Chapter 6

1. (*a*) $P = 1000(1.10)^{-10} = \$385.54$.

   (*b*) $P = 1000(1.09)^{-10} = \$422.41$.

   (*c*) The price increase percentage is $\dfrac{422.41 - 385.54}{385.54} = .0956$, or 9.56%.

2. The price is the present value of the accumulated value, so we have

$$P = 1000\left(1 + \frac{.08}{2}\right)^{20}(1.1)^{-10} = \$844.77.$$

3. (*a*) The day counting method is actual/360. In 26 weeks there are $26 \times 7 = 182$ days. Using the simple discount method, we have

$$9600 = 10,000\left(1 - \frac{182}{360}d\right) \text{ and } d = .0791, \text{ or } 7.91\%.$$

   (*b*) An equation of value with compound interest is

$$9600 = 10,000(1+i)^{-\frac{1}{2}} \text{ and } i = .0851, \text{ or } 8.51\%.$$

4. We have $F = 100$, $C = 105$, $r = .05$, $g = 5/105$, $i = .04$, $G = 5/.04 = 125$,

   $K = 105(1.04)^{-20} = 47.921$, and $n = 20$.

   Basic: $P = 5a_{\overline{20}|} + 105v^{20} = 5(13.59031) + 105(.45639) = \$115.87$.

   Premium/discount: $P = 105 + (5 - 4.2)a_{\overline{20}|} = \$115.87$.

   Base amount: $P = 125 + (105 - 125)(1.04)^{-20} = \$115.87$.

   Makeham: $P = 47.921 + \dfrac{5}{.04(105)}(105 - 47.921) = \$115.87$.

5. We apply the premium/discount formula to the first bond to obtain

$$1136.78 = 1000 + 1000(.025 - .02)a_{\overline{n}|}$$

which can be solved to obtain $a_{\overline{n}|} = 27.356$. Now apply the premium/discount formula to the second bond to obtain

$$P = 1000 + 1000(.0125 - .02)(27.356) = \$794.83.$$

6. Since the present value of the redemption value is given, we will use Makeham's formula. First, we find

$$g = \frac{Fr}{C} = \frac{45}{1125} = .04.$$

Now

$$P = K + \frac{g}{i}(C - K) = 225 + \frac{.04}{.05}(1125 - 225) = \$945.$$

7. Since $K = Cv^n$, we have $450 = 1000v^n$ and $v^n = .45$. Now we will apply the base amount formula

$$P = G + (C - G)v^n = G(1 - v^n) + Cv^n$$

and substituting values

$$1110 = G(1 - .45) + 450 \quad \text{and} \quad G = \$1200.$$

8. The price of the 10-year bond is

$$P = 1000(1.035)^{-20} + 50a_{\overline{20}|.035} = 1213.19.$$

The price of the 8-year bond is

$$P = F(1.035)^{-16} + .03Fa_{\overline{16}|.035} = 1213.19$$

and solving

$$F = \frac{1213.19}{.576706 + (.03)(12.09412)} = \$1291 \text{ to the nearest dollar.}$$

9. Since n is unknown, we should use an approach in which n only appears once. We will use the base amount formula. First, we have

$$G = \frac{Fr}{i} = \frac{1000(.06)}{.05} = 1200$$

and

$$P = 1200 + (1000 - 1200)v^n = 1200 - 200v^n.$$

If we double the term of the bond we have

$$P+50=1200+(1000-1200)v^{2n}=1250-200v^{n}.$$

Thus we have a quadratic which reduces to

$$200v^{2n}-200v^{n}+50=0$$

or

$$4v^{2n}-4v^{n}+1=0$$

and factoring

$$\left(2v^{n}-1\right)^{2}=0.$$

Thus, $v^n=.5$ and $P=1200-200(.5)=\$1100$.

10. (*a*) The nominal yield is the annualized coupon rate of 8.40%.

(*b*) Here we want the annualized modified coupon rate, so

$$2g=2\left(\frac{Fr}{C}\right)=2\left(\frac{42}{1050}\right)=8.00\%.$$

(*c*) Current yield is the ratio of annualized coupon to price or $\frac{84}{919.15}=9.14\%$.

(*d*) Yield to maturity is given as 10.00%.

11. Using the premium/discount formula, we have

$$P_1=1+p=1+(1.5i-i)a_{\overline{n}|}=1+.5ia_{\overline{n}|}$$

and

$$P_2=1+(.75i-i)a_{\overline{n}|}=1-.25ia_{\overline{n}|}=1-.5p.$$

12. Let *X* be the coupon amount and we have $X=5+.75X$ so $X=20$.

13. We have $n=20$ and are given that $P_{19}=C(i-g)v^2=8$. We know that the principal adjustment column is a geometric progression. Therefore, we have

$$\sum_{t=1}^{8}P_t=8\left(v^{11}+v^{12}+\cdots+v^{18}\right)\ \text{ at }\ 4.5\%$$

$$=8v^{10}a_{\overline{8}|}=8(1.045)^{-10}(6.59589)=\$33.98.$$

14. Since $i > g$, the bond is bought at a discount. Therefore, the total interest exceeds total coupons by the amount of the discount. We have

$$\Sigma I_t = n \cdot Cg + d = (10)(50) + 1000(.06 - .05)\, a_{\overline{10|}.06}$$
$$= 500 + 10(7.36009) = \$573.60.$$

15. We have semiannual yield rate *j*

(*i*) $X = (40 - 1000j)\, a_{\overline{20|}}$

(*ii*) $Y = -(45 - 1000j)\, a_{\overline{20|}}$

(*iii*) $2X = -(50 - 1000j)\, a_{\overline{20|}}$.

By inspection, we have $2(X+Y) = X + 2X$, so that $2Y = X$ and $Y = \dfrac{X}{2}$.

16. (*a*) The total premium is $1037.17 - 1000 = 37.17$ amortized over four periods, with each amortization equal to $37.17/4 = 9.2925$. Thus, we have

$$B_0 = 1037.17$$
$$B_1 = 1037.17 - 9.2925 = 1027.88$$
$$B_2 = 1027.8775 - 9.2925 = 1018.59$$
$$B_3 = 1018.585 - 9.2925 = 1009.29$$
$$B_4 = 1009.2925 - 9.2925 = 1000.00$$

(*b*) The total discount is $1000 - 964.54 = 35.46$ amortized over four periods, with each amortization equal to $35.46/4 = 8.865$. Thus, we have

$$B_0 = 964.54$$
$$B_1 = 965.54 + 8.865 = 973.41$$
$$B_2 = 973.405 + 8.865 = 982.27$$
$$B_3 = 982.27 + 8.865 = 991.14$$
$$B_4 = 991.135 + 8.865 = 1000.00$$

(*c*) For premium bonds the straight line values are less than true book values. For discount bonds the opposite is the case.

17. (*a*) Since $k < 1$, then $1 + ki > (1+i)^k$, so

Theoretical = Semi-Theoretical < Practical.

(*b*) Since $\dfrac{(1+i)^k - 1}{i} < k,$ then for the accrued coupon, we have

Theoretical < Semi-Theoretical = Practical.

Finally, $B^m = B^f - AC$ and combining results

Semi-Theoretical < Theoretical
Semi-Theoretical < Practical

but Practical $\lesseqgtr$ Theoretical is indeterminate.

18. Theoretical method:

$$B^f_{1/3} = 964.54(1.05)^{1/3} = 980.35$$

$$AC = 40\left[\frac{(1.05)^{1/3} - 1}{.05}\right] = 13.12$$

$$B^m_{1/3} = 980.35 - 13.12 = 967.23$$

Practical method:

$$B^f_{1/3} = 964.54\left[1 + \left(\frac{1}{3}\right)(.05)\right] = 980.62$$

$$AC = \frac{1}{3}(40) = 13.33$$

$$B^m_{1/3} = 980.62 - 13.33 = 967.29$$

Semi-Theoretical:

$$B^f_{1/3} = 964.54(1.05)^{1/3} = 980.35$$

$$AC = \frac{1}{3}(40) = 13.33$$

$$B^m_{1/3} = 980.35 - 13.33 = 967.02$$

19. From Appendix A

April 15	is	Day 105
June 28	is	Day 179
October 15	is	Day 288

The price on April 15, *Z* is

$$P = 1000 + (30 - 35)\,a_{\overline{31}|.035} = 906.32.$$

The price on June 25, *Z* is

$$906.32\left[1 + \frac{179 - 105}{288 - 105}(.035)\right] = \$919.15.$$

20. (*a*) Using a financial calculator

$$\text{N} = 12 \times 2 = 24$$
$$\text{PMT} = 100\left(\frac{.10}{2}\right) = 5$$
$$\text{FV} = 100$$
$$\text{PV} = -110$$
$$\text{and CPT I} = 4.322.$$

Answer $= 2(4.322) = 8.64\%$.

(*b*) Applying formula (6.24), we have

$$i \approx \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n}k} \quad \text{where} \quad k = \frac{P-C}{C} = \frac{110-100}{100} = .1$$
$$= \frac{.05 - .1/24}{1 + \frac{25}{48}(.1)} = .04356.$$

Answer = $2(.04356) = .0871$, or 8.71%.

21. Bond 1: $P = 500 + (45 - 500i)a_{\overline{40|}}$.

Bond 2: $P = 1000 + (30 - 1000i)a_{\overline{40|}}$.

We are given that

$$(45 - 500i)a_{\overline{40|}} = 2(1000i - 30)a_{\overline{40|}}$$

so

$$45 - 500i = 2000i - 60$$

and

$$i = \frac{105}{2500} = .042.$$

The answer is $2i = .084$, or 8.4%.

22. Using the premium/discount formula

$$92 = 100\left[1 - .01a_{\overline{15|}\,i}\right]$$

so that

$$a_{\overline{15|}\,i} = 8.$$

Using a financial calculator and the technique in Section 3.7 we have

$$i = 9.13\%.$$

23. Using the basic formula, we have

$$P = 1000v^n + 42a_{\overline{n}|}$$

$$(i)\ \ P + 100 = 1000v^n + 52.50a_{\overline{n}|}$$

$$(ii)\ \ 42a_{\overline{n}|} = 1000v^n.$$

Subtracting the first two above

$$10.50a_{\overline{n}|} = 100 \quad \text{or} \quad a_{\overline{n}|} = 9.52381.$$

From (*ii*)

$$42a_{\overline{n}|} = 42(9.52381) = 400 = 1000v^n$$
$$= 1000\left(1 - ia_{\overline{n}|}\right) = 1000 - 9523.81i$$

so that $i = \dfrac{1000 - 400}{9523.81} = .063,$ or 6.3%.

24. (*a*) Premium bond, assume early:

$$P = 1000 + (40 - 30)\,a_{\overline{20}|.03} = \$1148.77\,.$$

(*b*) Discount bond, assume late:

$$P = 1000 + (40 - 50)\,a_{\overline{30}|.05} = \$846.28\,.$$

(*c*) Use a financial calculator:

$\text{N} = 20 \quad \text{PMT} = 40 \quad \text{FV} = 1000 \quad \text{PV} = -846.28 \quad \text{and} \quad \text{CPT I} = 5.261.$

Answer $= 2(5.261) = 10.52\%.$

(*d*) Premium bond, assume late:

$$P = 1000 + (40 - 30)\,a_{\overline{30}|.03} = \$1196.00\,.$$

(*e*) Discount bond, assume early:

$$P = 1000 + (40 - 50)\,a_{\overline{20}|.05} = \$875.38\,.$$

25. Note that this bond has a quarterly coupon rate and yield rate. The price assuming no early call is

$$P = 1000(1.015)^{-40} + 20a_{\overline{40}|.015} = 1149.58.$$

The redemption value at the end of five years to produce the same yield rate would have to be

$$1149.58 = C(1.015)^{-20} + 20a_{\overline{20|}.015}$$

$$\text{and } C = 1149.58(1.015)^{20} - 20s_{\overline{20|}.015}$$

$$= \$1086 \text{ to the nearest dollar.}$$

26. In Example 6.8 we had a premium bond and used the earliest possible redemption date in each interval. In this Exercise we have a discount bond and must use the latest possible redemption date in each interval:

At year 6: $P = 1050 + (20 - 26.25)a_{\overline{12|}.025} = 985.89$

At year 9: $P = 1025 + (20 - 25.625)a_{\overline{18|}.025} = 944.26$

At year 10: $P = 1000 + (20 - 25)a_{\overline{20|}.025} = 922.05$

Assume no early call, so the price is \$922.05. If the bond is called early, the yield rate will be higher than 5%.

27. Using Makeham's formula $g = \frac{1000(.045)}{1100} = \frac{.045}{1.1}$.

Now, $P = K + \frac{g}{i}(C - K)$ and we have

$$918 = 1100v^n + \frac{.045}{(1.1)(.05)}(1100 - 1100v^n)$$

$$= 200v^n + 900$$

$$v^n = \frac{18}{200} = .09 \quad \text{and} \quad n = \frac{-\ln(.09)}{\ln(1.05)} = 49.35.$$

The number of years to the nearest integer $= \frac{49.35}{2} = 25.$

28. The two calculated prices define the endpoints of the range of possible prices. Thus, to guarantee the desired yield rate the investor should pay no more than \$897.
The bond is then called at the end of 20 years at 1050. Using a financial calculator, we have

$$\text{N} = 20 \quad \text{PMT} = 80 \quad \text{FV} = 1050 \quad \text{PV} = -897 \quad \text{and} \quad \text{CPT I} = 9.24\%.$$

29. Use Makeham's formula

$$P = \sum_{t=1}^{10} 1000 v_{.04}^t + \frac{.06}{.04}\left[10,000 - \sum_{t=1}^{10} 1000 v_{.04}^t\right]$$
$$= 1000 a_{\overline{10}|.04} + \frac{3}{2}\left[10,000 - 1000 a_{\overline{10}|.04}\right]$$
$$= 15,000 - 500 a_{\overline{10}|.04} = \$10,945 \text{ to the nearest dollar.}$$

30. Use Makeham's formula

$P = K + \frac{.06}{.10}[10,000 - K]$ where $K = 500\left(a_{\overline{25}|} - a_{\overline{5}|}\right) = 2643.13$ and

$P = 6000 + .4(2643.13) = \$7057$ to the nearest dollar.

31. Use Makeham's formula

$$P = K + \frac{g}{i}(C - K) = \frac{g}{i}C + \left(1 - \frac{g}{i}\right)K$$

where

$$\frac{g}{i} = \frac{g}{1.25g} = .8 \qquad C = 100,000$$

and

$$K = 10,000\left(v^{10} + v^{16} + v^{22} + 2v^{28} + 2v^{34} + 3v^{40}\right).$$

Applying formula (4.3) in combination with the technique presented in Section 3.4 we obtain

$$K = 10,000\left[\frac{3a_{\overline{46}|} - a_{\overline{40}|} - a_{\overline{28}|} - a_{\overline{10}|}}{a_{\overline{6}|}}\right].$$

Thus, the answer is

$$80,000 + 2000\left[\frac{3a_{\overline{46}|} - a_{\overline{40}|} - a_{\overline{28}|} - a_{\overline{10}|}}{a_{\overline{6}|}}\right].$$

32. From the first principles we have

$$P = 105v^n + 8a_{\overline{n}|}^{(2)} = 105v^n + \frac{8(1 - v^n)}{i^{(2)}}$$
$$= \left(105 - \frac{8}{i^{(2)}}\right)v^n + \frac{8}{i^{(2)}}.$$

Thus, $A = 105i^{(2)} - 8$ and $B = 8$.

33. From first principles we have

$$P = 1000(1.06)^{-20} + 40a_{\overline{20}|.06} + 10a_{\overline{10}|.06}$$
$$= 311.8047 + 458.7968 + 73.6009 = \$844.20.$$

34. From first principles we have

$$P = 1050(1.0825)^{-20} + 75\left[\frac{1}{1.0825} + \frac{1.03}{(1.0825)^2} + \cdots + \frac{(1.03)^{19}}{(1.0825)^{20}}\right]$$
$$= 1050(1.0825)^{-20} + \frac{75}{1.0825}\left[\frac{1-\left(\frac{1.03}{1.0825}\right)^{20}}{1-\frac{1.03}{1.0825}}\right]$$
$$= \$1115 \text{ to the nearest dollar.}$$

35. Applying formula (6.28)

$$P = \frac{D}{i-g} = \frac{10}{.12-.05} = 142.857.$$

The level dividend that would be equivalent is denoted by D and we have

$$142.857 = Da_{\overline{\infty}|} = \frac{D}{.12} \text{ or } D = \$17.14.$$

36. Modifying formula (6.28) we have

$$P = v^5 \frac{D}{i-g} = (1.15)^{-5}\frac{(.5)(6)(1.08)^6}{.15-.08} = \$33.81.$$

37. If current earnings are E, then the earnings in 6 years will be $1.6E$. The stock price currently is $10E$ and in 6 years will be $15(1.6E) = 24E$. Thus, the yield rate can be determined from

$$10E(1+i)^6 = 24E$$

which reduces to

$$i = (2.4)^{1/6} - 1 = .157, \text{ or } 15.7\%.$$

38. The price at time $t = 0$ would be

$$2.50a_{\overline{\infty}|.02} = \frac{2.50}{.02} = 125.$$

The bond is called at the end of 10 years. Using a financial calculator we have

$$\text{N} = 40 \quad \text{PMT} = 2.50 \quad \text{FV} = 100 \quad \text{PV} = -125 \quad \text{and} \quad \text{CPT I} = .016424.$$

The answer is $4(.016424) = .0657$, or 6.57%.

39. (*a*) MV for the bonds $= 1000(900) = 900,000.$
MV for the stocks $= 10,000(115) = 1,150,000.$
Total MV = \$2,050,000.

(*b*) BV for the bonds = 1,000,000, since the yield rate equals the coupon rate.
BV for the stocks = 1,000,000, their cost.
Total BV = \$2,000,000.

(*c*) $\text{BV}_B + \text{MV}_S = 1,000,000 + 1,150,000 = \$2,150,000.$

(*d*) $\text{PV}_B = 40,000 a_{\overline{15}|.05} + 1,000,000 v_{.05}^{15} = 896,208.$

$$\text{PV}_S = 60,000 a_{\overline{\infty}|.05} = \frac{60,000}{.05} = 1,2000,000.$$

Total $\text{PV} = \$2,096,200$ to the nearest \$100.

40. From first principles we have

$$P = 9\bar{a}_{\overline{12}|} + 100v^{12} = 9\left(\frac{1-v^{12}}{\delta}\right) + 100v^{12}$$
$$= 9\left(\frac{1-e^{-12\delta}}{\delta}\right) + 100e^{-12\delta}$$
$$= \frac{1}{\delta}\left[(100\delta - 9)e^{-12\delta} + 9\right].$$

41. From the premium/discount formula we have

$$p = (g-i)a_{\overline{n}|} \quad \text{and} \quad q = \left(\frac{1}{2}g - i\right)a_{\overline{n}|}.$$

We then have

$$(2g-i)a_{\overline{n}|} = Ap + Bq = A(g-i)a_{\overline{n}|} + B\left(\frac{1}{2}g - i\right)a_{\overline{n}|}.$$

Equating coefficients gives

$$A+\frac{1}{2}B=2$$
$$A+B=1.$$

Solving these simultaneous equations gives $A=3$ and $B=-2.$

42. Using Makeham's formula for the first bond

$$P=K+\frac{g}{i}(C-K)=Cv_{.04}^{5}+\frac{.06}{.04}\left(C-Cv_{.04}^{5}\right)$$
$$=C\left[1.5-.5(1.04)^{-5}\right]=1.089036C.$$

Using Makeham's formula again for the second bond

$$1.089036C=C\left[1.25-.25(1.04)^{-n}\right].$$

Thus $(1.04)^{-n}=.643854$ and $n=\dfrac{-\ln(.643854)}{\ln 1.04}=11.23$ or 11 years to the nearest year.

43. Since $r=g>i,$ the bond is a premium bond. Therefore $B_{19}>C=1000.$ We then have $P_{20}=B_{19}-1000$ and $I_{20}=iB_{19}$ so that

$$Fr=1000r=P_{20}+I_{20}$$
$$=B_{19}-1000+iB_{19}=B_{19}(1+i)-1000.$$

Thus, we have

$$B_{19}=1000\frac{1+r}{1+i}=1000\frac{1.03+i}{1+i}.$$

We are also given

$$i\cdot B_{19}=.7\left(B_{19}-1000\right)\text{ so that }B_{19}=\frac{700}{.7-i}.$$

Therefore

$$1000\frac{1.03+i}{1+i}=\frac{700}{.7-i}$$

which can be solved to obtain $i=.02$. Finally, we can obtain the price of the bond as

$$P=1000+1000(.05-.02)a_{\overline{20|}.02}$$
$$=1000+30(16.35149)=\$1490.54.$$

44. If suspended coupon interest accrues at the yield rate, then there is no difference between the restructured bond and the original bond. We have

$$\begin{aligned} P &= 33.75a_{\overline{20}|.037} + 1000(1.037)^{-20} \\ &= 33.75(13.95861) + 1000(.483532) \\ &= \$955 \text{ to the nearest dollar.} \end{aligned}$$

45. The redemption value C is the same for both bonds.
Bond X: Use the base amount formula. We have $Fr = Gi,$ so that

$$G = F\frac{r}{i} = 1000(1.03125) = \$1031.25$$

and $K = Cv^n = 381.50.$

Bond Y: We have

$$Cv^{n/2} = 647.80.$$

Taking the ratio $\dfrac{Cv^n}{Cv^{n/2}} = v^{n/2} = \dfrac{381.50}{647.80} = .5889163$

so $v^n = (.5889163)^2 = .3468224$

and $C = \dfrac{381.50}{.3468224} = 1100$.

Finally,

$$\begin{aligned} P_X &= G + (C - G)v^n \\ &= 1031.25 + (1100 - 1031.25)(.3468224) \\ &= \$1055 \text{ to the nearest dollar.} \end{aligned}$$

46. (*a*) Prospectively, $B_t = C + (Fr - Ci)a_{\overline{n-t}|}$ so that

$$\begin{aligned} i\sum_{t=0}^{n-1} B_t &= \sum_{t=0}^{n-1}\left[Ci + (Fr - Ci)(1 - v^{n-t})\right] \\ &= \sum_{t=0}^{n-1}\left[Civ^{n-t} + Fr(1 - v^{n-t})\right] \\ &= Cia_{\overline{n}|} + nFr - Fra_{\overline{n}|}. \end{aligned}$$

However $P = C + (Fr - Ci)a_{\overline{n}|}$ so that

$$P + i\sum_{t=0}^{n-1} B_t = C + n \cdot Fr.$$

(*b*) In a bond amortization schedule

- $i\sum_{t=0}^{n-1} B_t$ is the sum of the interest earned column.
- $P - C = C(g-i)a_{\overline{n}|}$ is the sum of the principal adjustment column.
- $n \cdot Fr$ is the sum of coupon column.

The sum of the first two is equal to the third.

47. (*a*) From Exercise 50 in Chapter 4

$$\frac{d}{di}a_{\overline{n}|} = -v(Ia)_{\overline{n}|}.$$

Then $\frac{dP}{di} = \frac{d}{di}\left[Cga_{\overline{n}|} + Cv^n\right] = Cg\left[-v\left(Ia_{\overline{n}|}\right)\right] - Cv^{n+1} = -Cv\left[g(Ia)_{\overline{n}|} + nv^n\right].$

(*b*) $\frac{dP}{dg} = \frac{d}{dg}\left[Cga_{\overline{n}|} + Cv^n\right] = Ca_{\overline{n}|}.$