}{1+i}\right)^t}{\ln\left(\frac{1+k}{1+i}\right)}\right]_0^\infty$$

$$= -\frac{1}{\ln\left(\frac{1+k}{1+i}\right)} = \frac{1}{\ln(1+i) - \ln(1+k)} = \frac{1}{\delta_i - \delta_k}.$$

Note that the upper limit is zero since $i > k$.

42. (*a*) $(\bar{D}\bar{a})_{\overline{n}|} = \int_0^n (n-t)v^t dt.$

(*b*) $\int_0^n (n-t)v^t dt = n\bar{a}_{\overline{n}|} - (\bar{I}\bar{a})_{\overline{n}|}$

$$= \frac{n(1-v^n)}{\delta} - \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}.$$

The similarity to the discrete annuity formula (4.25) for $(Da)_{\overline{n}|}$ is apparent.

43. In this exercise we must adapt and apply formula (4.44). The present value is

$$\int_1^{14} (t^2 - 1)e^{-\int_0^t (1+r)^{-1} dr} dt.$$

The discounting function was seen to be equal to $(1+t)^{-1}$ in Exercise 19. Thus, the answer is

$$\int_1^{14} \frac{t^2-1}{t+1} dt = \int_1^{14} \frac{(t-1)(t+1)}{t+1} dt = \int_1^{14} (t-1)\,dt$$

$$= \left[\frac{1}{2}t^2 - t\right]_1^{14} = (98 - 14) - \left(\frac{1}{2} - 1\right) = 84.5.$$

44. For perpetuity #1 we have

$$1+v^{.5}+v+v^{1.5}+\cdots=\frac{1}{1-v^{.5}}=20$$

$$\text{so that}\quad 1-v^{.5}=.05 \quad\text{and}\quad v^{.5}=.95.$$

For perpetuity #2, we have

$$X\left[1+v^2+v^4+\cdots\right]=X\,\frac{1}{1-v^2}=20$$

$$\text{so that}\quad X=20\left(1-v^2\right)=20\left[1-(.95)^4\right]=3.71.$$

45. We have

$$\int_0^n \bar{a}_{\overline{t}|}dt=\frac{1}{\delta}\int_0^n\left(1-v^t\right)\Big]dt=\frac{n-\bar{a}_{\overline{n}|}}{\delta}=\frac{n-(n-4)}{.1}=40.$$

46. For each year of college the present value of the payments for the year evaluated at the beginning of the year is

$$1200a^{(12)}_{\overline{9/12}|}.$$

The total present value for the payments for all four years of college is

$$1200a^{(12)}_{\overline{9/12}|}\left(1+v+v^2+v^3\right)=1200\ddot{a}_{\overline{4}|}a^{(12)}_{\overline{9/12}|}.$$

47. For annuity #1, we have $\mathrm{PV}_1=\dfrac{P}{i}$.

For annuity #2, we have $\mathrm{PV}_2=q\left(\dfrac{1}{i}+\dfrac{1}{i^2}\right)$.

Denote the difference in present values by *D*.

$$D=\mathrm{PV}_1-\mathrm{PV}_2=\frac{p-q}{i}-\frac{q}{i^2}.$$

(*a*) If $D=0$, then

$$\frac{p-q}{i}-\frac{q}{i^2}=0 \quad\text{or}\quad p-q=\frac{q}{i} \quad\text{or}\quad i=\frac{q}{p-q}.$$

(*b*) We seek to maximize *D*.

$$\frac{dD}{di}=\frac{d}{di}\left[(p-q)i^{-1}-qi^{-2}\right]$$

$$=-(p-q)i^{-2}+2qi^{-3}=0.$$

Multiply through by i^3 to obtain

$$-(p-q)i+2q=0 \quad \text{or} \quad i=\frac{2q}{p-q}.$$

48. We must set soil (S) posts at times 0,9,18,27. We must set concrete posts (C) at times 0,15,30. Applying formula (4.3) twice we have

$$\text{PV}_S = 2\frac{a_{\overline{36}|}}{a_{\overline{9}|}} \quad \text{and} \quad \text{PV}_C = (2+X)\frac{a_{\overline{45}|}}{a_{\overline{15}|}}.$$

Equating the two present values, we have

$$2\frac{a_{\overline{36}|}}{a_{\overline{9}|}} = (2+X)\frac{a_{\overline{45}|}}{a_{\overline{15}|}} \quad \text{so that}$$

$$X = 2\left[\frac{a_{\overline{36}|}}{a_{\overline{9}|}} - \frac{a_{\overline{45}|}}{a_{\overline{15}|}}\right]\cdot\frac{a_{\overline{45}|}}{a_{\overline{15}|}} = 2\left(\frac{a_{\overline{36}|}a_{\overline{15}|}}{a_{\overline{9}|}a_{\overline{45}|}}-1\right).$$

49. We know $\bar{a}_{\overline{n}|} = \dfrac{1-v^n}{\delta} = a$, so that $v^n = 1-a\delta$. Similarly, $\bar{a}_{\overline{2n}|} = \dfrac{1-v^{2n}}{\delta} = b$, so that $v^{2n} = 1-b\delta$. Therefore, $1-b\delta = (1-a\delta)^2 = 1-2a\delta + a^2\delta^2$, or $a^2\delta^2 = (2a-b)\delta$ so that $\delta = \dfrac{2a-b}{a^2}$. Also we see that $n\ln v = \ln(1-a\delta) - n\delta = \ln(1-a\delta)$ so that $n = \dfrac{\ln(1-a\delta)}{-\delta}$. From formula (4.42) we know that $(\bar{I}\bar{a})_{\overline{n}|} = \dfrac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$. We now substitute the identities derived above for $\bar{a}_{\overline{n}|}, n, v^n$, and δ. After several steps of tedious, but routine, algebra we obtain the answer

$$\frac{a^3}{(2a-b)^2}\left[2a-b-(b-a)\ln\left(\frac{a}{b-a}\right)\right].$$

50. (a) (1) $\dfrac{d}{di}a_{\overline{n}|} = \dfrac{d}{di}\displaystyle\sum_{t=1}^{n} v^t = \dfrac{d}{di}\sum_{t=1}^{n}(1+i)^{-t} = -\sum_{t=1}^{n} t(1+i)^{-t-1} = -v\sum_{t=1}^{n} tv^t = -v(Ia)_{\overline{n}|}.$

(2) $\dfrac{d}{di}a_{\overline{n}|}\Big|_{i=0} = -v(Ia)_{\overline{n}|}\Big|_{i=0} = -\displaystyle\sum_{t=1}^{n} t = -\frac{n(n+1)}{2}.$

(b) (1) $\dfrac{d}{di}\bar{a}_{\overline{n}|} = \dfrac{d}{di}\displaystyle\int_0^n v^t dt = \dfrac{d}{di}\int_0^n (1+i)^{-t}dt = -\int_0^n t(1+i)^{-t-1}dt = -v\int_0^n tv^t dt = -v(\bar{I}\bar{a})_{\overline{n}|}.$

(2) $\dfrac{d}{di}\bar{a}_{\overline{n}|}\Big|_{i=0} = -v(\bar{I}\bar{a})_{\overline{n}|}\Big|_{t=0} = -\displaystyle\int_0^n t\,dt = -\frac{n^2}{2}.$