Additional Information - Shape Factors in Materials Selection

The discussion in Sec. 11.8 focused attention on the last term in Eq. (11.5), $P = f_1(F) \times f_2(G) \times f_3(M)$, as a key factor for material selection to enhance the performance metric *P*. However, since all three terms in Eq. (11.5) are multiplied to determine *P,* it is instructive to see how changes in the geometry affect the performance metric. For example, we know from mechanics of materials that much greater stiffness can be achieved in a beam if it is in the shape of an I-section compared with a square section of equal weight. The following discussion builds on this idea to develop the more general concept of shape factor in design and in the selection of materials. The shape factor indicates how much more load or torque can be carried by the structural shape than by the same mass of material with a square cross section.

Shaped sections carry loads in bending, torsion, or axial compression much more efficiently than solid sections do. A shaped section is a body whose cross-sectional area has been formed into a shape like a tube, I-beam, or box section. More complex shapes are sandwich and corrugated structures. An efficient structural shape is one that requires much less material than other shapes for a given load. Thus, a design with a high shape factor is less costly since less material is used.

We have talked exclusively about properties when referring to materials. But when we think about using a material to make a component or structure then we also need to define its shape. The component's shape factor, ϕ, is a measure of the efficiency of material usage.

Machine elements, for simplicity, can be divided into beams, shafts, and axial elements.

- Beams carry bending moments and are designed for strength and stiffness.
- Shafts resist torques and are designed for strength and resistance to twist.
- Columns carry axial loads and must resist buckling.

Each is used to withstand different kinds of deformations and loads. In the following discussion, we describe how to find the shape factor ϕ, first in terms of failure by excessive deformation (lack of stiffness) and then by failure in terms of strength limitations.

Elastic Bending of Beams

The bending stiffness *S* of a beam is proportional to *EI*, where *E* is Young's modulus and *I* is the second moment of area of the beam about its bending axis.

$$I_{xx} = \int_{section} y^2 dA \qquad (1)$$

where *y* is measured normal to the x-axis (bending axis) and *dA* is the differential element of area at *y*. The shape factor [1] is defined with reference to a square cross section (indicated by subscript *o*). $I_o = b_o^4/12 = A^2/12$, where b_o is the length of an edge. The shape factor for elastic bending differs from that of a solid section by

1. M.F. Ashby, *op. cit.,* Chap. 11.

$$\phi_B^e = \frac{S}{S_o} = \frac{EI}{EI_o} = \frac{12I}{A^2} \tag{2}$$

Equation (2) is dimensionless. *It depends only on the shape, not on the scale.* The symbol for shape factor in Eq. (2) has special meaning. The subscript B denotes it was determined in bending, while the superscript e indicates that it is based on elastic deformation.

EXAMPLE

Determine the shape factor for a tube of circular cross section, with outside radius r_o equal to 40 mm and the inner radius r_i equal to 36 mm. The area A and moment of inertia I are given by

$$A = \pi\left(r_o^{\,2} - r_i^{\,2}\right) \simeq 2\pi r_o t \text{ where } t = r_o - r_i \quad \textit{and} \quad I = \frac{\pi}{4}\left(r_o^{\,4} - r_i^{\,4}\right)\pi r_o^{\,3} t$$

$$\text{Substituting in Eq. (2)} \qquad \phi_B^e = \frac{12\pi r_0^3 t}{4\pi^2 r_0^2 t^2} = \frac{3r_0}{\pi t} \tag{3}$$

Substituting values into (3) gives $\phi_B^e = 9.5$, meaning that the tubular section has 9.5 times the resistance to deflection in elastic bending (stiffness) of a square section with the same area.

A designer can use this information in one of two ways. She can take pride in designing a much stiffer structure than if routine design practice was followed, or more likely, she will find out how much the size of the beam, and hence the cost, can be reduced to still give the same stiffness as the beam with a square section. The tubular beam has an area of 1005 mm^2, which is equivalent to a square beam 31.7 mm on a side. Remembering that for the same material, the stiffness is directly proportional to the moment of inertia, we find that $I = 8.415 \times 10^4$ mm^4. A tubular beam with the same value of I and $t = 4$ mm has an outside radius of 18.8 mm. This results in about a 30% saving in material cost, assuming that the tubular shape cost is approximately twice the cost of the rolled square bar.

Failure in Bending a Beam

Failure in bending a ductile material starts when the stress at some point exceeds the yield strength of the material. Failure in a brittle material occurs when the stress reaches the fracture strength. In both instances, plastic yielding or actual fracture, the engineering usefulness of the component is reached and the condition constitutes a failure. Following the nomenclature by Ashby, this stress is denoted σ_f.

The maximum stress in bending a beam is given by

$$\sigma = \frac{M_b y_{\max}}{I} = \frac{M_b}{Z} \tag{4}$$

where M_b is the bending moment, and $Y_{\max}$ is the point on the beam surface furthest from the neutral axis. $Z = I/Y_{\max}$ is defined as the *section modulus*. From Eq. (4) we see that for a given material the strength of the beam is determined by the section modulus. As with beam stiffness, the efficiency of the strength of a section is measured by the ratio Z/Z_o, where Z_o is the section modulus of a square beam with the same cross-sectional area A.

$$Z_0 = \frac{I_0}{y_{max}} = \frac{b_0^4}{12} \frac{1}{b_0/2} = \frac{1}{6} b_0^3 = \frac{A^{3/2}}{6} \tag{5}$$

$$\text{Therefore,} \quad \phi_B^f = \frac{Z}{Z_0} = \frac{6Z}{A^{3/2}} \tag{6}$$

Equation (6) was constructed so that the shape factor of a square beam will be one, regardless of its dimensions. A beam with a shape factor of 10 is 10 times stronger in bending than a solid square section of the same weight and material.

Twisting of Shafts

Shafts are common machine elements that must resist torque. The stiffness of a shaft is given by the torque, T, divided by the angle of twist, θ.

$$S_T = \frac{T}{\theta} = \frac{GK}{L} \tag{7}$$

where G is the shear modulus[2], L is the length of the shaft, and K is a torsional moment of area. $K = J$, the polar moment of inertia, for circular cross sections. For noncircular sections K is less than J. From Eq. (7) and in analogy to Eq. (2), the shape factor for elastic stiffness in torsion can be written

$$\phi_T^e = \frac{S_T}{S_{To}} = \frac{K}{K_o} \tag{8}$$

K_o for a solid square section is $K_o = 0.14A^2$. Substituting into Eq. (8) gives

$$\phi_T^e = \frac{K}{0.14A^2} = \frac{7.14K}{A^2} \tag{9}$$

Relationships for finding K for common shapes are given by Ashby.[3]

Failure in Twisting a Shaft

Finding a shape factor to relate to failure in twisting is more complicated than in bending a beam. For circular bars or tubes subjected to a torque T, the shear stress is a maximum at the outer radius according to

$$\tau = \frac{Tr_{max}}{J} \tag{10}$$

where J is the polar moment of inertia. The quantity J/r_{max} is directly analogous to the term Z in Eq. (4). For noncircular sections, Eq. (10) is written as

2. $G = \frac{E}{2(1+\upsilon)}$ for an elastic material, and υ is Poisson's ratio.

3. M. F. Ashby, op. cit, pp. 288–89.

$$\tau = \frac{T}{Q} \tag{11}$$

where Q has units of m^3. Following the method used with developing the other shape factors,

$$\phi_T^f = \frac{Q}{Q_0} = 4.8\frac{Q}{A^{3/2}} \tag{12}$$

where the number 4.8 comes from the relation between Q and A. Shafts with solid symmetric sections all have values of shape factor $\phi_T^f \pi$ close to 1. Values of Q and ϕ_T^f can be found in Ashby.

Buckling of an Axial Loaded Member

The compressive load P to cause an axially loaded member like a column to buckle is given by the Euler equation

$$P_{cr} = \frac{\pi^2 E I_{\min}}{L_0^2} \tag{13}$$

where $I_{\min}$ is the smallest value of moment of inertia around any of the axes in the column and L_e is the equivalent length of the column. It is equal to the actual length of the column L if its ends are hinged, but for more severe end constraints it can be less than L.[4]

The shape factor for elastic buckling in axial loading is the same as for elastic bending, Eq. (2), where I, the largest moment of inertia for the cross section is replaced with $I_{\min}$, the moment of inertia taken with respect to the axis that gives the smallest value.

Material Performance Index with Shape Factor Included

Having developed the shape factor for bending and torsion, we now show how the shape factor can be included in the material performance index.

Equation (6) shows that the shape factor for yielding of a beam in bending is

$$\phi_B^f = \frac{Z}{Z_0} = \frac{6Z}{A^{3/2}} \quad \text{Also, } \sigma_f = \frac{M_b}{Z} \text{ and } Z = \frac{M_b}{\sigma_f}$$

so that

$$\phi_B^f = \frac{6M_b}{\sigma_f A^{3/2}} \tag{14}$$

If the objective is to find a material performance index to select a material for a beam with minimum mass and maximum resistance to yielding,

4. F. P. Beer, E. R. Johnston, and J. T. DeWolf, *Mechanics of Materials,* 4th ed., Chap. 10, McGraw-Hill, New York, 2006.

$$m = AL\rho. \text{ Solving Eq.(14) for } \quad A \text{ gives} A = \left(\frac{6M_b}{\phi_B^f \sigma_f} \right)^{2/3}$$

$$\text{Therefore, } m = (6M_b)^{2/3} (L) \left(\frac{\rho^{3/2}}{\phi_B^f \sigma_f} \right)^{2/3} \tag{15}$$

In agreement with Eq.(11.5) in the textbook, the first term in Eq. (15) expresses the function, the second the geometry, and the last term is the material index. The material performance index is formed by inverting this term so that the best material and shape combination is the greatest value of the index M.

$$M = \frac{(\phi_B^f \sigma_f)^{2/3}}{\rho} \tag{16}$$

By referring to Table 11.5 in the textbook, note that the formulation of M is identical to that given without considering shape factor, but now the shape factor has been simply included as a multiplier in the numerator. The same will be true for values of M for stiffness, and those involving torques instead of bending moments.

There are two practical limits to achieving highly efficient structural shapes. The first is the difficulty and expense of manufacturing complex shapes or box-like shapes with high width-to-thickness ratios. Then, when efficient shapes are achieved, the limits are set by the competition between different failure modes. Often a shape is chosen that raises the load to cause yielding only to find that failure by buckling ensues before reaching the yielding load.