

Rosen, Discrete Mathematics and Its Applications, 7th edition
Extra Examples
Section 9.5—Equivalence Relations



— Page references correspond to locations of Extra Examples icons in the textbook.

p.609, icon at Example 2

#1. (a) Verify that the following is an equivalence relation on the set of real numbers:

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{ where } \lfloor x \rfloor \text{ is the floor of } x.$$

(b) Describe the equivalence classes arising from the equivalence relation in part (a).

Solution:

(a) R is reflexive: $\lfloor a \rfloor = \lfloor a \rfloor$ is true for all real numbers.

R is symmetric: suppose $\lfloor a \rfloor = \lfloor b \rfloor$; then $\lfloor b \rfloor = \lfloor a \rfloor$.

R is transitive: suppose $\lfloor a \rfloor = \lfloor b \rfloor$ and $\lfloor b \rfloor = \lfloor c \rfloor$; from transitivity of equality of real numbers, it follows that $\lfloor a \rfloor = \lfloor c \rfloor$.

(b) Two real numbers, a and b , are related if they have the same floor. This happens if and only if a and b lie in the same interval $[n, n + 1)$ where n is an integer. That is, the equivalence classes are the intervals $\dots, [-2, -1), [-1, 0), [0, 1), [1, 2), [2, 3), \dots$

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#2. Let A be the set of all points in the plane with the origin removed. That is,

$$A = \{(x, y) \mid x, y \in \mathbf{R}\} - \{(0, 0)\}.$$

Define a relation on A by the rule:

$$(a, b)R(c, d) \leftrightarrow (a, b) \text{ and } (c, d) \text{ lie on the same line through the origin.}$$

(a) Prove that R is an equivalence relation.

(b) Describe the equivalence classes arising from the equivalence relation R in part (a).

(c) If A is replaced by the entire plane, is R an equivalence relation?

Solution:

(a) R is reflexive: (a, b) and (a, b) lie on the same line through the origin, namely on the line $y = bx/a$ (if $a \neq 0$), or else on the line $x = 0$ (if $a = 0$).

R is symmetric: if (a, b) and (c, d) lie on the same line through the origin, then (c, d) and (a, b) lie on the same line through the origin.

R is transitive: suppose (a, b) and (c, d) lie on the same line L through the origin and (c, d) and (e, f) lie on the same line M through the origin. Then L and M both contain the two distinct points $(0, 0)$ and (c, d) . Therefore L and M are the same line, and this line contains (a, b) and (e, f) . Therefore (a, b) and (e, f) lie on the same line through the origin.

Note: The proof that R is an equivalence relation can be carried out using analytic geometry: if (a, b) and (c, d) lie on the same nonvertical line through the origin, then the slope must equal b/a because the line passes through $(0, 0)$ and (a, b) and the slope must also equal d/c because the line passes through $(0, 0)$ and

(c, d) ; thus, $b/a = d/c$, or $ad = bc$. If (a, b) and (c, d) lie on the same vertical line through the origin, then the points must have the form $(0, b)$ and $(0, d)$, and again it must happen that $ad = bc$. Therefore, $(a, b)R(c, d)$ means that $ad = bc$. This equation can be used to verify that R is reflexive, symmetric, and transitive.

(b) Each equivalence class is the set of points of A on a line of the form $y = mx$ or the vertical line $x = 0$.

(c) If A is replaced by the entire plane, R is not an equivalence relation. It fails to satisfy the transitive property; for example, $(1, 2)R(0, 0)$ and $(0, 0)R(2, 2)$, but $(1, 2) \not R(2, 2)$ because the line passing through $(1, 2)$ and $(2, 2)$ does not pass through the origin.
