APPENDIX B

Fourier Series

The Fourier series is used to represent a periodic function as a sum of sines, cosines, and a constant. When used with the principle of superposition, which lets you obtain the total response as the sum of the individual responses, the Fourier series enables you to obtain the response of a linear system to any periodic function. All that is needed is the system response to a sine, a cosine, and a constant input. Although the Fourier series is an infinite series, in practice it can be truncated to a small number of terms whose frequencies lie within the bandwidth of the system.

If a function $f(t)$ is periodic with period P, then $f(t+P) = f(t)$. The Fourier series for this function defined on the interval $t_1 \leq t \leq t_1 + P$, where t_1 and P are constants and $P > 0$, is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{P} + b_n \sin \frac{2n\pi t}{P} \right)$$

where

$$a_n = \frac{2}{P} \int_{t_1}^{t_1+P} f(t) \cos \frac{2n\pi t}{P} \, dt \qquad n = 0, 1, 2, \dots$$

$$b_n = \frac{2}{P} \int_{t_1}^{t_1+P} f(t) \sin \frac{2n\pi t}{P} \, dt \qquad n = 1, 2, 3, \dots$$

If $f(t)$ is defined outside the specified interval $[t_1, t_1 + P]$ by a periodic extension of period P, and if $f(t)$ and df/dt are piecewise continuous, then the Fourier series converges to $f(t)$ if t is a point of continuity, and to the average value $[f(t_+) + f(t_-)]/2$ otherwise.

As an example, consider the train of unit pulses of width π and alternating in sign, as shown in Figure B.1. The function is described by

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

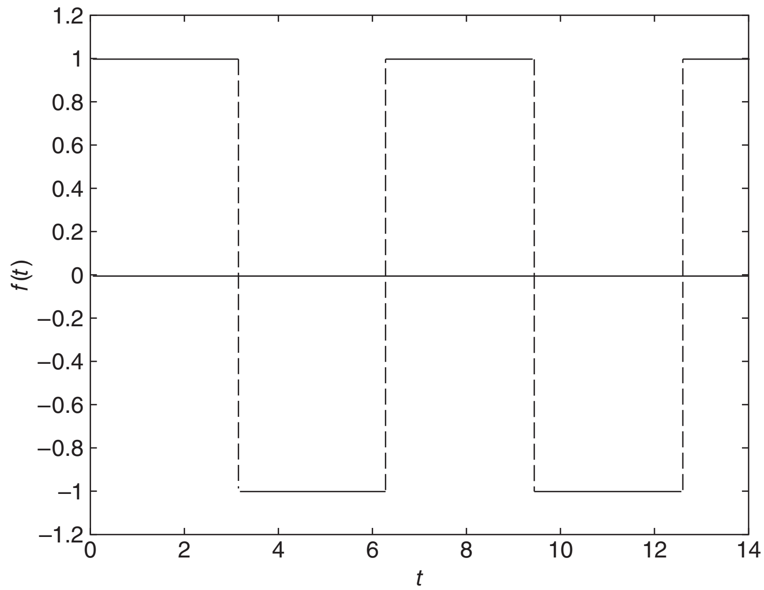
The period is $P = 2\pi$, and we can take the constant t_1 to be 0. Using a table of integrals, we find that

$$a_n = 0 \qquad \text{for all } n$$

$$b_n = \frac{4}{n\pi} \qquad \text{for } n \text{ odd}$$

$$b_n = 0 \qquad \text{for } n \text{ even}$$

Figure B.1 The function used for the Fourier series example.



The Fourier series is

$$f(t) = \frac{4}{\pi}\left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$$

In general, the constant term a_0 and the cosine terms will not appear in the series if the function is odd; that is, if $f(-t) = -f(t)$. If the function is even, then $f(-t) = f(t)$, and no sine terms will appear in the series.