

Rosen, Discrete Mathematics and Its Applications, 7th edition
Extra Examples
Section 1.4—Predicates and Quantifiers



— Page references correspond to locations of Extra Examples icons in the textbook.

p.38, icon at Example 3

#1. Let $P(x)$ be the statement

$$x^2 < x$$

where the universe for x is all real numbers.

- (a) Determine the truth value of $P(0)$.
- (b) Determine the truth value of $P(1/3)$.
- (c) Determine the truth value of $P(2)$.
- (d) Determine the set of all real numbers for which $P(x)$ is true.

Solution:

- (a) The proposition $P(0)$ states that $0^2 < 0$, which is false.
 - (b) The proposition $P(1/3)$ states that $(1/3)^2 < 1/3$, which is true.
 - (c) The proposition $P(2)$ states that $4 < 2$, which is false.
 - (d) If $x \geq 1$, then $x^2 \geq x$, so $P(x)$ is false. If $x \leq 0$, then $x^2 \geq 0$ and hence $x^2 \geq x$, so $P(x)$ is false. If $0 < x < 1$, then $x^2 < x$ is true (because this inequality can be rewritten as $x^2 - x < 0$, or $x(x - 1) < 0$, which is true because the product is negative — it is the product of a positive and a negative number). Therefore, $P(x)$ is true if and only if $0 < x < 1$.
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#2. Let $Q(x, y)$ be the statement

$$x + y = x - y$$

where the universe for x and y is the set of all real numbers. Determine the truth value of:

- (a) $Q(5, -2)$.
- (b) $Q(4.7, 0)$.
- (c) Determine the set of all pairs of numbers, x and y , such that $Q(x, y)$ is true.

Solution:

- (a) $Q(5, -2)$ says that $5 + (-2) = 5 - (-2)$, or $3 = 7$, which is false.
 - (b) $Q(4.7, 0)$ says that $4.7 + 0 = 4.7 - 0$, which is true.
 - (c) $x + y = x - y$ if and only if $x + 2y = x$, which is true if and only if $y = 0$. Therefore, x can be any real number and y must be zero.
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#3. Find all real numbers x and y such that $R(x, y)$ is true, where $R(x, y)$ is the predicate “ $xy = y$.”

Solution:

The given predicate can be rewritten as $xy - y = 0$, or $y(x - 1) = 0$. This is true if and only if either $y = 0$ or $x - 1 = 0$. That is, $R(x, y)$ is true if and only if $y = 0$ or $x = 1$.

p.41, icon at Example 8

#1. Suppose $P(x)$ is the predicate “ $x < |x|$.” Determine the truth value of $\forall x P(x)$, where the universe for x is:

(a) the three numbers $-3, -2, -1$.

(b) all real numbers.

Solution:

(a) $P(-3)$, $P(-2)$, and $P(-1)$ are all true because the numbers $-3, -2, -1$ are negative but their absolute values are positive. Therefore, $\forall x P(x)$ is true.

(b) The predicate $x < |x|$ is false for every nonnegative number. For example, $P(1)$ is false because $1 = |1|$. Having one value of x that makes the predicate false is enough to guarantee that $\forall x P(x)$ is false.

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#2. Find a universe for x such that $\forall x (x^2 < x)$ is true.

Solution:

We need to select numbers such that the square of the number is less than the number. We could take the universe for x to consist of any numbers greater than 0 but less than 1. For example, one such universe would consist of the single number $1/2$. Another universe is $\{1/5, 2/3, 9/11\}$. The largest universe we could choose is the entire interval $(0, 1)$.

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#1. Suppose $P(x)$ is the predicate “ $x < |x|$.” Determine the truth value of $\exists x P(x)$ where the universe for x is:

(a) the three numbers $1, 2, 3$.

(b) the six numbers $-2, -1, 0, 1, 2, 3$.

Solution:

(a) $P(1)$, $P(2)$, and $P(3)$ are all false because in each case $x = |x|$. Therefore, $\exists x P(x)$ is false for this universe.

(b) If we begin checking the six values of x , we find $P(-2)$ is true — it states that $-2 < |-2|$, or $-2 < 2$. We need check no further; having one case that makes the predicate true is enough to guarantee that $\exists x P(x)$ is

true.

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#2. Determine whether $\exists t (t^2 + 12 = 7t)$ is true, where the universe for t consists of all real numbers.

Solution:

The equation $t^2 + 12 = 7t$ can be rewritten as $t^2 - 7t + 12 = 0$, which factors as $(t - 3)(t - 4) = 0$. This is true for the numbers 3 and 4. Hence the given proposition is true.

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#3. Write the following statement in English, using the predicates

$$F(x): \text{“}x \text{ is a Freshman”}$$
$$T(x, y): \text{“}x \text{ is taking } y\text{”}$$

where x represents students and y represents courses:

$$\exists x (F(x) \wedge T(x, \text{Calculus 3})).$$

Solution:

The statement $\exists x (F(x) \wedge T(x, \text{Calculus 3}))$ says that there is a student x with two properties: x is a freshman and x is taking Calculus 3. In English, “Some Freshman is taking Calculus 3.”

p.47, icon at Example 20

#1. Negate “There is a person who walked on the moon.”

Solution:

We can always obtain the negation of a statement by placing the phrase “it is not the case that” in front of the statement. Thus, the negation is “It is not the case that there is a person who walked on the moon.” That is, “No person walked on the moon.” (Incidentally, the original statement is true because 12 astronauts walked on the moon between 1969 and 1972.)

We can also work with symbols. Let $W(x)$ be the statement “ x walked on the moon” where the universe for x consists of all people. The statement claims that there is an x such that x walked on the moon; that is, $\exists x W(x)$. The negation is $\neg \exists x W(x)$, which is equivalent to $\forall x \neg W(x)$. Therefore, the negative states that for every x that can be chosen, x did not walk on the moon. That is, “No person walked on the moon.”

Note: The negation is not “There is a person who did not walk on the moon.” You cannot have a statement and its negation both true, which would be the case if we took this statement as the negation of the original statement.

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#2. Negate “Everyone in the class has a laptop computer.”

Solution:

If we take $L(x)$ to be “ x has a laptop computer” where the universe for x consists of all people in this class, then the given statement is $\forall x L(x)$. The negation is $\neg\forall x L(x)$, which is equivalent to $\exists x \neg L(x)$. In words, “There is someone in this class who does not have a laptop computer.”

Note: The negation is not “No one in this class has a laptop computer.” You cannot have a statement and its negation both false, which would be the case in a class where at least one person has a laptop computer and at least one person does not have a laptop computer.

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#3. Negate “Some integer x is positive and all integers y are negative.”

Solution:

Using all integers as the universe for x and y , the statement is $\exists x (x > 0) \wedge \forall y (y < 0)$. The negation is

$$\begin{aligned} \neg[\exists x (x > 0) \wedge \forall y (y < 0)] &\equiv \neg\exists x (x > 0) \vee \neg\forall y (y < 0) && \text{De Morgan's law} \\ &\equiv \forall x \neg(x > 0) \vee \exists y \neg(y < 0) && \text{properties of negation} \\ &\equiv \forall x (x \leq 0) \vee \exists y (y \geq 0). \end{aligned}$$

Therefore, the negation is “Every integer x is nonpositive or there is an integer y that is nonnegative.”

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#4. Negate “There is a student who came late to class and there is a student who is absent from class.”

Solution:

In symbols, if $L(x)$ means “ x came late to class” and $A(x)$ means “ x is absent from class,” this statement can be written as

$$\exists x L(x) \wedge \exists y A(y).$$

Note that we must use a second variable y . By one of De Morgan’s laws the negation can be written as

$$\neg\exists x L(x) \vee \neg\exists y A(y),$$

which is equivalent to

$$\forall x \neg L(x) \vee \forall y \neg A(y).$$

In English this is “No student came late to class or no student is absent from class.”

p.48, icon at Example 23

#1. Write in symbols using predicates and quantifiers: “Everyone who visited France stayed in Paris.”

Solution:

The solution depends on the universe for the variable. If we take as the universe all people who visited France, we can write the proposition as $\forall x P(x)$, where $P(x)$ is the predicate “ x stayed in Paris.”

However, if we take all people as the universe, then we need to introduce a second predicate $F(x)$ for “ x visited France.” In this case, the proposition can be written as $\forall x (F(x) \rightarrow P(x))$.

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#2. Express this statement in symbols, using predicates and any needed quantifiers:

“Every freshman at the College is taking CS 101.”

Solution:

There are various ways to answer this question, depending on the universe.

If we take as our universe all freshmen at the College and use the predicate $T(x)$ to mean “ x is taking CS 101”, then the statement can be written as $\forall x T(x)$.

However, we may wish to use the universe of all students at the College, not only freshmen. In this case we introduce a second predicate, $F(x)$, to mean “ x is at the freshman level”.

We are making a conditional statement: “If the student is a freshman, then the student is taking CS 101;” that is,

$$\forall x (F(x) \rightarrow T(x)).$$

Note that we cannot say $\forall x (F(x) \wedge T(x))$, because this says that every student is a freshman, which is not something we can assume here.

We can enlarge the realm of discussion still further, by considering the universe of all people. In this case we introduce another predicate to restrict our attention to students at the College, $C(x)$, to mean “ x is enrolled at the College”.

Now we are saying that “If the person is enrolled at the College and is a freshman, then the person is taking CS 101,” or

$$\forall x (C(x) \wedge F(x) \rightarrow T(x)).$$

Now suppose that we also need to discuss other courses at the College. We need another variable, y , to denote courses and we need to change the predicate $T(x)$ to reflect the fact that we can be talking about other courses. We introduce $T(x, y)$ to mean “ x is taking course y ”. If we take as the universe all freshmen at the College, we have

$$\forall x T(x, \text{CS 101}).$$

However, if we take as universe all students at the College, the original statement becomes

$$\forall x (F(x) \rightarrow T(x, \text{CS 101})).$$

Finally, if we take as universe all people, the original statement becomes

$$\forall x (C(x) \wedge F(x) \rightarrow T(x, \text{CS 101})).$$

p.48, icon at Example 23

#3. Express this statement in symbols, using predicates and any needed quantifiers:

“Every freshman at the College is taking some Computer Science course.”

Solution:

There are many solutions, depending on the universes chosen for people and for courses.

If we take as our universe for people all freshmen at the College and our universe for courses all Computer Science courses, then we can use the predicate

$T(x, y)$: “ x is taking y ”

and hence the statement can be written as $\forall x \exists y T(x, y)$.

However, suppose we enlarge the universe of people to include all students at the College, not only freshmen. Also, suppose we enlarge the range of courses to include all courses offered at the College. In this case we need to introduce a predicate to restrict the students and a predicate to restrict the courses:

$F(x)$: “ x is a freshman” $C(y)$: “ y is a Computer Science course”.

We are making a conditional statement: “If the student is a freshman, then the student is taking some Computer Science course;” that is

$$\forall x (F(x) \rightarrow \exists y (C(y) \wedge T(x, y))),$$

which we can rewrite as

$$\forall x \exists y (F(x) \rightarrow C(y) \wedge T(x, y)),$$

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#4. Consider this sentence, which is the final sentence of 12th Amendment of U. S. Constitution: “No person constitutionally ineligible to the office of President shall be eligible to the office of Vice President of the United States.”

(a) Rewrite the sentence in English in the form “If . . . , then”

(b) Using the predicates $P(x)$: “ x is constitutionally eligible to the office of President” and $V(x)$: “ x is constitutionally eligible to the office of Vice President of the United States,” where the universe for x consists of all people, write the sentence using quantifiers and these predicates.

Solution:

(a) “If a person is constitutionally ineligible to the office of the President, then the person is ineligible to the office of Vice President of the United States.”

(b) $\forall x (\neg P(x) \rightarrow \neg V(x))$, or, equivalently, $\neg \exists x (\neg P(x) \wedge V(x))$. (You should check to see that the first statement can be rewritten as the second statement.)

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#5. Consider this sentence, which is Section 2 of Article I of the U. S. Constitution: “No person shall be a Representative who shall not have attained the age of twenty-five years, and been seven years a citizen

of the United States, and who shall not, when elected, be an inhabitant of that state in which he shall be chosen.”

(a) Rewrite the sentence in English in the form “If . . . , then . . .”.

(b) Using the predicates $A(x)$: “ x is at least twenty-five years old,” $C(x)$: “ x has been a citizen of the United States for at least seven years,” $I(x)$: “ x , when elected, is an inhabitant of the state in which he is chosen,” and $R(x)$: “ x can be a Representative,” where the universe for x in all four predicates consists of all people, rewrite the sentence using quantifiers and these predicates. [Note: At the time at which the U. S. Constitution was ratified, the universe for x consisted of landowning males.]

Solution:

(a) The sentence from the Constitution has the form “If the person has not attained . . . , then the person shall not be a Representative.” That is, “If a person shall not have attained the age of twenty-five years or shall not be a citizen of the U. S. for seven years or shall not, when elected, be an inhabitant of that state in which he shall be chosen, then the person shall not be a Representative.”

(b) The sentence states that if the person fails to meet one or more of the conditions $A(x)$, $C(x)$, or $I(x)$, then the person fails to meet $R(x)$. That is,

$$\forall x ((\neg A(x) \vee \neg C(x) \vee \neg I(x)) \rightarrow \neg R(x)).$$

p.50, icon at Example 25

#1. Express the specification “Whenever at least one network link is operating, a 10 megabyte file can be transmitted” using predicates and quantifiers.

Solution:

We begin by determining predicates and variables that can be used to express this specification. We begin by translating “at least one network link is operating.” Suppose we are concerned with all network links and the possible states in which they can be. We can use the predicate $S(x, y)$ to mean “network link x is in state y ”. The statement “at least one network link is operating” can be expressed using the existential quantifier as

$$\exists x S(x, \text{operating}).$$

Note that if we used a predicate with a single variable, such as $O(x)$, to mean “link x is operating,” we would need other predicates to describe other possible states that the links could be in. However, if we did not care about this, “at least one network link is operating” could be expressed as $\exists x O(x)$.

Because we are also concerned with the size of a file that can be transmitted, we let

$$T(y) \text{ mean “a file of } y \text{ megabytes can be transmitted,”}$$

where the universe for y is the set of all nonnegative numbers.

It follows that $T(10)$ denotes that a file of 10 megabytes can be transmitted. Because “whenever p , q ” is a way of expressing the conditional statement $p \rightarrow q$, it follows that the specification can be expressed as

$$\exists x S(x, \text{operating}) \rightarrow T(10).$$