Chapter 12

1. $$\mathrm{E}\left[a^{-1}(n)\right]=\mathrm{E}\left[\prod_{t=1}^{n}(1+i_t)^{-1}\right]$$
$$=\prod_{t=1}^{n}\mathrm{E}\left[1+i_t\right]^{-1} \quad \text{from independence}$$
$$=(1+\bar{i})^{-n}.$$

2. $$\mathrm{E}\left[a_{\overline{n}|}\right]=\mathrm{E}\left[\sum_{t=1}^{n}\prod_{s=1}^{t}(1+i_s)^{-1}\right]$$
$$=\sum_{t=1}^{n}\prod_{s=1}^{t}\mathrm{E}\left[1+i_s\right]^{-1} \quad \text{from independence}$$
$$=\sum_{t=1}^{n}(1+\bar{i})^{-t}=a_{\overline{n}|\bar{i}}.$$

3. (*a*) Year 1: 8% given.
Year 2: $.5(.07+.09)=.08$, or 8%.
Year 3: $.25[.06+2(.08)+.10]=.08$, or 8%.

(*b*) Year 1: $\sigma=0$, no variance.
Year 2: $\sigma^2=.5\left[(.07-.08)^2+(.09-.08)^2\right]=.0001$
$\sigma=\sqrt{.0001}=.01.$
Year 3: $\sigma^2=.25\left[(.06-.08)^2+2(.08-.08)^2+(.10-.08)^2\right]$
$=.0002$
$\sigma=\sqrt{.0002}=.01\sqrt{2}$

(*c*) $1000(1.08)(1.09)(1.10)=\$1294.92.$

(*d*) $1000(1.08)(1.07)(1.06)=\$1224.94.$

(*e*) $1000(1.08)^3=\$1259.71.$

(*f*) $.25(1000)[(1.08)(1.09)(1.10)+(1.08)(1.09)(1.08)$
$+(1.08)(1.07)(1.08)+(1.08)(1.07)(1.06)]$
$=.25[1294.92+1271.38+1248.05+1224.94]$
$=\$1259.82$

$$(g)\ \sigma^2 = .25\left[(1294.92-1259.82)^2 + (1271.38-1259.82)^2 + (1248.05-1259.82)^2 + (1224.94-1259.82)^2\right]$$
$$= 2720.79$$
$$\sigma = \sqrt{2720.79} = 52.16.$$

4. (a) $\mathrm{E}\left[(1+i_t)^{-1}\right] = \frac{1}{.09-.07}\int_{.07}^{.09}\frac{1}{1+t}dt$
$$= \frac{1}{.09-.07}\ln(1+t)\Big]_{.07}^{.09} = .925952.$$
Then set $(1+\bar{i})^{-1} = .925952$ and solve $\bar{i} = .07997.$

(b) We have $a^{-1}(3) = (1.07997)^{-3} = .79390.$

(c) $\mathrm{E}\left[(1+i_t)^{-2}\right] = \frac{1}{.09-.07}\int_{.07}^{.09}\frac{1}{(1+t)^2}\,dt$
$$= \left[\frac{-1}{.09-.07}\cdot\frac{1}{1+t}\right]_{.07}^{.09} = .857412.$$
Then set $(1+\bar{k})^{-1} = .857412$ and solve $\bar{k} = .16630.$

(d) Applying formula (12.10), we have
$$\mathrm{Var}\left[a^{-1}(3)\right] = (.857412)^3 - (.925952)^6 = .0000549$$
and the standard deviation is $\sqrt{.0000549} = .00735.$

5. (b) Applying formula (12.11), we have
$$\mathrm{E}\left[a_{\overline{3}|}\right] = a_{\overline{3}|\,\bar{i}} = a_{\overline{3}|.07997} = 2.5772.$$

(d) Applying formula (12.14), we have
$$\mathrm{Var}\left[a_{\overline{3}|}\right] = \frac{m_2^a + m_1^a}{m_2^a - m_1^a}a_{\overline{3}|\,\bar{k}} - \frac{2m_2^a}{m_2^a - m_1^a}a_{\overline{3}|\,\bar{i}} - \left(a_{\overline{3}|\,\bar{i}}\right)^2$$
$$= \frac{.857412+.925952}{.857412-.925952}(2.2229) - \frac{(2)(.857412)}{.857412-.925952}(2.5772) - (2.5772)^2$$
$$= .005444$$
and the standard deviation is $\sqrt{.005444} = .0735.$

6. The random variable $i_t^{(2)}/2$ will be normal with $\mu = 3\%$ and $\sigma = .25\%$.

   (*a*) Applying formula (12.1), we have

$$\mathrm{E}\left[100a(4)\right] = 100(1.03)^4 = 112.55.$$

   Applying formula (12.3), we have

$$\begin{aligned}\mathrm{Var}\left[100a_{\overline{4|}}\right] &= 10,000\left[\left(1+2\bar{i}+\bar{i}^2+s^2\right)^4-\left(1+\bar{i}\right)^8\right]\\ &= 10,000\left[\left\{1+(2)(.03)+(.03)^2+.0025\right\}^4-(1.03)^8\right]\\ &= 119.828\end{aligned}$$

   and the standard deviation is $\sqrt{119.828} = 10.95$.

   (*b*) Applying formula (12.5), we have

$$\mathrm{E}\left[100\ddot{s}_{\overline{4|}}\right] = 100\ddot{s}_{\overline{4|}.03} = 430.91.$$

   Applying formula (12.8), we have

$$\begin{aligned}m_1^s &= 1.03\\ m_2^s &= 1+2(.03)+(.03)^2+.0025 = 1.0634\end{aligned}$$

   and

$$\begin{aligned}\mathrm{Var}\left[100\ddot{s}_4\right] &= 10,000\left[\frac{1.0634+1.03}{1.0634-1.03}(4.67549)-\frac{(2)(1.0634)}{1.0634-1.03}(4.3091)-(4.3091)^2\right]\\ &= 944.929\end{aligned}$$

   and the standard deviation is $\sqrt{944.929} = 30.74$.

7. (*a*) $\mathrm{E}\left[s_{\overline{n|}}\right] = \mathrm{E}\left[\ddot{s}_{\overline{n+1|}}-1\right] = \ddot{s}_{\overline{n+1|}\,\bar{i}}-1.$

   (*b*) $\mathrm{Var}\left[s_{\overline{n|}}\right] = \mathrm{Var}\left[\ddot{s}_{\overline{n+1|}}-1\right] = \mathrm{Var}\left[\ddot{s}_{\overline{n+1|}}\right].$

   (*c*) $\mathrm{E}\left[\ddot{a}_{\overline{n|}}\right] = \mathrm{E}\left[1+a_{\overline{n-1|}}\right] = 1+a_{\overline{n-1|}\,\bar{i}}.$

   (*d*) $\mathrm{Var}\left[\ddot{a}_{\overline{n|}}\right] = \mathrm{Var}\left[1+a_{\overline{n-1|}}\right] = \mathrm{Var}\left[a_{\overline{n-1|}}\right].$

8. We know that $1+i$ is lognormal with $\mu = .06$ and $\sigma^2 = .01$. From the solution to Example 12.3(1), we have $\bar{i} = .067159$ and then

$$\begin{aligned}s^2 &= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) = e^{2(.06)+.01}\left(e^{.01}-1\right)\\ &= e^{.13}\left(e^{.01}-1\right) = .011445.\end{aligned}$$

We then apply formula (12.4*a*) to obtain

$$\begin{aligned}\text{Var}[a(n)] &= (1+2\bar{i}+\bar{i}^2+s^2)^n-(1+\bar{i})^{2n}\\ &= [1+2(.067159)+(.067159)^2+.011445]^5-(1.067159)^{10}\\ &= .09821\end{aligned}$$

and the standard deviation $=\sqrt{.09821}=.3134$ agreeing with the other approach.

9. (*a*) Formula (12.5) with $\bar{i}=e^{\mu+\sigma^2/2}-1$.

   (*b*) Formulas (12.6), (12.7) and (12.8) with $\bar{j}=e^{2\mu+2\sigma}$.

   (*c*) Formula (12.11) with $\bar{i}=e^{\mu-\sigma^2/2}-1$.

   (*d*) Formulas (12.12), (12.13) and (12.14) with $\bar{k}=e^{-2\mu+2\sigma^2}$.

10. (*a*) $\text{E}[1+i_t]=e^{.06+.0001/2}=1.06189$

    mean $=\text{E}[a(10)]=(1.06189)^{10}=1.823.$

    $\text{Var}[a(10)]=e^{(2)(10)(.06)+(10)(.0001)}\left(e^{(10)(.0001)}-1\right)$

    $=e^{1.201}\left(e^{.001}-1\right)=.003325$

    and $\text{s.d.}=\sqrt{.003325}=.058$.

    (*b*) Mean $=\text{E}\left[\ddot{s}_{\overline{10}|}\right]=\ddot{s}_{\overline{10}|.06189}=14.121$

    s.d. using formula (12.8) = .297.

    (*c*) $\text{E}\left[(1+i_t)^{-1}\right]=e^{-.06+.0001/2}=.941812$

    mean $=\text{E}[a^{-1}(10)]=(.941812)^{10}=.549$

    $\text{Var}[a^{-1}(10)]=e^{-1.2+.001}\left(e^{.001}-1\right)=.000302$

    and s.d. $=\sqrt{.000302}=.017$.

    (*d*) We have $(1+\bar{i})^{-1}=.941812$ or $\bar{i}=.06178$

    and $(1+\bar{k})^{-1}=e^{-.12+.0001}e^{.0001}=e^{-.1198}$

    $=.887098$ or $\bar{k}=.12727.$

    Mean $=\text{E}\left[a_{\overline{10}|}\right]=a_{\overline{10}|.06178}=7.298.$

    s.d. using formula (12.14) = .134.

11. $\mathrm{E}\left[1+i_t\right]=e^{\mu+\sigma^2/2}=1.067.$

$\mathrm{Var}\left[1+i_t\right]=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)=.011445.$

Solving two equations in two unknowns gives

$$\mu=.06 \quad \sigma^2=.01$$

Therefore $\delta_{[t]}$ follows a normal distribution with mean $=.06$ and $\mathrm{var}=.01.$

12. $\mathrm{E}\left[1+i_t\right]=1.08=e^{\mu+\sigma^2/2}=e^{\mu+.0001/2}$ so that $\mu=.07691.$

$\mathrm{Var}\left[1+i_t\right]=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)=(1.08)^2\left(e^{.0001}-1\right)=.00011665.$

$\mathrm{E}\left[a(3)\right]=(1.08)^3=1.25971.$

$\mathrm{Var}\left[a(3)\right]=\left[1+2(.08)+(.08)^2+.00011665\right]^3-(1.08)^6$

$=.0004762$ and $\mathrm{s.d.}=\sqrt{.0004762}=.02182.$

The 95% confidence interval is

$1.25971\pm1.96(.02182)$ or $(1.21693, 1.30247).$

13. $\mathrm{E}\left[s_{\overline{3|}}\right]=\mathrm{E}\left[\ddot{s}_{\overline{4|}}-1\right]=\ddot{s}_{\overline{4|}.08}-1=s_{\overline{3|}.08}=3.246=\text{mean}.$

$\mathrm{Var}\left[s_{\overline{3|}}\right]=\mathrm{Var}\left[\ddot{s}_{\overline{4|}}-1\right]=\mathrm{Var}\left[\ddot{s}_{\overline{4|}}\right].$

$\mathrm{Var}=65.62$ using formula (12.8).

14. $\mathrm{E}\left[\ln\left(1+i_t\right)\right]=\dfrac{.07+.09}{2}=.08=\mu.$

$\mathrm{Var}\left[\ln\left(1+i_t\right)\right]=\dfrac{(.09-.07)^2}{2}=\dfrac{.0001}{3}=\sigma^2.$

$\mathrm{E}\left[\ln a^{-1}(30)\right]=-30\mu=-30(.08)=-2.4.$

$\mathrm{Var}\left[\ln a^{-1}(30)\right]=30\sigma^2=30\left(\dfrac{.0001}{3}\right)=.001.$

The 95th percentile of $\ln a^{-1}(30)$ is

$$-2.4+1.645\sqrt{.001}=-2.34798.$$

Thus, $100{,}000e^{-2.34798}=\$9556.20.$

15. Continuing Example 12.7:

$$\delta_{[6]} = .08 + .6(.091 - .08) + .2(.095 - .08) = .0896$$
$$\delta_{[7]} = .08 + .6(.0896 - .08) + .2(.091 - .08) = .0880$$
$$\delta_{[8]} = .08 + .6(.0880 - .08) + .2(.0896 - .08) = .0867.$$

16. (*a*) Formula (12.33)

$$\text{Var}\left[\delta_{[t]}\right] = \frac{1-k_2}{1+k_2} \cdot \frac{\sigma^2}{(1-k_2)^2 k_1^2}$$
$$= \frac{\sigma^2}{1-k_1} \quad \text{if } k_2 = 0$$

which is formula (12.30) with $k_1 = k$.

(*b*) Formula (12.34)

$$\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \text{Var}\left[\delta_{[t]}\right]\left[\tau g_1^{t-s} + (1-\tau)\, g_2^{t-s}\right].$$

We set $k_2 = 0,$ so that

$$\tau = 1 \qquad g_1 = k_1 \qquad g_2 = 0$$

from formula (12.35). We also substitute the result from part (*a*).

Thus, $\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \dfrac{\sigma^2}{1-k_1^2} k_1^{t-s}$

which is formula (12.31) with $k_1 = k$.

17. Use formula (12.33) with $k_1 = .6$ and $k_2 = .2$. Find the empirical estimate for $\text{Var}\left[\delta_{[t]}\right]$ based upon the sample data for $\delta_{[t]}$ given in Example 12.6. This will result in one equation in one unknown that can be solved for σ^2.

18. (*a*) Applying formula (12.33)

$$\text{Var}\left[\delta_{[t]}\right] = \frac{1-k_2}{1+k_2} \cdot \frac{\sigma^2}{(1-k_2)^2 - k_1^2}$$
$$= \frac{1-.2}{1+.2} \cdot \frac{.0002}{(1-.2)^2 - (.6)^2} = .0004762.$$

(*b*) Applying formulas (12.34), (12.35) and (12.36) with $k_1 = .6$ and $k_2 = .2$ and with $t - s = 2$ gives the answer .0001300.

19. (*a*) Applying formula (12.29) twice, we have

$$.096 = \delta + k(.100 - \delta)$$
$$.100 = \delta + k(.105 - \delta).$$

Solving these two equations in two unknowns, we have

$$k = .08 \quad \text{and} \quad \delta = .08.$$

Therefore

$$\delta_{[4]}^{E} = .08 + .8(.095 - .08) = .092.$$

(*b*) Applying formula (12.31), we have

$$\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \frac{\sigma^2}{1-k_2} k^{t-s} = (.0001)(.8)^{6-3} = .0000512.$$

20. There are 9 paths each with probability 1/9:

$$\begin{array}{lll} .06/.02/.02-.04k & .06/.06/.02 & .06/.10/.02+.04k \\ .06/.02/.06-.04k & .06/.06/.06 & .06/.10/.06+.04k \\ .06/.02/.10-.04k & .06/.06/.10 & .06/.10/.10+.04k \end{array}$$

(*a*) $$\text{E}[a(2)] = \frac{1}{9}[(1.02)(1.02-.04k) + (1.02)(1.06-.04k) + 7 \text{ more terms}]$$
$$= \frac{1}{3}\left[(1.02)(1.06-.04k) + (1.06)^2 + (1.10)(1.06-.04k)\right]$$
$$= (1.06)^2 + \frac{1}{3}(.0032)k.$$

(*b*) $$\text{E}\left[a(2)^2\right] = \frac{1}{9}\left[(1.02)^2(1.02-.04k)^2 + (1.02)^2(1.06-.04k)^2 + 7 \text{ more terms}\right]$$
$$= \frac{1}{9}\left[(1.02)^2 + (1.06)^2 + (1.10)^2\right] + \frac{(1.10)^2 - (1.02)^2}{3}(.08)(1.06)k$$
$$\left. + \frac{(1.10)^2 + (1.02)^2}{3}(.0016)k^2 \right]$$
$$= \frac{1}{9}(11.383876 + .04314624k + .01080192k^2)$$

and

$$\text{Var}[a(2)] = \text{E}\left[a(2)^2\right] - \text{E}[a(2)]^2$$
$$= \frac{1}{9}(.02158336 + .02157312k + .01079168k^2).$$

21. At time $t = 2$:

$$i = .144 \qquad V = \frac{(.5)(1000) + (.5)(1000)}{1.144} = 874.126$$

$$i = .10 \qquad V = \frac{(.5)(1000) + (.5)(1000)}{1.1} = 909.091$$

$$i = .06944 \quad V = \frac{(.5)(1000) + (.5)(1000)}{1.06944} = 935.069$$

At time $t = 1$:

$$i = .12 \qquad V = \frac{(.5)(874.126) + (.5)(909.091)}{1.12} = 796.079$$

$$i = .08333 \quad V = \frac{(.5)(909.091) + (.5)(935.069)}{1.08333} = 851.153$$

At time $t = 0$:

$$i = .10 \quad V = \frac{(.5)(796.079) + (.5)(851.153)}{1.1} = 748.74$$

obtaining the same answer as obtained with the other method.

22. (*a*)

Path	Probability	PV	PV^2
10/11/12	.25	.73125	.53473
10/11/10	.25	.74455	.55435
10/9/10	.25	.75821	.57488
10/9/8	.25	.77225	.59637

Value of the bond is

$$1000(.25)[.73125 + .74455 + .75821 + .77225] = 751.57.$$

(*b*) $\text{Var} = (1000)^2 (.25)[.53473 + .55435 + .57488 + .59637]$
$= 232.5664$

and the s.d. $= \sqrt{232.5664} = 15.25$.

(*c*) The mean interest rate is $i = .10$ so the value is $1000(1.1)^{-3} = 751.31.$

23. (*a*) At time $t = 1/2$:

$$i^{(2)} = .10 \qquad V = \frac{(.3)(1045) + (.7)(1045)}{1.05} = 995.238$$

$$i^{(2)} = .08 \qquad V = \frac{(.3)(1045) + (.7)(1045)}{1.04} = 1004.808.$$

At time $t=0$:

$$i^{(2)}=.09 \quad V=\frac{(.3)(995.238+45)+(.7)(1004.808+45)}{1.045} = \$1001.85.$$

(*b*) The equation of value is

$$1001.854 = 45v + 1045v^2$$

and solving the quadratic $v=.95789$. Then we have $v=.95789=e^{-.5\delta}$ and $\delta=.0861$, or 8.61%.

24. If the interest rate moves down, then call the bond, which gives

$$V=\frac{(.3)(995.238+45)+(.7)(1000+45)}{1.045}=\$998.63.$$

25. At time $t=\frac{1}{2}$:

$$j=.0288 \quad V=\frac{(.4)(1038/1.03458)+(.6)(1038/1.024)}{1.0288}=981.273$$

$$j=.02 \quad V=\frac{(.4)(1038/1.024)+(.6)(1038/1.01667)}{1.02}=998.095$$

$$j=.01389 \quad V=\frac{(.4)(1038/1.01667)+(.6)(1038/1.011575)}{1.01389}=1010.036$$

At time $t=\frac{1}{4}$:

$$j=.024 \quad V=\frac{(.4)(981.273+38)+(.6)(998.095+38)}{1.024}=1005.24$$

$$j=.01667 \quad V=\frac{(.4)(998.095+38)+(.6)(1010.036+38)}{1.01667}=1026.1536.$$

At time $t=0$:

$$j=.02 \quad V=\frac{(.4)(1005.24)+(.6)(1026.1536)}{1.02}=\$997.83.$$

26.

Path	Probability	PV	CV	CV from time 1
10/12/14.4	.16	.7095	1.4094	1.28128
10/12/10	.24	.7379	1.3552	1.23200
10/8.333/10	.24	.7629	1.3108	1.19170
10/8.333/6.944	.36	.7847	1.2744	1.15860

(*a*) $\mathrm{E}[a(3)] = (.16)(1.4094) + (.24)(1.3552) + (.24)(1.3108) + (.36)(1.2744) = 1.326.$

(*b*) $\mathrm{E}[a^{-1}(3)] = .749$.

(*c*) $\mathrm{E}[a(3)] = \mathrm{E}[a^{-1}(3)] + \mathrm{E}[a^{-1}(2)] + \mathrm{E}[a^{-1}(1)]$

$$= .749 + \left[(.4)(1.12)^{-1}(1.1)^{-1} + (.6)(1.08333)^{-1}(1.1)^{-1}\right] + (1.1)^{-1}$$
$$= 2.486.$$

(*d*) $\mathrm{E}\left[\ddot{s}_{\overline{3}|}\right] = 1.326 + 1.2038 + 1.096 = 3.626.$

27. Rendleman – Bartter:

mean

$$\mathrm{E}[\delta_t] = \mathrm{E}[\delta_t - \delta_0 + \delta_0] = \mathrm{E}[\Delta\delta_0] + \mathrm{E}[\delta_0]$$
$$= a\delta_0 t + \delta_0 = \delta_0(1 + at)$$

variance

$$\mathrm{Var}[\delta_t] = a^2\delta^2 t$$

Vasicek:

mean

$$\mathrm{E}[\delta_t] = \mathrm{E}[\Delta\delta_0] + \mathrm{E}[\delta_0]$$
$$= c(b - \delta_0) + \delta_0 = cb + (1 - c)\delta_0$$

variance

$$\mathrm{Var}[\delta_t] = \sigma^2 t$$

Cox – Ingersoll – Ross:

mean

$$\mathrm{E}[\delta_t] = cb + (1 - c)\delta_0$$

variance

$\mathrm{Var}[\delta_t] = \sigma^2\delta_0 t,$ since the process error is proportional to $\sqrt{\delta}$ which squares in computing variances.

28. (*a*) We have

$$d\delta = c(b - \delta)\,dt + \sigma dz$$
$$= 0 + \delta dz \quad \text{if} \quad c = 0$$
$$= adt + \sigma dz \quad \text{where} \quad a = 0$$

which is the process for a random walk.

(*b*) We have

$$d\delta = c(b-\delta)dt + \sigma dz$$
$$= (b-\delta)dt + \sigma dz \text{ if } c = 1$$

which is the process for a normal distribution with $\mu = b$.

29. For the random walk model

$$\Delta\delta = a\Delta t + \sigma\Delta z$$

and for the Rendleman-Bartter model

$$\Delta\delta = a\delta\Delta t + \sigma\delta\Delta z$$

Random walk		Rendleman - Bartter	
$\delta_0 = .06$			$\delta_0 = .06$
$\delta_{.5} = .0675$	$\Delta\delta_{.5} = .0075$	$\Delta\delta_{.5} = (.0075)(.06)$	$\delta_{.5} = .06045$
$\delta_1 = .065$	$\Delta\delta_1 = -.0025$	$\Delta\delta_1 = (-.0025)(.06045)$	$\delta_1 = .06030$
$\delta_{1.5} = .063$	$\Delta\delta_{1.5} = -.0020$	$\Delta\delta_{1.5} = (-.002)(.06030)$	$\delta_{1.5} = .06018$
$\delta_2 = .0685$	$\Delta\delta_2 = .0055$	$\Delta\delta_2 = (.0055)(.06018)$	$\delta_2 = .06051$

30. (*a*) We have $\delta_0 = .08$

$$E[\delta_{.5}] = \delta_0 + at = .08 + (.006)(.5) = .083$$

and

$$P = 39e^{-(.08)(.5)} + 1039e^{-(.08)(.5)-(.083)(.5)} = \$995.15.$$

(*b*) We have

$$995.151 = 39v + 1039v^2$$

and solving the quadratic

$$i^{(2)}/2 = .0606 \text{ so that } i^{(2)} = .1212.$$

(*c*) We have

$$\delta_{.5} = .08 + .006(.5) + (.01)(.5)\sqrt{.5} = .08654$$

and

$$P = 39e^{-(.08)(.5)} + 1039e^{-(.08)(.5)-(.08654)(.5)} = \$993.46.$$

31. Rework Examples 12.11-12.14 using ± 2 standard deviations. The following results are obtained:

Random walk	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0880$	$\delta_{.50} = .0480$
	$\delta_{.75} = .0970$	$\delta_{.75} = .0370$
	$\delta_{1} = .106$	$\delta_{1} = .026$

Rendleman – Bartter	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0892$	$\delta_{.50} = .0497$
	$\delta_{.75} = .1007$	$\delta_{.75} = .0419$
	$\delta_{1} = .114$	$\delta_{1} = .035$

Vasicek	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0876$	$\delta_{.50} = .0486$
	$\delta_{.75} = .0957$	$\delta_{.75} = .0386$
	$\delta_{1} = .103$	$\delta_{1} = .029$

Cox-Ingersoll-Ross	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0923$	$\delta_{.50} = .0494$
	$\delta_{.75} = .1017$	$\delta_{.75} = .0410$
	$\delta_{1} = .111$	$\delta_{1} = .034$

32. (*a*) $(.08)(1.1)^{10} = .2075,$ or 20.75%.

(*b*) $(.08)(.9)^{10} = .0279,$ or 2.79%.

(*c*) $(.08)(1.1)^5(.9)^5 = .0761,$ or 7.61%.

(*d*) A 10% increase followed by a 10% decrease results in a result that is $(1.1)(.9) = 99\%$ of the starting value.

(*e*) $\binom{10}{5}(.5)^{10} = .2461$ using the binomial distribution.

(*f*) 10 increases $(.08)(1.1)^{10} = .2075$

9 increases $(.08)(1.1)^{9}(.9) = .1698$

Probability $= \left[\binom{10}{10} + \binom{10}{9}\right](.5)^{10} = 11(.5)^{10} = .0107.$

33. One year spot rates s_1:

$$i_0 = .070000$$
$$i_1 = .070000e^{1.65(.1)} = .082558$$
$$i_2 = .082558e^{-.26(.1)} = .080439$$
$$i_3 = .080439e^{.73(.1)} = .086530$$
$$i_4 = .086530e^{1.17(.1)} = .097270$$
$$i_5 = .097270e^{.98(.1)} = .1073, \text{ or } 10.73\%.$$

Five year spot rates s_5:

$$i_0 = .080000$$
$$i_1 = .080000e^{1.65(.05)} = .086880$$
$$i_2 = .086880e^{-.26(.05)} = .085758$$
$$i_3 = .085758e^{.73(.05)} = .088946$$
$$i_4 = .088946e^{1.17(.05)} = .094304$$
$$i_5 = .094304e^{.98(.05)} = .0990, \text{ or } 9.90\%.$$

The yield curve became invested, since 10.73% > 9.90%.