



— Page references correspond to locations of Extra Examples icons in the textbook.

**p.206, icon at Example 1**

#1. Give a big- $O$  estimate for each of these functions. Use a simple function in the big- $O$  estimate.

(a)  $3n + n^3 + 4$ .

(b)  $1 + 2 + 3 + \cdots + n$ .

(c)  $\log_{10}(2^n) + 10^{10}n^2$ .

**Solution:**

(a)  $3n + n^3 + 4 \leq 3n^3 + n^3 + 4n^3 = 8n^3$  for  $n > 1$ . Therefore  $3n + n^3 + 4$  is  $O(n^3)$ . (It is also  $O(n^4)$ ,  $O(n^5)$ , etc.)

(b) We have  $1 + 2 + 3 + \cdots + n \leq n + n + n + \cdots + n = n \cdot n$ . Therefore  $1 + 2 + 3 + \cdots + n$  is  $O(n^2)$ . (It is also  $O(n^3)$ ,  $O(n^4)$ , etc.)

(c)  $\log_{10}(2^n) + 10^{10}n^2 = n \log_{10} 2 + 10^{10}n^2 \leq (\log_{10} 2 + 10^{10})n^2$  if  $n \geq 1$ . But  $\log_{10} 2 + 10^{10}$  is a constant. Therefore  $\log_{10}(2^n) + 10^{10}n^2$  is  $O(n^2)$ . (It is also  $O(n^3)$ ,  $O(n^4)$ , etc.)

**p.206, icon at Example 1**

#2. Use the definition of big- $O$  to prove that  $5x^4 - 37x^3 + 13x - 4 = O(x^4)$

**Solution:**

We must find integers  $C$  and  $k$  such that

$$5x^4 - 37x^3 + 13x - 4 \leq C|x^4|$$

for all  $x \geq k$ . We can proceed as follows:

$$|5x^4 - 37x^3 + 13x - 4| \leq |5x^4 + 37x^3 + 13x + 4| \leq |5x^4 + 37x^4 + 13x^4 + 4x^4| = 59|x^4|,$$

where the first inequality is satisfied if  $x \geq 0$  and the second inequality is satisfied if  $x \geq 1$ . Therefore

$$|5x^4 - 37x^3 + 13x - 4| \leq 59|x^4|$$

if  $x \geq 1$ , so we have  $C = 59$  and  $k = 1$ .

Note that the solution we have given is by no means the only possible one. Here is a second solution. It makes the value  $C$  smaller, but requires us to make the value  $k$  larger:

$$|5x^4 - 37x^3 + 13x - 4| \leq |5x^4 + 37x^3 + 13x + 4| \leq |5x^4 + 4x^4 + x^4 + x^4| = 11|x^4|$$

In the first inequality we changed from subtraction to addition of two terms (which is valid if  $x \geq 0$ ). In the second inequality we replaced the term  $37x^3$  by  $4x^4$  (which is valid if  $x \geq 10$ ), replaced  $13x$  by  $x^4$  (which is valid if  $x \geq 3$ ) and replaced  $4$  by  $x^4$  (which is valid if  $x \geq 2$ ). Therefore,



