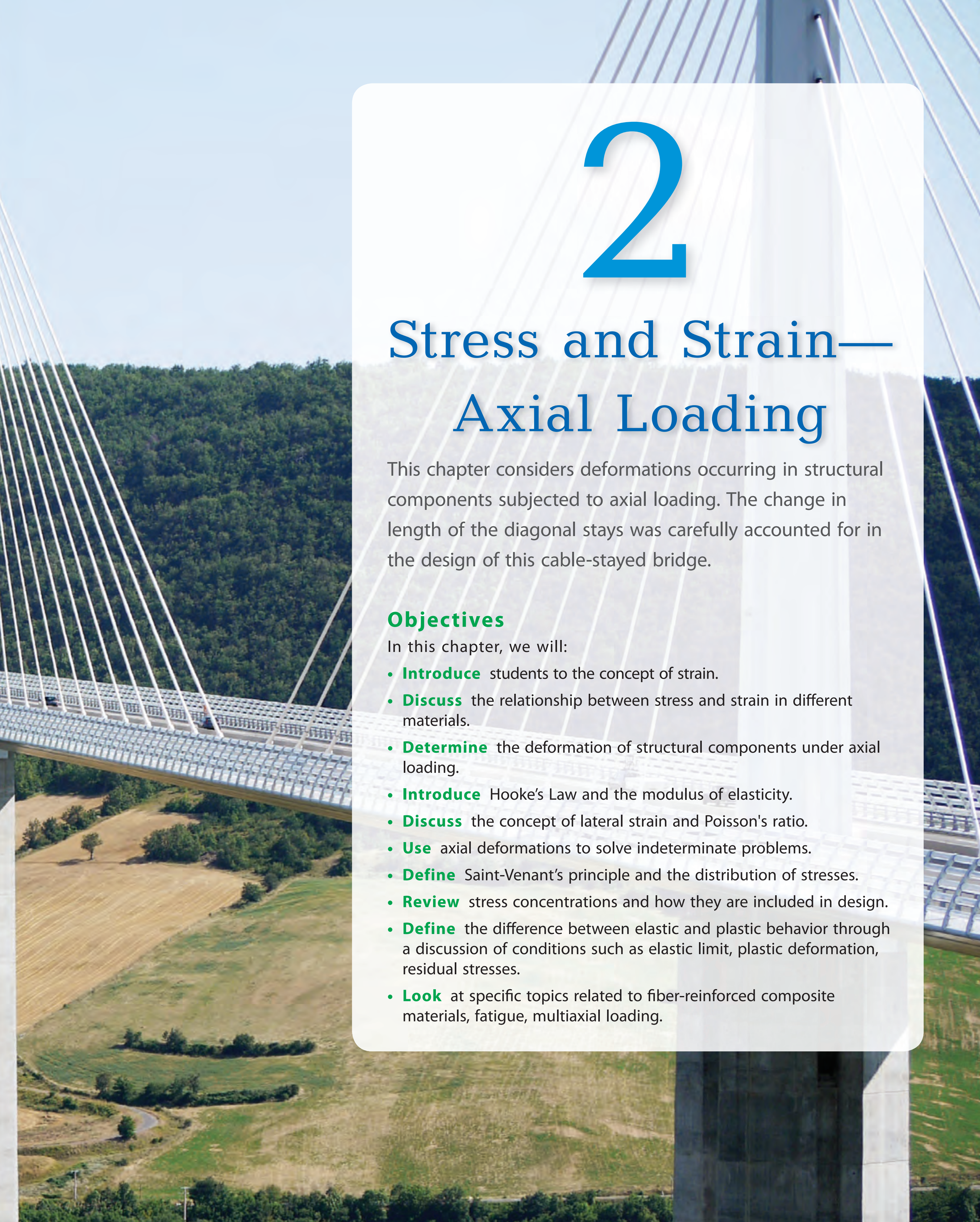


2

Stress and Strain—Axial Loading

This chapter considers deformations occurring in structural components subjected to axial loading. The change in length of the diagonal stays was carefully accounted for in the design of this cable-stayed bridge.

Objectives

In this chapter, we will:

- **Introduce** students to the concept of strain.
- **Discuss** the relationship between stress and strain in different materials.
- **Determine** the deformation of structural components under axial loading.
- **Introduce** Hooke's Law and the modulus of elasticity.
- **Discuss** the concept of lateral strain and Poisson's ratio.
- **Use** axial deformations to solve indeterminate problems.
- **Define** Saint-Venant's principle and the distribution of stresses.
- **Review** stress concentrations and how they are included in design.
- **Define** the difference between elastic and plastic behavior through a discussion of conditions such as elastic limit, plastic deformation, residual stresses.
- **Look** at specific topics related to fiber-reinforced composite materials, fatigue, multiaxial loading.

Introduction

An important aspect of the analysis and design of structures relates to the *deformations* caused by the loads applied to a structure. It is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it was intended. But the analysis of deformations also helps us to determine stresses. Indeed, it is not always possible to determine the forces in the members of a structure by applying only the principles of statics. This is because statics is based on the assumption of undeformable, rigid structures. By considering engineering structures as *deformable* and analyzing the deformations in their various members, it will be possible for us to compute forces that are *statically indeterminate.* The distribution of stresses in a given member is statically indeterminate, even when the force in that member is known.

In this chapter, you will consider the deformations of a structural member such as a rod, bar, or plate under *axial loading*. First, the *normal strain* ϵ in a member is defined as the *deformation of the member per unit length*. Plotting the stress σ versus the strain ε as the load applied to the member is increased produces a *stress-strain diagram* for the material used. From this diagram, some important properties of the material, such as its *modulus of elasticity*, and whether the material is *ductile* or *brittle* can be determined. While the behavior of most materials is independent of the direction of the load application, you will see that the response of fiber-reinforced composite materials depends upon the direction of the load.

From the stress-strain diagram, you also can determine whether the strains in the specimen will disappear after the load has been removed—when the material is said to behave *elastically*—or whether *a permanent set* or *plastic deformation* will result.

You will examine the phenomenon of *fatigue*, which causes structural or machine components to fail after a very large number of repeated loadings, even though the stresses remain in the elastic range.

Sections 2.2 and 2.3 discuss *statically indeterminate problems* in which the reactions and the internal forces *cannot* be determined from statics alone. Here the equilibrium equations derived from the free-body diagram of the member must be complemented by relationships involving deformations that are obtained from the geometry of the problem.

Additional constants associated with isotropic materials—i.e., materials with mechanical characteristics independent of direction—are introduced in Secs. 2.4 through 2.8. They include *Poisson's ratio*, relating lateral and axial strain, the *bulk modulus*, characterizing the change in volume of a material under hydrostatic pressure, and the *modulus of rigidity*, concerning the components of the shearing stress and shearing strain. Stress-strain relationships for an isotropic material under a multiaxial loading also are determined.

Stress-strain relationships involving modulus of elasticity, Poisson's ratio, and the modulus of rigidity are developed for fiber-reinforced composite materials under a multiaxial loading. While these materials are not isotropic, they usually display special *orthotropic* properties.

In Chap. 1, stresses were assumed uniformly distributed in any given cross section; they were also assumed to remain within the elastic range. The first assumption is discussed in Sec. 2.10, while *stress concentrations* near circular holes and fillets in flat bars are considered in Sec. 2.11.

Sections 2.12 and 2.13 discuss stresses and deformations in members made of a ductile material when the yield point of the material is exceeded, resulting in permanent *plastic deformations* and *residual stresses*.

2.1 AN INTRODUCTION TO STRESS AND STRAIN

2.1A Normal Strain Under Axial Loading

Consider a rod BC of length L and uniform cross-sectional area A, which is suspended from B (Fig. 2.1*a*). If you apply a load **P** to end C, the rod elongates (Fig. 2.1*b*). Plotting the magnitude P of the load against the deformation δ (Greek letter delta), you obtain a load-deformation diagram (Fig. 2.2). While this diagram contains information useful to the analysis of the rod under consideration, it cannot be used to predict the deformation of a rod of the same material but with different dimensions. Indeed, if a deformation δ is produced in rod BC by a load **P**, a load 2**P** is required to cause the same deformation in rod $B'C'$ of the same length L but cross-sectional area $2A$ (Fig. 2.3). Note that in both cases the value of the stress is the same: $\sigma = P/A$. On the other hand, when load **P** is applied to a rod $B''C''$ of the same cross-sectional area A but of length $2L$, a deformation 2δ occurs in that rod (Fig. 2.4). This is a deformation twice as large as the deformation δ produced in rod BC. In both cases, the ratio of the deformation over the length of the rod is the same at δ/L. This introduces the concept of *strain*. We define the *normal strain* in a rod under axial loading as the *deformation per unit length* of that rod. The normal strain, ϵ (Greek letter epsilon), is

$$\epsilon = \frac{\delta}{L} \tag{2.1}$$

Plotting the stress $\sigma = P/A$ against the strain $\epsilon = \delta/L$ results in a curve that is characteristic of the properties of the material but does not depend upon the dimensions of the specimen used. This curve is called a *stress-strain diagram*.

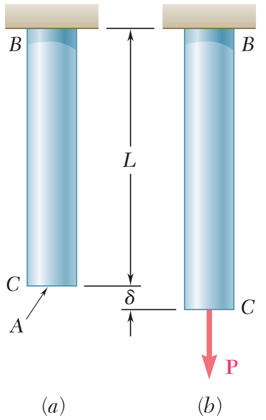


Fig. 2.1 Undeformed and deformed axially-loaded rod.

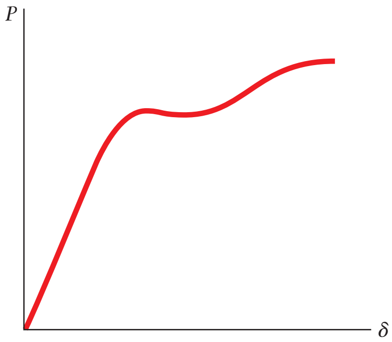


Fig. 2.2 Load-deformation diagram.

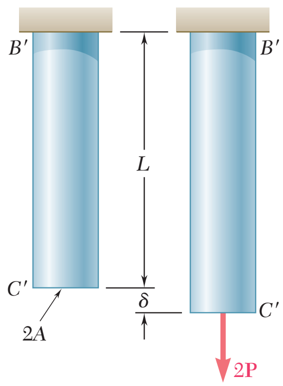


Fig. 2.3 Twice the load is required to obtain the same deformation δ when the cross-sectional area is doubled.

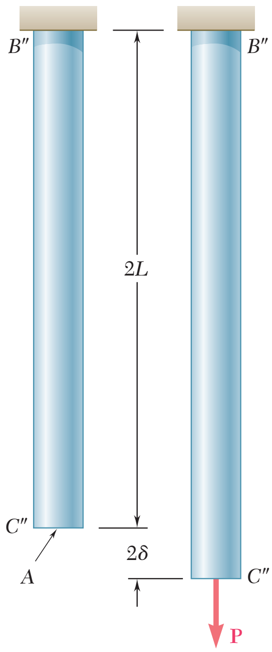


Fig. 2.4 The deformation is doubled when the rod length is doubled while keeping the load **P** and cross-sectional area A the same.

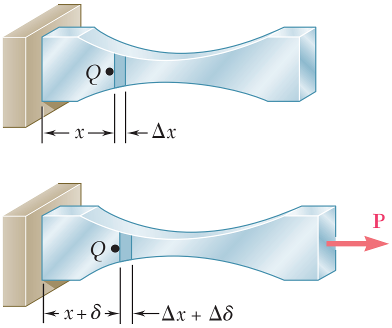


Fig. 2.5 Deformation of axially-loaded member of variable cross-sectional area.

Since rod BC in Fig. 2.1 has a uniform cross section of area A, the normal stress σ is assumed to have a constant value P/A throughout the rod. The strain ε is the ratio of the total deformation δ over the total length L of the rod. It too is consistent throughout the rod. However, for a member of variable cross-sectional area A, the normal stress $\sigma = P/A$ varies along the member, and it is necessary to define the strain at a given point Q by considering a small element of undeformed length Δx (Fig. 2.5). Denoting the deformation of the element under the given loading by $\Delta\delta$, the *normal strain at point Q* is defined as

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \quad \textbf{(2.2)}$$

Since deformation and length are expressed in the same units, the normal strain ϵ obtained by dividing δ by L (or $d\delta$ by dx) is a *dimensionless quantity*. Thus, the same value is obtained for the normal strain, whether SI metric units or U.S. customary units are used. For instance, consider a bar of length $L = 0.600$ m and uniform cross section that undergoes a deformation $\delta = 150 \times 10^{-6}$ m. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{150 \times 10^{-6}\,\text{m}}{0.600\,\text{m}} = 250 \times 10^{-6}\,\text{m/m} = 250 \times 10^{-6}$$

Note that the deformation also can be expressed in micrometers: $\delta = 150\,\mu\text{m}$ and the answer written in micros (μ):

$$\epsilon = \frac{\delta}{L} = \frac{150\,\mu\text{m}}{0.600\,\text{m}} = 250\,\mu\text{m/m} = 250\,\mu$$

When U.S. customary units are used, the length and deformation of the same bar are $L = 23.6$ in. and $\delta = 5.91 \times 10^{-3}$ in. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{5.91 \times 10^{-3}\,\text{in.}}{23.6\,\text{in.}} = 250 \times 10^{-6}\,\text{in./in.}$$

which is the same value found using SI units. However, when lengths and deformations are expressed in inches or microinches (μin.), keep the original units obtained for the strain. Thus, in the previous example, the strain would be recorded as either $\epsilon = 250 \times 10^{-6}$ in./in. or $\epsilon = 250\,\mu$in./in.

2.1B Stress-Strain Diagram

Tensile Test. To obtain the stress-strain diagram of a material, a *tensile test* is conducted on a specimen of the material. One type of specimen is shown in Photo 2.1. The cross-sectional area of the cylindrical central portion of the specimen is accurately determined and two gage marks are inscribed on that portion at a distance L_0 from each other. The distance L_0 is known as the *gage length* of the specimen.

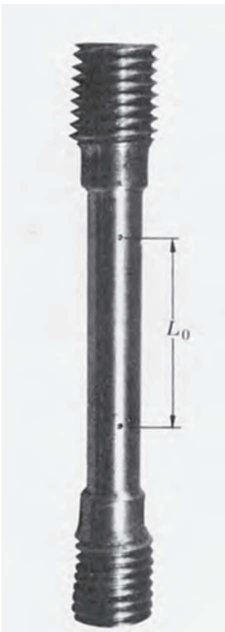


Photo 2.1 Typical tensile-test specimen. Undeformed gage length is L_0.

The test specimen is then placed in a testing machine (Photo 2.2), which is used to apply a centric load **P**. As load **P** increases, the distance L between the two gage marks also increases (Photo 2.3). The distance L is measured with a dial gage, and the elongation $\delta = L - L_0$ is recorded for each value of P. A second dial gage is often used simultaneously to measure and record the change in diameter of the specimen. From each pair of readings P and δ, the engineering stress σ is

$$\sigma = \frac{P}{A_0} \tag{2.3}$$

and the engineering strain ε is

$$\epsilon = \frac{\delta}{L_0} \tag{2.4}$$

The stress-strain diagram can be obtained by plotting ε as an abscissa and σ as an ordinate.

Stress-strain diagrams of materials vary widely, and different tensile tests conducted on the same material may yield different results, depending upon the temperature of the specimen and the speed of loading. However, some common characteristics can be distinguished from stress-strain diagrams to divide materials into two broad categories: *ductile* and *brittle* materials.

Ductile materials, including structural steel and many alloys of other materials are characterized by their ability to *yield* at normal temperatures. As the specimen is subjected to an increasing load, its length first increases linearly with the load and at a very slow rate. Thus, the initial portion of the stress-strain diagram is a straight line with a steep slope

Photo 2.2 Universal test machine used to test tensile specimens.

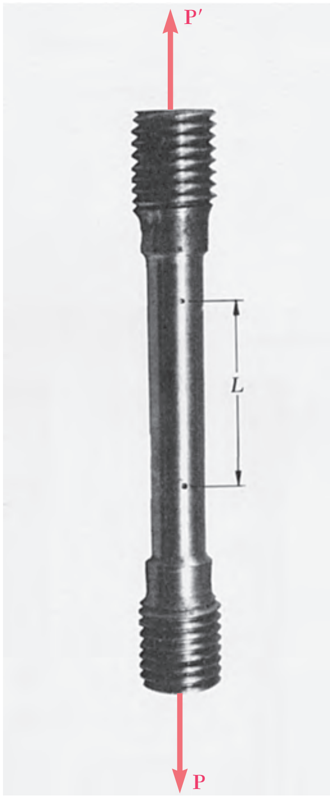


Photo 2.3 Elongated tensile test specimen having load P and deformed length $L > L_0$.

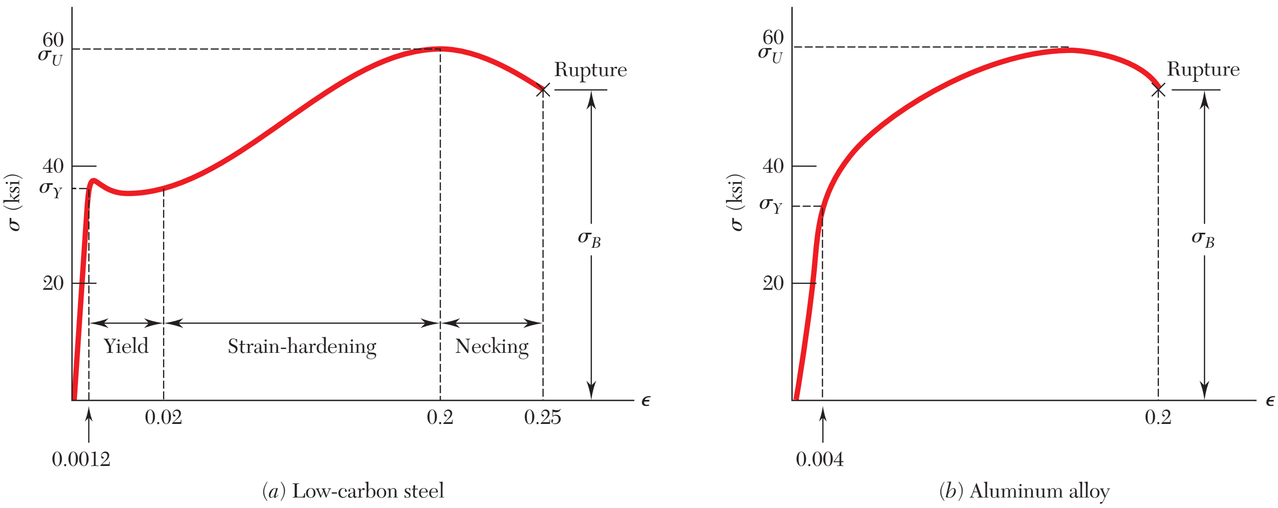


Fig. 2.6 Stress-strain diagrams of two typical ductile materials.

(Fig. 2.6). However, after a critical value σ_Y of the stress has been reached, the specimen undergoes a large deformation with a relatively small increase in the applied load. This deformation is caused by slippage along oblique surfaces and is due primarily to shearing stresses. After a maximum value of the load has been reached, the diameter of a portion of the specimen begins to decrease, due to local instability (Photo 2.4*a*). This phenomenon is known as *necking*. After necking has begun, lower loads are sufficient for specimen to elongate further, until it finally ruptures (Photo 2.4*b*). Note that rupture occurs along a cone-shaped surface that forms an angle of approximately 45° with the original surface of the specimen. This indicates that shear is primarily responsible for the failure of ductile materials, confirming the fact that shearing stresses under an axial load are largest on surfaces forming an angle of 45° with the load (see Sec. 1.3). Note from Fig. 2.6 that the elongation of a ductile specimen after it has ruptured can be 200 times as large as its deformation at yield. The stress σ_Y at which yield is initiated is called the *yield strength* of the material. The stress σ_U corresponding to the maximum load applied is known as the *ultimate strength.* The stress σ_B corresponding to rupture is called the *breaking strength.*

Brittle materials, comprising of cast iron, glass, and stone rupture without any noticeable prior change in the rate of elongation (Fig. 2.7). Thus, for brittle materials, there is no difference between the ultimate strength and the breaking strength. Also, the strain at the time of rupture is much smaller for brittle than for ductile materials. Note the absence of any necking of the specimen in the brittle material of Photo 2.5 and observe that rupture occurs along a surface perpendicular to the load. Thus, normal stresses are primarily responsible for the failure of brittle materials.[†]

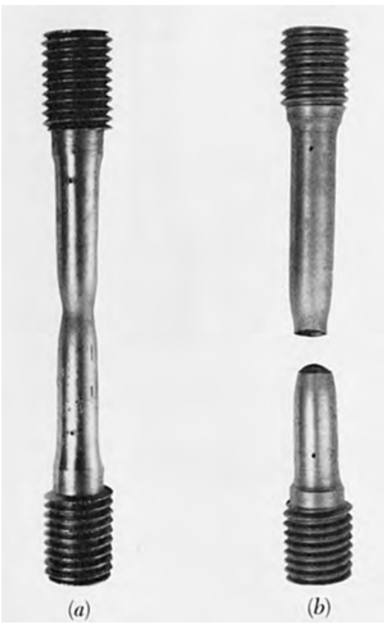


Photo 2.4 Ductile material tested specimens: (*a*) with cross-section necking, (*b*) ruptured.

[†]The tensile tests described in this section were assumed to be conducted at normal temperatures. However, a material that is ductile at normal temperatures may display the characteristics of a brittle material at very low temperatures, while a normally brittle material may behave in a ductile fashion at very high temperatures. At temperatures other than normal, therefore, one should refer to *a material in a ductile state* or to *a material in a brittle state,* rather than to a ductile or brittle material.

The stress-strain diagrams of Fig. 2.6 show that while structural steel and aluminum are both ductile, they have different yield characteristics. For structural steel (Fig. 2.6*a*), the stress remains constant over a large range of the strain after the onset of yield. Later, the stress must be increased to keep elongating the specimen until the maximum value σ_U has been reached. This is due to a property of the material known as *strain-hardening*. The *yield strength* of structural steel is determined during the tensile test by watching the load shown on the display of the testing machine. After increasing steadily, the load will suddenly drop to a slightly lower value, which is maintained for a certain period as the specimen keeps elongating. In a very carefully conducted test, one may be able to distinguish between the *upper yield point*, which corresponds to the load reached just before yield starts, and the *lower yield point*, which corresponds to the load required to maintain yield. Since the upper yield point is transient, the lower yield point is used to determine the yield strength of the material.

For aluminum (Fig. 2.6*b*) and of many other ductile materials, the stress keeps increasing—although not linearly—until the ultimate strength is reached. Necking then begins and eventually ruptures. For such materials, the yield strength σ_Y can be determined using the offset method. For example the yield strength at 0.2% offset is obtained by drawing through the point of the horizontal axis of abscissa $\epsilon = 0.2\%$ (or $\epsilon = 0.002$), which is a line parallel to the initial straight-line portion of the stress-strain diagram (Fig. 2.8). The stress σ_Y corresponding to the point Y is defined as the yield strength at 0.2% offset.

A standard measure of the ductility of a material is its *percent elongation*:

$$\text{Percent elongation} = 100\,\frac{L_B - L_0}{L_0}$$

where L_0 and L_B are the initial length of the tensile test specimen and its final length at rupture, respectively. The specified minimum elongation for a 2-in. gage length for commonly used steels with yield strengths up to 50 ksi is 21 percent. This means that the average strain at rupture should be at least 0.21 in./in.

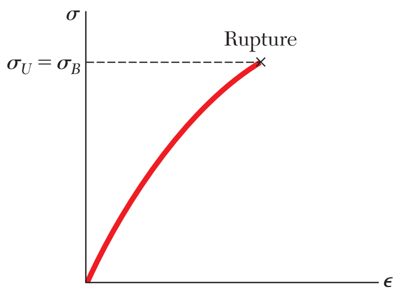


Fig. 2.7 Stress-strain diagram for a typical brittle material.

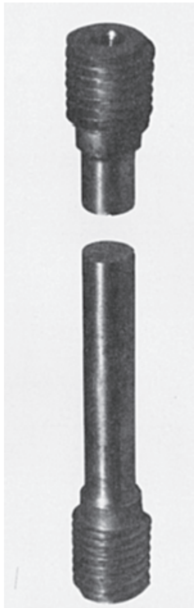


Photo 2.5 Ruptured brittle material specimen.

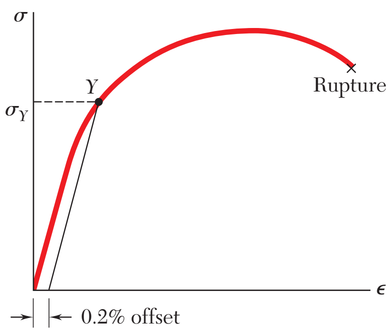


Fig. 2.8 Determination of yield strength by 0.2% offset method.

Another measure of ductility that is sometimes used is the *percent reduction in area*:

$$\text{Percent reduction in area} = 100\frac{A_0 - A_B}{A_0}$$

where A_0 and A_B are the initial cross-sectional area of the specimen and its minimum cross-sectional area at rupture, respectively. For structural steel, percent reductions in area of 60 to 70 percent are common.

Compression Test. If a specimen made of a ductile material is loaded in compression instead of tension, the stress-strain curve is essentially the same through its initial straight-line portion and through the beginning of the portion corresponding to yield and strain-hardening. Particularly noteworthy is the fact that for a given steel, the yield strength is the same in both tension and compression. For larger values of the strain, the tension and compression stress-strain curves diverge, and necking does not occur in compression. For most brittle materials, the ultimate strength in compression is much larger than in tension. This is due to the presence of flaws, such as microscopic cracks or cavities that tend to weaken the material in tension, while not appreciably affecting its resistance to compressive failure.

An example of brittle material with different properties in tension and compression is provided by *concrete*, whose stress-strain diagram is shown in Fig. 2.9. On the tension side of the diagram, we first observe a linear elastic range in which the strain is proportional to the stress. After the yield point has been reached, the strain increases faster than the stress until rupture occurs. The behavior of the material in compression is different. First, the linear elastic range is significantly larger. Second, rupture does not occur as the stress reaches its maximum value. Instead, the stress decreases in magnitude while the strain keeps increasing until rupture occurs. Note that the modulus of elasticity, which is represented by the slope of the stress-strain curve in its linear portion, is the same in tension and compression. This is true of most brittle materials.

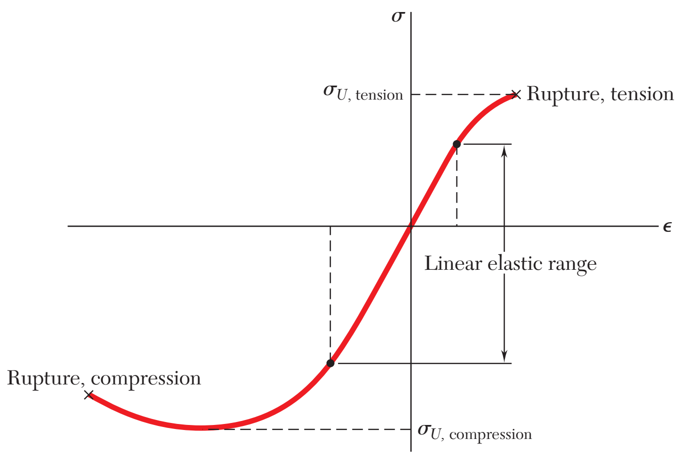


Fig. 2.9 Stress-strain diagram for concrete shows difference in tensile and compression response.

*2.1C True Stress and True Strain

Recall that the stress plotted in Figs. 2.6 and 2.7 was obtained by dividing the load P by the cross-sectional area A_0 of the specimen measured before any deformation had taken place. Since the cross-sectional area of the specimen decreases as P increases, the stress plotted in these diagrams does not represent the actual stress in the specimen. The difference between the *engineering stress* $\sigma = P/A_0$ and the *true stress* $\sigma_t = P/A$ becomes apparent in ductile materials after yield has started. While the engineering stress σ, which is directly proportional to the load P, decreases with P during the necking phase, the true stress σ_t, which is proportional to P but also inversely proportional to A, keeps increasing until rupture of the specimen occurs.

For *engineering strain* $\epsilon = \delta/L_0$, instead of using the total elongation δ and the original value L_0 of the gage length, many scientists use all of the values of L that they have recorded. Dividing each increment ΔL of the distance between the gage marks by the corresponding value of L, the elementary strain $\Delta\epsilon = \Delta L/L$. Adding the successive values of $\Delta\epsilon$, the *true strain* ϵ_t is

$$\epsilon_t = \Sigma\Delta\epsilon = \Sigma(\Delta L/L)$$

With the summation replaced by an integral, the true strain can be expressed as:

$$\epsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\frac{L}{L_0} \quad \textbf{(2.5)}$$

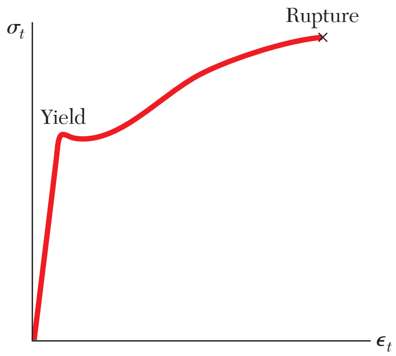


Fig. 2.10 True stress versus true strain for a typical ductile material.

Plotting true stress versus true strain (Fig. 2.10) more accurately reflects the behavior of the material. As already noted, there is no decrease in true stress during the necking phase. Also, the results obtained from either tensile or compressive tests yield essentially the same plot when true stress and true strain are used. This is not the case for large values of the strain when the engineering stress is plotted versus the engineering strain. However, in order to determine whether a load P will produce an acceptable stress and an acceptable deformation in a given member, engineers will use a diagram based on Eqs. (2.3) and (2.4) since these involve the cross-sectional area A_0 and the length L_0 of the member in its undeformed state, which are easily available.

2.1D Hooke's Law; Modulus of Elasticity

Modulus of Elasticity. Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram. For that initial portion of the diagram (Fig. 2.6), the stress σ is directly proportional to the strain ϵ:

$$\sigma = E\epsilon \quad \textbf{(2.6)}$$

This is known as *Hooke's law*, after Robert Hooke (1635–1703), an English scientist and one of the early founders of applied mechanics. The coefficient E of the material is the *modulus of elasticity* or *Young's modulus*, after the English scientist Thomas Young (1773–1829). Since the strain ϵ is a dimensionless quantity, E is expressed in the same units as stress σ—in pascals or one of its multiples for SI units and in psi or ksi for U.S. customary units.

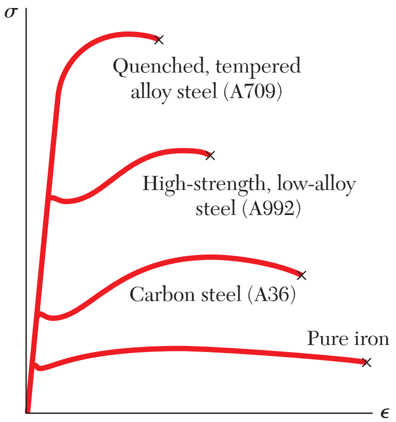


Fig. 2.11 Stress-strain diagrams for iron and different grades of steel.

The largest value of stress for which Hooke's law can be used for a given material is the *proportional limit* of that material. For ductile materials possessing a well-defined yield point, as in Fig. 2.6*a*, the proportional limit almost coincides with the yield point. For other materials, the proportional limit cannot be determined as easily, since it is difficult to accurately determine the stress σ for which the relation between σ and ϵ ceases to be linear. For such materials, however, using Hooke's law for values of the stress slightly larger than the actual proportional limit will not result in any significant error.

Some physical properties of structural metals, such as strength, ductility, and corrosion resistance, can be greatly affected by alloying, heat treatment, and the manufacturing process used. For example, the stress-strain diagrams of pure iron and three different grades of steel (Fig. 2.11) show that large variations in the yield strength, ultimate strength, and final strain (ductility) exist. All of these metals possess the same modulus of elasticity—their "stiffness," or ability to resist a deformation within the linear range is the same. Therefore, if a high-strength steel is substituted for a lower-strength steel and if all dimensions are kept the same, the structure will have an increased load-carrying capacity, but its stiffness will remain unchanged.

For the materials considered so far, the relationship between normal stress and normal strain, $\sigma = E\epsilon$, is independent of the direction of loading. This is because the mechanical properties of each material, including its modulus of elasticity E, are independent of the direction considered. Such materials are said to be *isotropic*. Materials whose properties depend upon the direction considered are said to be *anisotropic*.

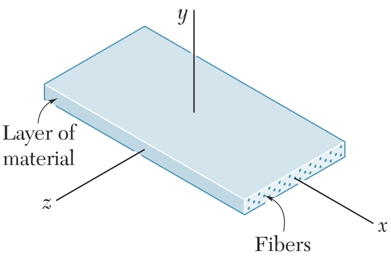


Fig. 2.12 Layer of fiber-reinforced composite material.

Fiber-Reinforced Composite Materials. An important class of anisotropic materials consists of *fiber-reinforced composite materials*. These are obtained by embedding fibers of a strong, stiff material into a weaker, softer material, called a *matrix*. Typical materials used as fibers are graphite, glass, and polymers, while various types of resins are used as a matrix. Figure 2.12 shows a layer, or *lamina*, of a composite material consisting of a large number of parallel fibers embedded in a matrix. An axial load applied to the lamina along the x axis, (in a direction parallel to the fibers) will create a normal stress σ_x in the lamina and a corresponding normal strain ϵ_x, satisfying Hooke's law as the load is increased and as long as the elastic limit of the lamina is not exceeded. Similarly, an axial load applied along the y axis, (in a direction perpendicular to the lamina) will create a normal stress σ_y and a normal strain ϵ_y, and an axial load applied along the z axis will create a normal stress σ_z and a normal strain ϵ_z, all satisfy Hooke's law. However, the moduli of elasticity E_x, E_y, and E_z corresponding, to each of these loadings will be different. Because the fibers are parallel to the x axis, the lamina will offer a much stronger resistance to a load directed along the x axis than to one directed along the y or z axis, and E_x will be much larger than either E_y or E_z.

A flat *laminate* is obtained by superposing a number of layers or laminas. If the laminate is subjected only to an axial load causing tension, the fibers in all layers should have the same orientation as the load in order to obtain the greatest possible strength. But if the laminate is in compression, the matrix material may not be strong enough to prevent the fibers from kinking or buckling. The lateral stability of the laminate can be increased by positioning some of the layers so that their fibers are perpendicular to the load. Positioning some layers so that their fibers are oriented at 30°, 45°, or 60° to the load also can be used to increase the resistance of the laminate to in-plane shear. Fiber-reinforced composite materials will be further discussed in Sec. 2.9, where their behavior under multiaxial loadings will be considered.

2.1E Elastic Versus Plastic Behavior of a Material

Material behaves *elastically* if the strains in a test specimen from a given load disappear when the load is removed. The largest value of stress causing this elastic behavior is called the *elastic limit* of the material.

If the material has a well-defined yield point as in Fig. 2.6*a*, the elastic limit, the proportional limit, and the yield point are essentially equal. In other words, the material behaves elastically and linearly as long as the stress is kept below the yield point. However, if the yield point is reached, yield takes place as described in Sec. 2.1B. When the load is removed, the stress and strain decrease in a linear fashion along a line *CD* parallel to the straight-line portion *AB* of the loading curve (Fig. 2.13). The fact that ϵ does not return to zero after the load has been removed indicates that a *permanent set* or *plastic deformation* of the material has taken place. For most materials, the plastic deformation depends upon both the maximum value reached by the stress and the time elapsed before the load is removed. The stress-dependent part of the plastic deformation is called *slip*, and the time-dependent part—also influenced by the temperature—is *creep*.

When a material does not possess a well-defined yield point, the elastic limit cannot be determined with precision. However, assuming the elastic limit to be equal to the yield strength using the offset method (Sec. 2.1B) results in only a small error. Referring to Fig. 2.8, note that the straight line used to determine point *Y* also represents the unloading curve after a maximum stress σ_Y has been reached. While the material does not behave truly elastically, the resulting plastic strain is as small as the selected offset.

If, after being loaded and unloaded (Fig. 2.14), the test specimen is loaded again, the new loading curve will follow the earlier unloading curve until it almost reaches point *C*. Then it will bend to the right and connect with the curved portion of the original stress-strain diagram. This straight-line portion of the new loading curve is longer than the corresponding portion of the initial one. Thus, the proportional limit and the

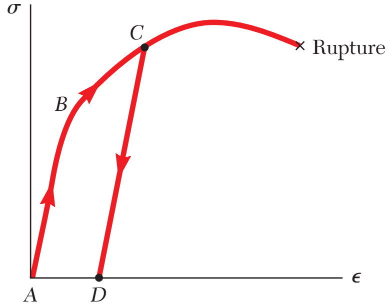


Fig. 2.13 Stress-strain response of ductile material loaded beyond yield and unloaded.

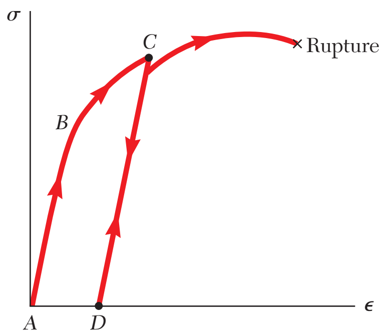


Fig. 2.14 Stress-strain response of ductile material reloaded after prior yielding and unloading.

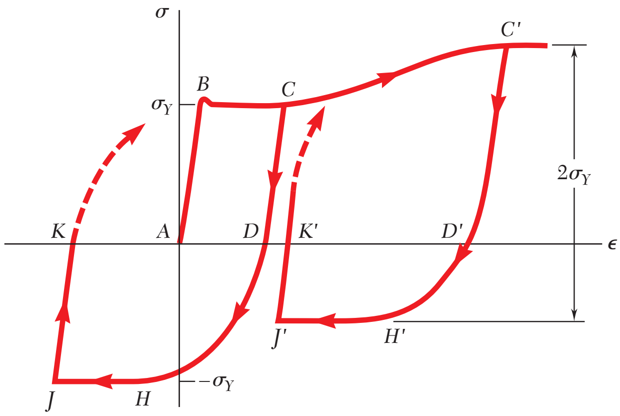


Fig. 2.15 Stress-strain response for mild steel subjected to two cases of reverse loading.

elastic limit have increased as a result of the strain-hardening that occurred during the earlier loading. However, since the point of rupture *R* remains unchanged, the ductility of the specimen, which should now be measured from point *D*, has decreased.

In previous discussions the specimen was loaded twice in the same direction (i.e., both loads were tensile loads). Now consider that the second load is applied in a direction opposite to that of the first one. Assume the material is mild steel where the yield strength is the same in tension and in compression. The initial load is tensile and is applied until point *C* is reached on the stress-strain diagram (Fig. 2.15). After unloading (point *D*), a compressive load is applied, causing the material to reach point *H*, where the stress is equal to $-\sigma_Y$. Note that portion *DH* of the stress-strain diagram is curved and does not show any clearly defined yield point. This is referred to as the *Bauschinger effect.* As the compressive load is maintained, the material yields along line *HJ*.

If the load is removed after point *J* has been reached, the stress returns to zero along line *JK*, and the slope of *JK* is equal to the modulus of elasticity *E*. The resulting permanent set *AK* may be positive, negative, or zero, depending upon the lengths of the segments *BC* and *HJ*. If a tensile load is applied again to the test specimen, the portion of the stress-strain diagram beginning at *K* (dashed line) will curve up and to the right until the yield stress σ_Y has been reached.

If the initial loading is large enough to cause strain-hardening of the material (point C'), unloading takes place along line $C'D'$. As the reverse load is applied, the stress becomes compressive, reaching its maximum value at H' and maintaining it as the material yields along line $H'J'$. While the maximum value of the compressive stress is less than σ_Y, the total change in stress between C' and H' is still equal to $2\sigma_Y$.

If point *K* or K' coincides with the origin *A* of the diagram, the permanent set is equal to zero, and the specimen may appear to have returned to its original condition. However, internal changes will have taken place and, the specimen will rupture without any warning after relatively few repetitions of the loading sequence. Thus, the excessive plastic deformations to which the specimen was subjected caused a radical change in the characteristics of the material. Therefore reverse loadings into the plastic range are seldom allowed, being permitted only under carefully controlled conditions such as in the straightening of damaged material and the final alignment of a structure or machine.

2.1F Repeated Loadings and Fatigue

You might think that a given load may be repeated many times, provided that the stresses remain in the elastic range. Such a conclusion is correct for loadings repeated a few dozen or even a few hundred times. However, it is not correct when loadings are repeated thousands or millions of times. In such cases, rupture can occur at a stress much lower than the static breaking strength; this phenomenon is known as *fatigue*. A fatigue failure is of a brittle nature, even for materials that are normally ductile.

Fatigue must be considered in the design of all structural and machine components subjected to repeated or fluctuating loads. The number of loading cycles expected during the useful life of a component varies greatly. For example, a beam supporting an industrial crane can be loaded as many as two million times in 25 years (about 300 loadings per working day), an automobile crankshaft is loaded about half a billion times if the automobile is driven 200,000 miles, and an individual turbine blade can be loaded several hundred billion times during its lifetime.

Some loadings are of a fluctuating nature. For example, the passage of traffic over a bridge will cause stress levels that will fluctuate about the stress level due to the weight of the bridge. A more severe condition occurs when a complete reversal of the load occurs during the loading cycle. The stresses in the axle of a railroad car, for example, are completely reversed after each half-revolution of the wheel.

The number of loading cycles required to cause the failure of a specimen through repeated loadings and reverse loadings can be determined experimentally for any given maximum stress level. If a series of tests is conducted using different maximum stress levels, the resulting data is plotted as a σ-n curve. For each test, the maximum stress σ is plotted as an ordinate and the number of cycles n as an abscissa. Because of the large number of cycles required for rupture, the cycles n are plotted on a logarithmic scale.

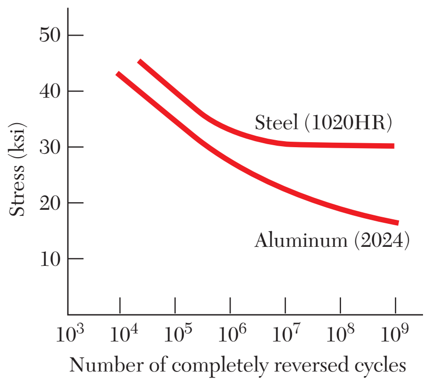


Fig. 2.16 Typical σ-n curves.

A typical σ-n curve for steel is shown in Fig. 2.16. If the applied maximum stress is high, relatively few cycles are required to cause rupture. As the magnitude of the maximum stress is reduced, the number of cycles required to cause rupture increases, until the *endurance limit* is reached. The endurance limit is the stress for which failure does not occur, even for an indefinitely large number of loading cycles. For a low-carbon steel, such as structural steel, the endurance limit is about one-half of the ultimate strength of the steel.

For nonferrous metals, such as aluminum and copper, a typical σ-n curve (Fig. 2.16) shows that the stress at failure continues to decrease as the number of loading cycles is increased. For such metals, the *fatigue limit* is the stress corresponding to failure after a specified number of loading cycles.

Examination of test specimens, shafts, springs, and other components that have failed in fatigue shows that the failure initiated at a microscopic crack or some similar imperfection. At each loading, the crack was very slightly enlarged. During successive loading cycles, the crack propagated through the material until the amount of undamaged material was insufficient to carry the maximum load, and an abrupt, brittle failure

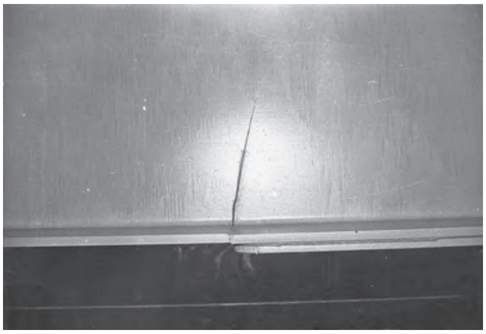


Photo 2.6 Fatigue crack in a steel girder of the Yellow Mill Pond Bridge, Connecticut, prior to repairs.

occurred. For example, Photo 2.6 shows a progressive fatigue crack in a highway bridge girder that initiated at the irregularity associated with the weld of a cover plate and then propagated through the flange and into the web. Because fatigue failure can be initiated at any crack or imperfection, the surface condition of a specimen has an important effect on the endurance limit obtained in testing. The endurance limit for machined and polished specimens is higher than for rolled or forged components or for components that are corroded. In applications in or near seawater or in other applications where corrosion is expected, a reduction of up to 50% in the endurance limit can be expected.

2.1G Deformations of Members Under Axial Loading

Consider a homogeneous rod BC of length L and uniform cross section of area A subjected to a centric axial load **P** (Fig. 2.17). If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, Hooke's law applies and

$$\sigma = E\epsilon \tag{2.6}$$

from which

$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE} \tag{2.7}$$

Recalling that the strain ϵ in Sec. 2.1A is $\epsilon = \delta/L$

$$\delta = \epsilon L \tag{2.8}$$

and substituting for ϵ from Eq. (2.7) into Eq.(2.8):

$$\delta = \frac{PL}{AE} \tag{2.9}$$

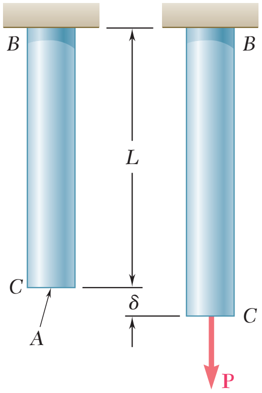


Fig. 2.17 Undeformed and deformed axially-loaded rod.

Equation (2.9) can be used only if the rod is homogeneous (constant E), has a uniform cross section of area A, and is loaded at its ends. If the rod is loaded at other points, or consists of several portions of various cross sections and possibly of different materials, it must be divided into component parts that satisfy the required conditions for the application of Eq. (2.9). Using the internal force P_i, length L_i, cross-sectional area A_i, and modulus of elasticity E_i, corresponding to part i, the deformation of the entire rod is

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{2.10}$$

In the case of a member of variable cross section (Fig. 2.18), the strain ϵ depends upon the position of the point Q, where it is computed as $\epsilon = d\delta/dx$ (Sec. 2.1A). Solving for $d\delta$ and substituting for ϵ from Eq. (2.7), the deformation of an element of length dx is

$$d\delta = \epsilon\, dx = \frac{P\, dx}{AE}$$

The total deformation δ of the member is obtained by integrating this expression over the length L of the member:

$$\delta = \int_0^L \frac{P\, dx}{AE} \tag{2.11}$$

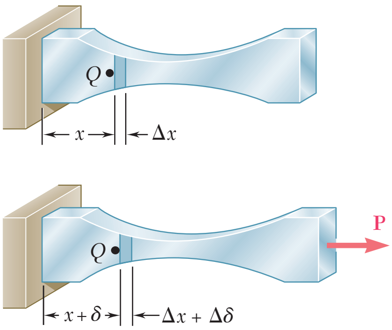


Fig. 2.18 Deformation of axially-loaded member of variable cross-sectional area.

Equation (2.11) should be used in place of (2.9) when both the cross-sectional area A is a function of x, or when the internal force P depends upon x, as is the case for a rod hanging under its own weight.

Concept Application 2.1

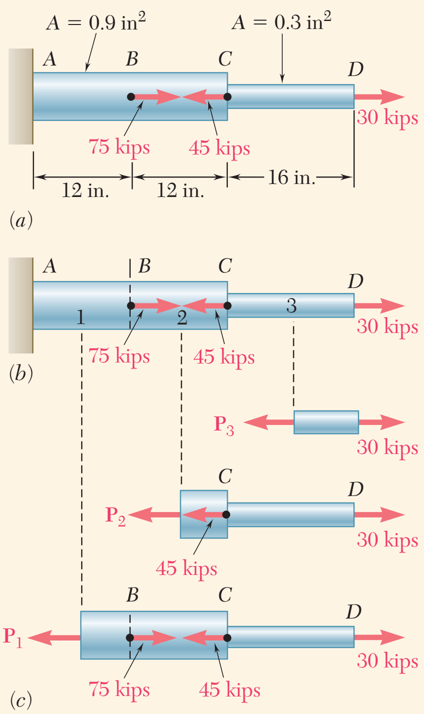


Fig. 2.19 (*a*) Axially-loaded rod. (*b*) Rod divided into three sections. (*c*) Three sectioned free-body diagrams with internal resultant forces P_1, P_2, and P_3.

Determine the deformation of the steel rod shown in Fig. 2.19*a* under the given loads ($E = 29 \times 10^6$ psi).

The rod is divided into three component parts in Fig. 2.19*b*, so

$$L_1 = L_2 = 12 \text{ in.} \qquad L_3 = 16 \text{ in.}$$
$$A_1 = A_2 = 0.9 \text{ in}^2 \qquad A_3 = 0.3 \text{ in}^2$$

To find the internal forces P_1, P_2, and P_3, pass sections through each of the component parts, drawing each time the free-body diagram of the portion of rod located to the right of the section (Fig. 2.19*c*). Each of the free bodies is in equilibrium; thus

$$P_1 = 60 \text{ kips} = 60 \times 10^3 \text{ lb}$$
$$P_2 = -15 \text{ kips} = -15 \times 10^3 \text{ lb}$$
$$P_3 = 30 \text{ kips} = 30 \times 10^3 \text{ lb}$$

Using Eq. (2.10)

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$
$$= \frac{1}{29 \times 10^6}\left[\frac{(60 \times 10^3)(12)}{0.9} + \frac{(-15 \times 10^3)(12)}{0.9} + \frac{(30 \times 10^3)(16)}{0.3}\right]$$
$$\delta = \frac{2.20 \times 10^6}{29 \times 10^6} = 75.9 \times 10^{-3} \text{ in.}$$

Rod BC of Fig. 2.17, used to derive Eq. (2.9), and rod AD of Fig. 2.19 have one end attached to a fixed support. In each case, the deformation δ of the rod was equal to the displacement of its free end. When both ends of a rod move, however, the deformation of the rod is measured by the *relative displacement* of one end of the rod with respect to the other. Consider the assembly shown in Fig. 2.20*a*, which consists of three elastic bars of length L connected by a rigid pin at A. If a load **P** is applied at B (Fig. 2.20*b*), each of the three bars will deform. Since the bars AC and AC' are attached to fixed supports at C and C', their common deformation is measured by the displacement δ_A of point A. On the other hand, since both ends of bar AB move, the deformation of AB is measured by the difference between the displacements δ_A and δ_B of points A and B, (i.e., by the relative displacement of B with respect to A). Denoting this relative displacement by $\delta_{B/A}$,

$$\delta_{B/A} = \delta_B - \delta_A = \frac{PL}{AE} \qquad \textbf{(2.12)}$$

where A is the cross-sectional area of AB and E is its modulus of elasticity.

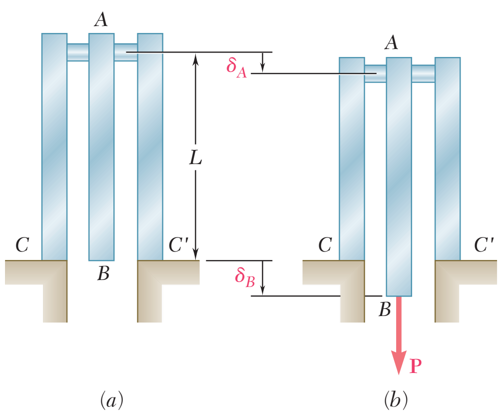
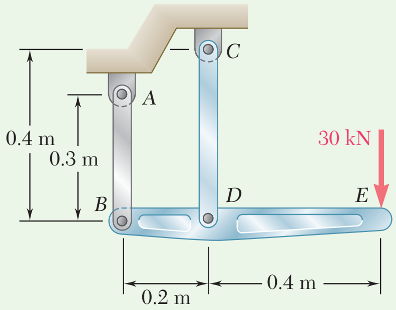


Fig. 2.20 Example of relative end displacement, as exhibited by the middle bar. (*a*) Unloaded. (*b*) Loaded, with deformation.



Sample Problem 2.1

The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of *B*, (*b*) of *D*, and (*c*) of *E*.

STRATEGY: Consider the free body of the rigid bar to determine the internal force of each link. Knowing these forces and the properties of the links, their deformations can be evaluated. You can then use simple geometry to determine the deflection of E.

MODELING: Draw the free body diagrams of the rigid bar (Fig. 1) and the two links (Fig. 2 and 3)

ANALYSIS:

Free Body: Bar BDE (Fig. 1)

$$+\circlearrowleft \Sigma M_B = 0: \qquad -(30\text{ kN})(0.6\text{ m}) + F_{CD}(0.2\text{ m}) = 0$$

$$F_{CD} = +90\text{ kN} \qquad F_{CD} = 90\text{ kN} \quad \textit{tension}$$

$$+\circlearrowleft \Sigma M_D = 0: \qquad -(30\text{ kN})(0.4\text{ m}) - F_{AB}(0.2\text{ m}) = 0$$

$$F_{AB} = -60\text{ kN} \qquad F_{AB} = 60\text{ kN} \quad \textit{compression}$$

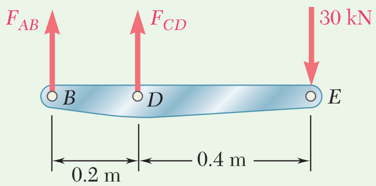


Fig. 1 Free-body diagram of rigid bar *BDE*.

a. Deflection of B. Since the internal force in link *AB* is compressive (Fig. 2), $P = -60$ kN and

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3\text{ N})(0.3\text{ m})}{(500 \times 10^{-6}\text{ m}^2)(70 \times 10^9\text{ Pa})} = -514 \times 10^{-6}\text{ m}$$

The negative sign indicates a contraction of member *AB*. Thus, the deflection of end *B* is upward:

$$\delta_B = \mathbf{0.514\ mm} \uparrow \blacktriangleleft$$

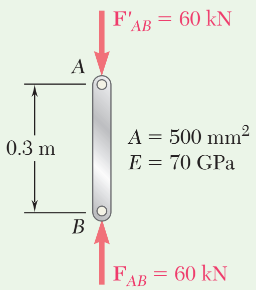


Fig. 2 Free-body diagram of two-force member *AB*.

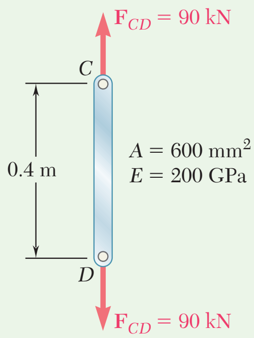


Fig. 3 Free-body diagram of two-force member *CD*.

(continued)

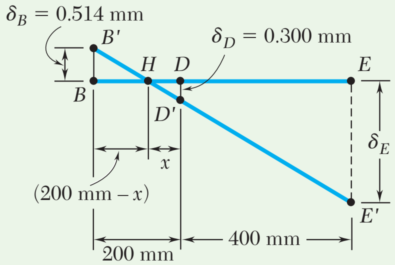


Fig. 4 Deflections at *B* and *D* of rigid bar are used to find δ_E.

b. Deflection of D. Since in rod *CD* (Fig. 3), $P = 90$ kN, write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3 \text{ N})(0.4 \text{ m})}{(600 \times 10^{-6} \text{m}^2)(200 \times 10^9 \text{ Pa})}$$

$$= 300 \times 10^{-6} \text{ m} \qquad \delta_D = \mathbf{0.300\ mm} \downarrow \blacktriangleleft$$

c. Deflection of E. Referring to Fig. 4, we denote by B' and D' the displaced positions of points *B* and *D*. Since the bar *BDE* is rigid, points B', D', and E' lie in a straight line. Therefore,

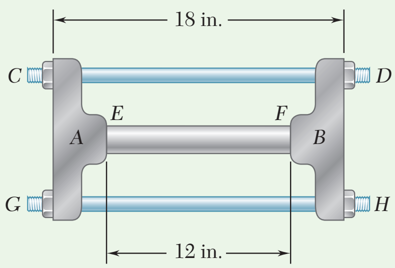
$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x} \qquad x = 73.7 \text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 \text{ mm}) + (73.7 \text{ mm})}{73.7 \text{ mm}}$$

$$\delta_E = \mathbf{1.928\ mm} \downarrow \blacktriangleleft$$

REFLECT and THINK: Comparing the relative magnitude and direction of the resulting deflections, you can see that the answers obtained are consistent with the loading and the deflection diagram of Fig. 4.

Sample Problem 2.2



The rigid castings *A* and *B* are connected by two $\frac{3}{4}$-in.-diameter steel bolts *CD* and *GH* and are in contact with the ends of a 1.5-in.-diameter aluminum rod *EF*. Each bolt is single-threaded with a pitch of 0.1 in., and after being snugly fitted, the nuts at *D* and *H* are both tightened one-quarter of a turn. Knowing that *E* is 29×10^6 psi for steel and 10.6×10^6 psi for aluminum, determine the normal stress in the rod.

STRATEGY: The tightening of the nuts causes a displacement of the ends of the bolts relative to the rigid casting that is equal to the difference in displacements between the bolts and the rod. This will give a relation between the internal forces of the bolts and the rod that, when combined with a free body analysis of the rigid casting, will enable you to solve for these forces and determine the corresponding normal stress in the rod.

MODELING: Draw the free body diagrams of the bolts and rod (Fig. 1) and the rigid casting (Fig. 2).

ANALYSIS:

Deformations.

Bolts CD and GH. Tightening the nuts causes tension in the bolts (Fig. 1). Because of symmetry, both are subjected to the same

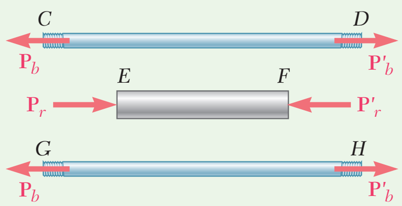


Fig. 1 Free-body diagrams of bolts and aluminum bar.

(continued)

internal force P_b and undergo the same deformation δ_b. Therefore,

$$\delta_b = +\frac{P_b L_b}{A_b E_b} = +\frac{P_b(18 \text{ in.})}{\frac{1}{4}\pi(0.75 \text{ in.})^2(29 \times 10^6 \text{ psi})} = +1.405 \times 10^{-6} P_b \quad \textbf{(1)}$$

Rod EF. The rod is in compression (Fig. 1), where the magnitude of the force is P_r and the deformation δ_r:

$$\delta_r = -\frac{P_r L_r}{A_r E_r} = -\frac{P_r(12 \text{ in.})}{\frac{1}{4}\pi(1.5 \text{ in.})^2(10.6 \times 10^6 \text{ psi})} = -0.6406 \times 10^{-6} P_r \quad \textbf{(2)}$$

Displacement of D Relative to B. Tightening the nuts one-quarter of a turn causes ends D and H of the bolts to undergo a displacement of $\frac{1}{4}$(0.1 in.) *relative to casting B*. Considering end D,

$$\delta_{D/B} = \tfrac{1}{4}(0.1 \text{ in.}) = 0.025 \text{ in.} \quad \textbf{(3)}$$

But $\delta_{D/B} = \delta_D - \delta_B$, where δ_D and δ_B represent the displacements of D and B. If casting A is held in a fixed position while the nuts at D and H are being tightened, these displacements are equal to the deformations of the bolts and of the rod, respectively. Therefore,

$$\delta_{D/B} = \delta_b - \delta_r \quad \textbf{(4)}$$

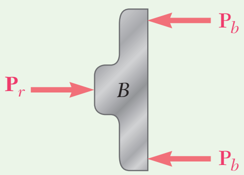


Fig. 2 Free-body diagram of rigid casting.

Substituting from Eqs. (1), (2), and (3) into Eq. (4),

$$0.025 \text{ in.} = 1.405 \times 10^{-6} P_b + 0.6406 \times 10^{-6} P_r \quad \textbf{(5)}$$

Free Body: Casting B (Fig. 2)

$$\xrightarrow{+} \Sigma F = 0: \qquad P_r - 2P_b = 0 \qquad P_r = 2P_b \quad \textbf{(6)}$$

Forces in Bolts and Rod Substituting for P_r from Eq. (6) into Eq. (5), we have

$$0.025 \text{ in.} = 1.405 \times 10^{-6} P_b + 0.6406 \times 10^{-6}(2P_b)$$

$$P_b = 9.307 \times 10^3 \text{ lb} = 9.307 \text{ kips}$$

$$P_r = 2P_b = 2(9.307 \text{ kips}) = 18.61 \text{ kips}$$

Stress in Rod

$$\sigma_r = \frac{P_r}{A_r} = \frac{18.61 \text{ kips}}{\frac{1}{4}\pi(1.5 \text{ in.})^2} \qquad \sigma_r = \mathbf{10.53 \text{ ksi}} \quad \blacktriangleleft$$

REFLECT and THINK: This is an example of a *statically indeterminate* problem, where the determination of the member forces could not be found by equilibrium alone. By considering the relative displacement characteristics of the members, you can obtain additional equations necessary to solve such problems. Situations like this will be examined in more detail in the following section.

Problems

2.1 A nylon thread is subjected to a 8.5-N tension force. Knowing that $E = 3.3$ GPa and that the length of the thread increases by 1.1%, determine (*a*) the diameter of the thread, (*b*) the stress in the thread.

2.2 A 4.8-ft-long steel wire of $\frac{1}{4}$-in.-diameter is subjected to a 750-lb tensile load. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the elongation of the wire, (*b*) the corresponding normal stress.

2.3 An 18-m-long steel wire of 5-mm diameter is to be used in the manufacture of a prestressed concrete beam. It is observed that the wire stretches 45 mm when a tensile force **P** is applied. Knowing that $E = 200$ GPa, determine (*a*) the magnitude of the force **P**, (*b*) the corresponding normal stress in the wire.

2.4 Two gage marks are placed exactly 250 mm apart on a 12-mm-diameter aluminum rod with $E = 73$ GPa and an ultimate strength of 140 MPa. Knowing that the distance between the gage marks is 250.28 mm after a load is applied, determine (*a*) the stress in the rod, (*b*) the factor of safety.

2.5 An aluminum pipe must not stretch more than 0.05 in. when it is subjected to a tensile load. Knowing that $E = 10.1 \times 10^6$ psi and that the maximum allowable normal stress is 14 ksi, determine (*a*) the maximum allowable length of the pipe, (*b*) the required area of the pipe if the tensile load is 127.5 kips.

2.6 A control rod made of yellow brass must not stretch more than 3 mm when the tension in the wire is 4 kN. Knowing that $E = 105$ GPa and that the maximum allowable normal stress is 180 MPa, determine (*a*) the smallest diameter rod that should be used, (*b*) the corresponding maximum length of the rod.

2.7 A steel control rod is 5.5 ft long and must not stretch more than 0.04 in. when a 2-kip tensile load is applied to it. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the smallest diameter rod that should be used, (*b*) the corresponding normal stress caused by the load.

2.8 A cast-iron tube is used to support a compressive load. Knowing that $E = 10 \times 10^6$ psi and that the maximum allowable change in length is 0.025%, determine (*a*) the maximum normal stress in the tube, (*b*) the minimum wall thickness for a load of 1600 lb if the outside diameter of the tube is 2.0 in.

2.9 A 4-m-long steel rod must not stretch more than 3 mm and the normal stress must not exceed 150 MPa when the rod is subjected to a 10-kN axial load. Knowing that $E = 200$ GPa, determine the required diameter of the rod.

2.10 A nylon thread is to be subjected to a 10-N tension. Knowing that $E = 3.2$ GPa, that the maximum allowable normal stress is 40 MPa, and that the length of the thread must not increase by more than 1%, determine the required diameter of the thread.

2.11 A block of 10-in. length and 1.8×1.6-in. cross section is to support a centric compressive load **P**. The material to be used is a bronze for which $E = 14 \times 10^6$ psi. Determine the largest load that can be applied, knowing that the normal stress must not exceed 18 ksi and that the decrease in length of the block should be at most 0.12% of its original length.

2.12 A square yellow-brass bar must not stretch more than 2.5 mm when it is subjected to a tensile load. Knowing that $E = 105$ GPa and that the allowable tensile strength is 180 MPa, determine (*a*) the maximum allowable length of the bar, (*b*) the required dimensions of the cross section if the tensile load is 40 kN.

2.13 Rod *BD* is made of steel ($E = 29 \times 10^6$ psi) and is used to brace the axially compressed member *ABC*. The maximum force that can be developed in member *BD* is $0.02P$. If the stress must not exceed 18 ksi and the maximum change in length of *BD* must not exceed 0.001 times the length of *ABC*, determine the smallest-diameter rod that can be used for member *BD*.

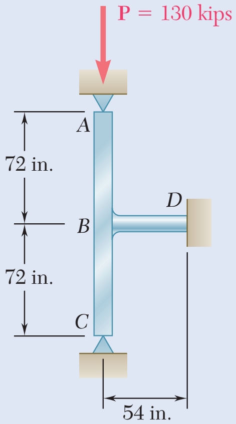


Fig. P2.13

2.14 The 4-mm-diameter cable *BC* is made of a steel with $E = 200$ GPa. Knowing that the maximum stress in the cable must not exceed 190 MPa and that the elongation of the cable must not exceed 6 mm, find the maximum load **P** that can be applied as shown.

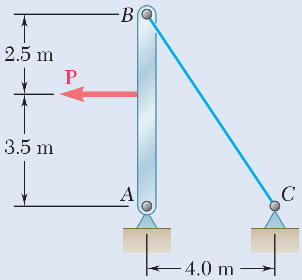


Fig. P2.14

2.15 A single axial load of magnitude $P = 15$ kips is applied at end *C* of the steel rod *ABC*. Knowing that $E = 30 \times 10^6$ psi, determine the diameter *d* of portion *BC* for which the deflection of point *C* will be 0.05 in.

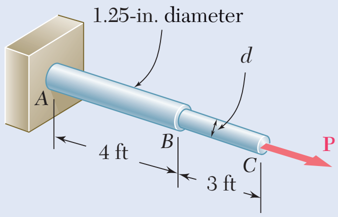


Fig. P2.15

2.16 A 250-mm-long aluminum tube ($E = 70$ GPa) of 36-mm outer diameter and 28-mm inner diameter can be closed at both ends by means of single-threaded screw-on covers of 1.5-mm pitch. With one cover screwed on tight, a solid brass rod ($E = 105$ GPa) of 25-mm diameter is placed inside the tube and the second cover is screwed on. Since the rod is slightly longer than the tube, it is observed that the cover must be forced against the rod by rotating it one-quarter of a turn before it can be tightly closed. Determine (*a*) the average normal stress in the tube and in the rod, (*b*) the deformations of the tube and of the rod.

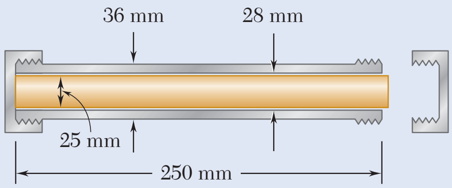


Fig. *P2.16*

2.17 The specimen shown has been cut from a $\frac{1}{4}$-in.-thick sheet of vinyl ($E = 0.45 \times 10^6$ psi) and is subjected to a 350-lb tensile load. Determine (*a*) the total deformation of the specimen, (*b*) the deformation of its central portion *BC*.

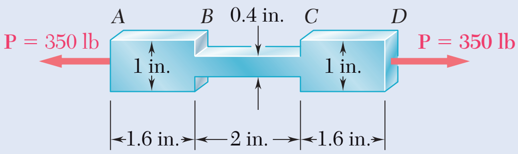


Fig. P2.17

2.18 The brass tube *AB* ($E = 105$ GPa) has a cross-sectional area of 140 mm^2 and is fitted with a plug at *A*. The tube is attached at *B* to a rigid plate that is itself attached at *C* to the bottom of an aluminum cylinder ($E = 72$ GPa) with a cross-sectional area of 250 mm^2. The cylinder is then hung from a support at *D*. In order to close the cylinder, the plug must move down through 1 mm. Determine the force **P** that must be applied to the cylinder.

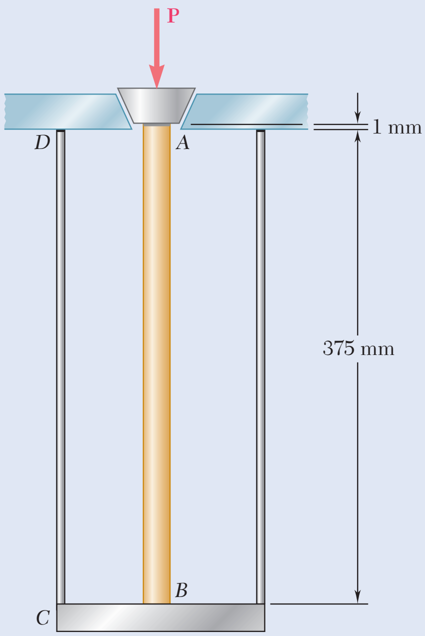


Fig. *P2.18*

2.19 Both portions of the rod *ABC* are made of an aluminum for which $E = 70$ GPa. Knowing that the magnitude of **P** is 4 kN, determine (*a*) the value of **Q** so that the deflection at *A* is zero, (*b*) the corresponding deflection of *B*.

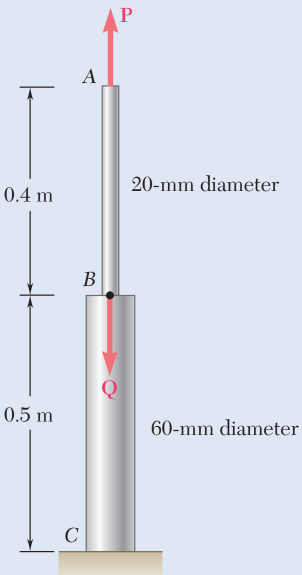


Fig. P2.19 and P2.20

2.20 The rod *ABC* is made of an aluminum for which $E = 70$ GPa. Knowing that $P = 6$ kN and $Q = 42$ kN, determine the deflection of (*a*) point *A*, (*b*) point *B*.

2.21 For the steel truss ($E = 200$ GPa) and loading shown, determine the deformations of members *AB* and *AD*, knowing that their cross-sectional areas are 2400 mm^2 and 1800 mm^2, respectively.

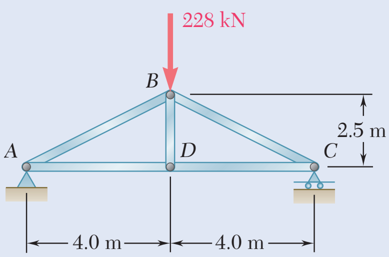


Fig. P2.21

2.22 For the steel truss ($E = 29 \times 10^6$ psi) and loading shown, determine the deformations of members *BD* and *DE*, knowing that their cross-sectional areas are 2 in^2 and 3 in^2, respectively.

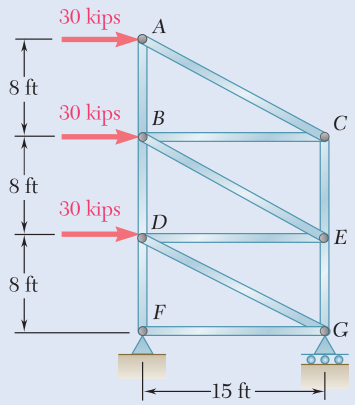


Fig. P2.22

2.23 Members *AB* and *BC* are made of steel ($E = 29 \times 10^6$ psi) with cross-sectional areas of 0.80 in^2 and 0.64 in^2, respectively. For the loading shown, determine the elongation of (*a*) member *AB*, (*b*) member *BC*.

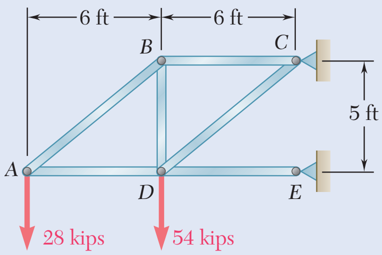


Fig. P2.23

2.24 The steel frame ($E = 200$ GPa) shown has a diagonal brace *BD* with an area of 1920 mm^2. Determine the largest allowable load **P** if the change in length of member *BD* is not to exceed 1.6 mm.

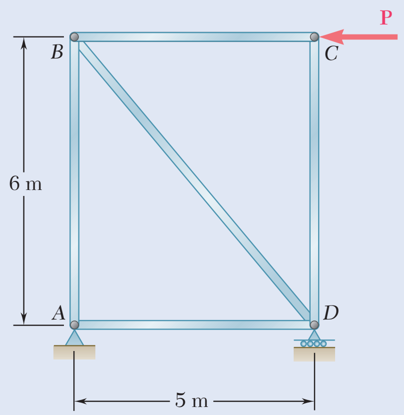


Fig. P2.24

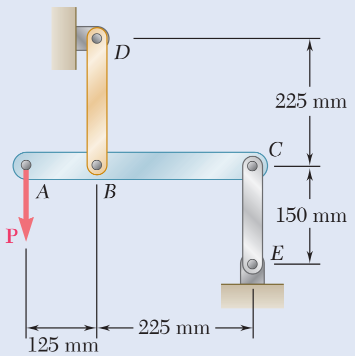


Fig. P2.25

2.25 Link *BD* is made of brass ($E = 105$ GPa) and has a cross-sectional area of 240 mm^2. Link *CE* is made of aluminum ($E = 72$ GPa) and has a cross-sectional area of 300 mm^2. Knowing that they support rigid member *ABC*, determine the maximum force **P** that can be applied vertically at point *A* if the deflection of *A* is not to exceed 0.35 mm.

2.26 Members *ABC* and *DEF* are joined with steel links ($E = 200$ GPa). Each of the links is made of a pair of 25×35-mm plates. Determine the change in length of (*a*) member *BE*, (*b*) member *CF*.

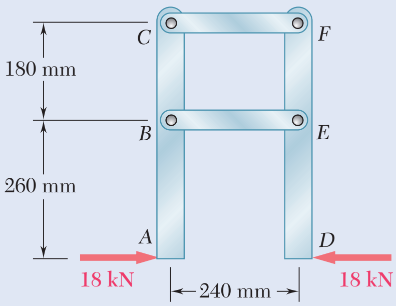


Fig. P2.26

2.27 Each of the links *AB* and *CD* is made of aluminum ($E = 10.9 \times 10^6$ psi) and has a cross-sectional area of 0.2 in^2. Knowing that they support the rigid member *BC*, determine the deflection of point *E*.

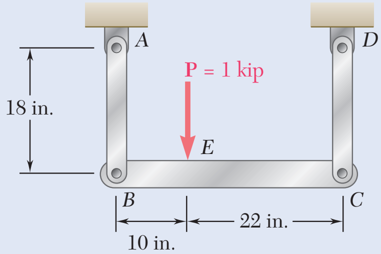


Fig. P2.27

2.28 The length of the $\frac{3}{32}$-in.-diameter steel wire *CD* has been adjusted so that with no load applied, a gap of $\frac{1}{16}$ in. exists between the end *B* of the rigid beam *ACB* and a contact point *E*. Knowing that $E = 29 \times 10^6$ psi, determine where a 50-lb block should be placed on the beam in order to cause contact between *B* and *E*.

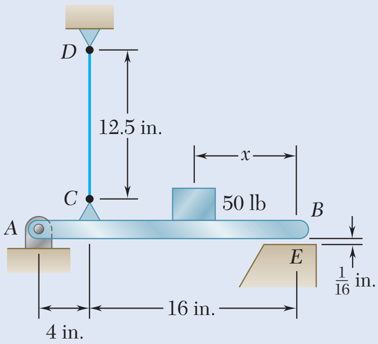


Fig. *P2.28*

2.29 A homogenous cable of length *L* and uniform cross section is suspended from one end. (*a*) Denoting by ρ the density (mass per unit volume) of the cable and by *E* its modulus of elasticity, determine the elongation of the cable due to its own weight. (*b*) Show that the same elongation would be obtained if the cable were horizontal and if a force equal to half of its weight were applied at each end.

2.30 The vertical load **P** is applied at the center *A* of the upper section of a homogeneous frustum of a circular cone of height *h*, minimum radius *a*, and maximum radius *b*. Denoting by *E* the modulus of elasticity of the material and neglecting the effect of its weight, determine the deflection of point *A*.

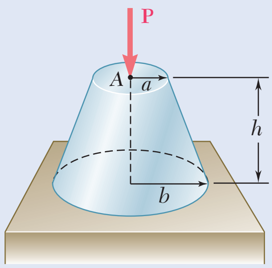


Fig. P2.30

2.31 Denoting by ϵ the "engineering strain" in a tensile specimen, show that the true strain is $\epsilon_t = \ln(1 + \epsilon)$.

2.32 The volume of a tensile specimen is essentially constant while plastic deformation occurs. If the initial diameter of the specimen is d_1, show that when the diameter is d, the true strain is $\epsilon_t = 2 \ln(d_1/d)$.

2.2 STATICALLY INDETERMINATE PROBLEMS

In the problems considered in the preceding section, we could always use free-body diagrams and equilibrium equations to determine the internal forces produced in the various portions of a member under given loading conditions. There are many problems, however, where the internal forces cannot be determined from statics alone. In most of these problems, the reactions themselves—the external forces—cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relationships involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are called *statically indeterminate*. The following concept applications show how to handle this type of problem.

Concept Application 2.2

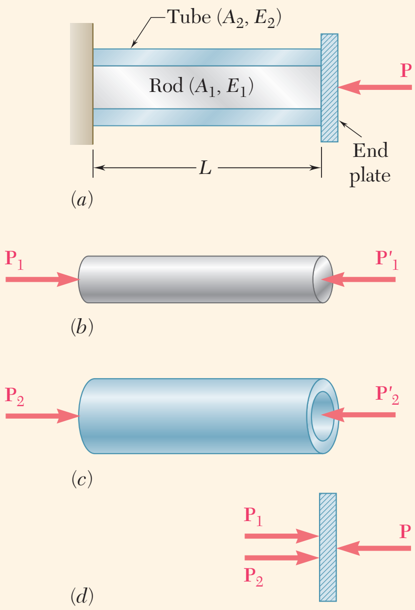


Fig. 2.21 (*a*) Concentric rod and tube, loaded by force *P*. (*b*) Free-body diagram of rod. (*c*) Free-body diagram of tube. (*d*) Free-body diagram of end plate.

A rod of length L, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length L, but of cross-sectional area A_2 and modulus of elasticity E_2 (Fig. 2.21*a*). What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?

The axial forces in the rod and in the tube are P_1 and P_2, respectively. Draw free-body diagrams of all three elements (Fig. 2.21*b*, *c*, *d*). Only Fig. 2.21*d* yields any significant information, as:

$$P_1 + P_2 = P \tag{1}$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal. Recalling Eq. (2.9), write

$$\delta_1 = \frac{P_1 L}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L}{A_2 E_2} \tag{2}$$

Equating the deformations δ_1 and δ_2,

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \tag{3}$$

Equations (1) and (3) can be solved simultaneously for P_1 and P_2:

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

Either of Eqs. (2) can be used to determine the common deformation of the rod and tube.

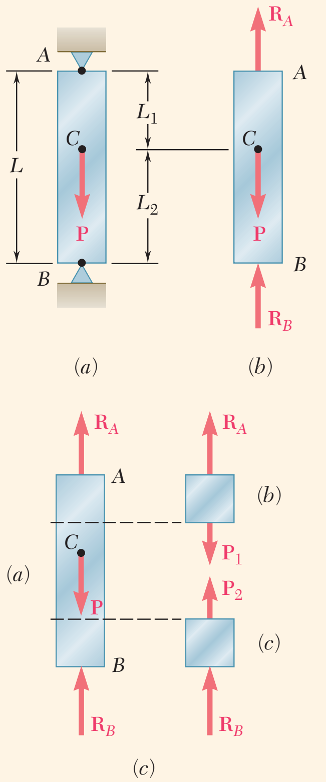


Fig. 2.22 (*a*) Restrained bar with axial load. (*b*) Free-body diagram of bar. (*c*) Free-body diagrams of sections above and below point *C* used to determine internal forces P_1 and P_2.

Concept Application 2.3

A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 2.22*a*)?

Drawing the free-body diagram of the bar (Fig. 2.22*b*), the equilibrium equation is

$$R_A + R_B = P \tag{1}$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B, the problem is statically indeterminate.

However, the reactions can be determined if observed from the geometry that the total elongation δ of the bar must be zero. The elongations of the portions AC and BC are respectively δ_1 and δ_2, so

$$\delta = \delta_1 + \delta_2 = 0$$

Using Eq. (2.9), δ_1 and δ_2 can be expressed in terms of the corresponding internal forces P_1 and P_2,

$$\delta = \frac{P_1 L_1}{AE} + \frac{P_2 L_2}{AE} = 0 \tag{2}$$

Note from the free-body diagrams shown in parts *b* and *c* of Fig. 2.22*c* that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into Equation (2),

$$R_A L_1 - R_B L_2 = 0 \tag{3}$$

Equations (1) and (3) can be solved simultaneously for R_A and R_B, as $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses σ_1 in AC and σ_2 in BC are obtained by dividing $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar:

$$\sigma_1 = \frac{PL_2}{AL} \qquad \sigma_2 = -\frac{PL_1}{AL}$$

Superposition Method. A structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support. Since the stated conditions of the problem cannot be changed, the redundant reaction must be maintained in the solution. It will be treated as an *unknown load* that, together with the other loads, must produce deformations compatible with the original constraints. The actual solution of the problem considers separately the deformations caused by the given loads and the redundant reaction, and by adding—or *superposing*—the results obtained. The general conditions under which the combined effect of several loads can be obtained in this way are discussed in Sec. 2.5.

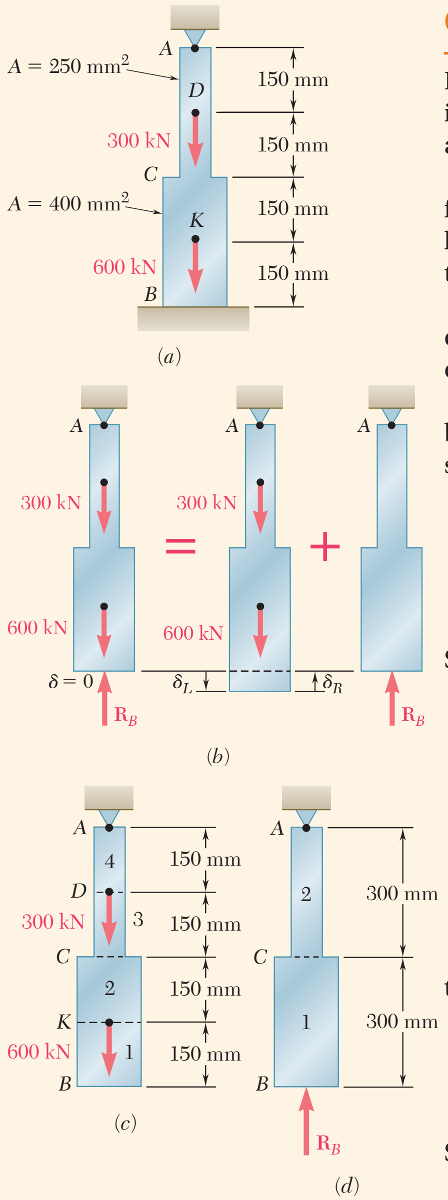


Fig. 2.23 (*a*) Restrained axially-loaded bar. (*b*) Reactions will be found by releasing constraint at point *B* and adding compressive force at point *B* to enforce zero deformation at point *B*. (*c*) Free-body diagram of released structure. (*d*) Free-body diagram of added reaction force at point *B* to enforce zero deformation at point *B*.

Concept Application 2.4

Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.23a, assuming a close fit at both supports before the loads are applied.

We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is considered to be an unknown load and is determined from the condition that the deformation δ of the bar equals zero.

The solution is carried out by considering the deformation δ_L caused by the given loads and the deformation δ_R due to the redundant reaction $\mathbf{R}_B$ (Fig. 2.23*b*).

The deformation δ_L is obtained from Eq. (2.10) after the bar has been divided into four portions, as shown in Fig. 2.23*c*. Follow the same procedure as in Concept Application 2.1:

$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3\,\text{N} \qquad P_4 = 900 \times 10^3\,\text{N}$$

$$A_1 = A_2 = 400 \times 10^{-6}\,\text{m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\,\text{m}$$

Substituting these values into Eq. (2.10),

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3\,\text{N}}{400 \times 10^{-6}\,\text{m}^2} + \frac{600 \times 10^3\,\text{N}}{250 \times 10^{-6}\,\text{m}^2} + \frac{900 \times 10^3\,\text{N}}{250 \times 10^{-6}\,\text{m}^2}\right)\frac{0.150\,\text{m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \tag{1}$$

Considering now the deformation δ_R due to the redundant reaction R_B, the bar is divided into two portions, as shown in Fig. 2.23*d*

$$P_1 = P_2 = -R_B$$

$$A_1 = 400 \times 10^{-6}\,\text{m}^2 \qquad A_2 = 250 \times 10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = 0.300\,\text{m}$$

Substituting these values into Eq. (2.10),

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \tag{2}$$

Express the total deformation δ of the bar as zero:

$$\delta = \delta_L + \delta_R = 0 \tag{3}$$

and, substituting for δ_L and δ_R from Eqs. (1) and (2) into Eqs. (3),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

(continued)

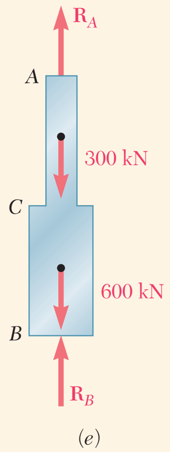


Fig. 2.23 (cont.) (*e*) Complete free-body diagram of *ACB*.

Solving for R_B,

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.23*e*),

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$

$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN}$$

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. Note that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions.

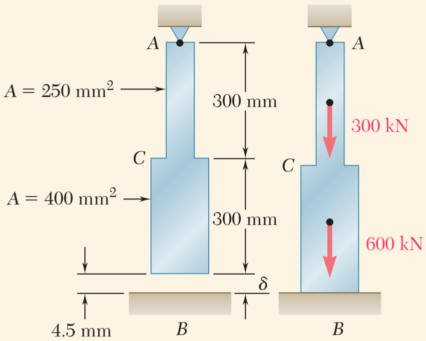


Fig. 2.24 Multi-section bar of Concept Application 2.4 with initial 4.5-mm gap at point *B*. Loading brings bar into contact with constraint.

Concept Application 2.5

Determine the reactions at *A* and *B* for the steel bar and loading of Concept Application 2.4, assuming now that a 4.5-mm clearance exists between the bar and the ground before the loads are applied (Fig. 2.24). Assume $E = 200$ GPa.

Considering the reaction at *B* to be redundant, compute the deformations δ_L and δ_R caused by the given loads and the redundant reaction $\mathbf{R}_B$. However, in this case, the total deformation is $\delta = 4.5$ mm. Therefore,

$$\delta = \delta_L + \delta_R = 4.5 \times 10^{-3} \text{ m} \qquad \textbf{(1)}$$

Substituting for δ_L and δ_R into (Eq. 1), and recalling that $E = 200$ GPa $= 200 \times 10^9$ Pa,

$$\delta = \frac{1.125 \times 10^9}{200 \times 10^9} - \frac{(1.95 \times 10^3)R_B}{200 \times 10^9} = 4.5 \times 10^{-3} \text{ m}$$

Solving for R_B,

$$R_B = 115.4 \times 10^3 \text{ N} = 115.4 \text{ kN}$$

The reaction at *A* is obtained from the free-body diagram of the bar (Fig. 2.23*e*):

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$

$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 115.4 \text{ kN} = 785 \text{ kN}$$

2.3 PROBLEMS INVOLVING TEMPERATURE CHANGES

Consider a homogeneous rod *AB* of uniform cross section that rests freely on a smooth horizontal surface (Fig. 2.25*a*). If the temperature of the rod is raised by ΔT, the rod elongates by an amount δ_T that is proportional to both the temperature change ΔT and the length L of the rod (Fig. 2.25*b*). Here

$$\delta_T = \alpha(\Delta T)L \qquad \textbf{(2.13)}$$

where α is a constant characteristic of the material called the *coefficient of thermal expansion*. Since δ_T and L are both expressed in units of length, α represents a quantity *per degree C* or *per degree F*, depending whether the temperature change is expressed in degrees Celsius or Fahrenheit.

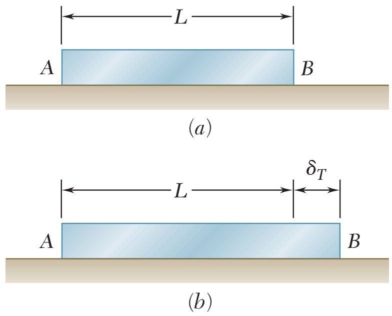


Fig. 2.25 Elongation of an unconstrained rod due to temperature increase.

Associated with deformation δ_T must be a strain $\epsilon_T = \delta_T/L$. Recalling Eq. (2.13),

$$\epsilon_T = \alpha\Delta T \qquad \textbf{(2.14)}$$

The strain ϵ_T is called a *thermal strain*, as it is caused by the change in temperature of the rod. However, there is *no stress associated with the strain* ϵ_T.

Assume the same rod *AB* of length L is placed between two fixed supports at a distance L from each other (Fig. 2.26*a*). Again, there is neither stress nor strain in this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the restraints imposed on its ends; the elongation δ_T of the rod is zero. Since the rod is homogeneous and of uniform cross section, the strain ϵ_T at any point is $\epsilon_T = \delta_T/L$ and thus is also zero. However, the supports will exert equal and opposite forces **P** and **P**′ on the rod after the temperature has been raised, to keep it from elongating (Fig. 2.26*b*). It follows that a state of stress (with no corresponding strain) is created in the rod.

The problem created by the temperature change ΔT is statically indeterminate. Therefore, the magnitude P of the reactions at the supports is determined from the condition that the elongation of the rod is zero.

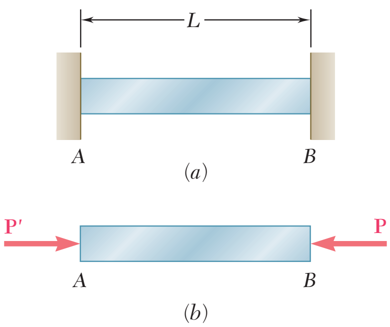


Fig. 2.26 Force *P* develops when the temperature of the rod increases while ends *A* and *B* are restrained.

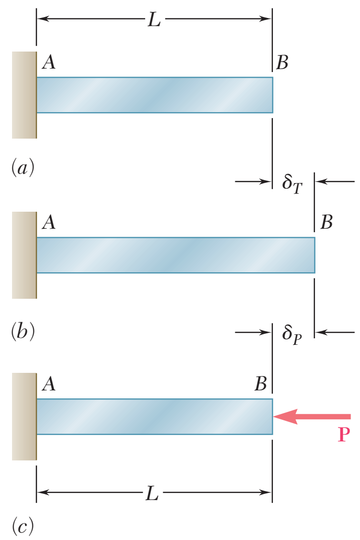


Fig. 2.27 Superposition method to find force at point *B* of restrained rod *AB* undergoing thermal expansion. (*a*) Initial rod length; (*b*) thermally expanded rod length; (*c*) force *P* pushes point *B* back to zero deformation.

Using the superposition method described in Sec. 2.2, the rod is detached from its support *B* (Fig. 2.27*a*) and elongates freely as it undergoes the temperature change ΔT (Fig. 2.27*b*). According to Eq. (2.13), the corresponding elongation is

$$\delta_T = \alpha(\Delta T)L$$

Applying now to end *B* the force **P** representing the redundant reaction, and recalling Eq. (2.9), a second deformation (Fig. 2.27*c*) is

$$\delta_P = \frac{PL}{AE}$$

Expressing that the total deformation δ must be zero,

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

from which

$$P = -AE\alpha(\Delta T)$$

The stress in the rod due to the temperature change ΔT is

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T) \tag{2.15}$$

The absence of any strain in the rod *applies only in the case of a homogeneous rod of uniform cross section.* Any other problem involving a restrained structure undergoing a change in temperature must be analyzed on its own merits. However, the same general approach can be used by considering the deformation due to the temperature change and the deformation due to the redundant reaction separately and superposing the two solutions obtained.

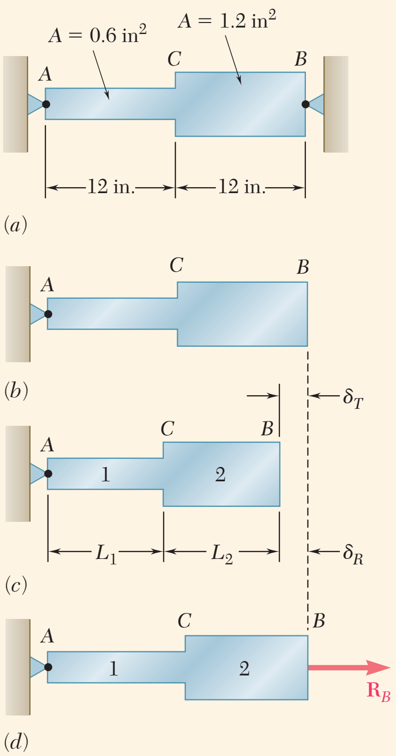


Fig. 2.28 (*a*) Restrained bar. (*b*) Bar at +75°F temperature. (*c*) Bar at lower temperature. (*d*) Force R_B needed to enforce zero deformation at point B.

Concept Application 2.6

Determine the values of the stress in portions *AC* and *CB* of the steel bar shown (Fig. 2.28*a*) when the temperature of the bar is −50°F, knowing that a close fit exists at both of the rigid supports when the temperature is +75°F. Use the values $E = 29 \times 10^6$ psi and $\alpha = 6.5 \times 10^{-6}/°\text{F}$ for steel.

Determine the reactions at the supports. Since the problem is statically indeterminate, detach the bar from its support at *B* and let it undergo the temperature change

$$\Delta T = (-50°\text{F}) - (75°\text{F}) = -125°\text{F}$$

The corresponding deformation (Fig. 2.28*c*) is

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})(24 \text{ in.})\\ &= -19.50 \times 10^{-3} \text{ in.}\end{aligned}$$

Applying the unknown force $\mathbf{R}_B$ at end *B* (Fig. 2.28*d*), use Eq. (2.10) to express the corresponding deformation δ_R. Substituting

$$L_1 = L_2 = 12 \text{ in.}$$
$$A_1 = 0.6 \text{ in}^2 \qquad A_2 = 1.2 \text{ in}^2$$
$$P_1 = P_2 = R_B \qquad E = 29 \times 10^6 \text{ psi}$$

into Eq. (2.10), write

$$\begin{aligned}\delta_R &= \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E}\\ &= \frac{R_B}{29 \times 10^6 \text{ psi}}\left(\frac{12 \text{ in.}}{0.6 \text{ in}^2} + \frac{12 \text{ in.}}{1.2 \text{ in}^2}\right)\\ &= (1.0345 \times 10^{-6} \text{ in./lb})R_B\end{aligned}$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, write

$$\begin{aligned}\delta &= \delta_T + \delta_R = 0\\ &= -19.50 \times 10^{-3} \text{ in.} + (1.0345 \times 10^{-6} \text{ in./lb})R_B = 0\end{aligned}$$

from which

$$R_B = 18.85 \times 10^3 \text{ lb} = 18.85 \text{ kips}$$

The reaction at *A* is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 18.85$ kips, obtain the following values of the stress in portions *AC* and *CB* of the bar:

(continued)

$$\sigma_1 = \frac{P_1}{A_1} = \frac{18.85 \text{ kips}}{0.6 \text{ in}^2} = +31.42 \text{ ksi}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{18.85 \text{ kips}}{1.2 \text{ in}^2} = +15.71 \text{ ksi}$$

It cannot emphasized too strongly that, while the *total deformation* of the bar must be zero, the deformations of the portions AC and CB *are not zero*. A solution of the problem based on the assumption that these deformations are zero would therefore be wrong. Neither can the values of the strain in AC or CB be assumed equal to zero. To amplify this point, determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. 2.28*c*). From Eq. (2.14),

$$\begin{aligned}\epsilon_T = \alpha\,\Delta T &= (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})\\ &= -812.5 \times 10^{-6} \text{ in./in.}\end{aligned}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force $\mathbf{R}_B$ applied to the bar (Fig. 2.28*d*). From Hooke's law, express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{+31.42 \times 10^3 \text{ psi}}{29 \times 10^6 \text{ psi}} = +1083.4 \times 10^{-6} \text{ in./in.}$$

Add the two components of the strain in AC to obtain

$$\begin{aligned}\epsilon_{AC} = \epsilon_T + \frac{\sigma_1}{E} &= -812.5 \times 10^{-6} + 1083.4 \times 10^{-6}\\ &= +271 \times 10^{-6} \text{ in./in.}\end{aligned}$$

A similar computation yields the strain in portion CB of the bar:

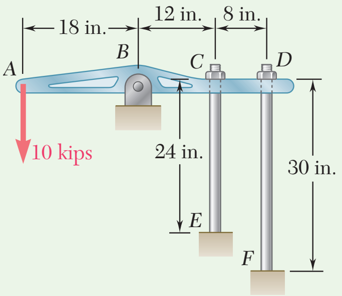
$$\begin{aligned}\epsilon_{CB} = \epsilon_T + \frac{\sigma_2}{E} &= -812.5 \times 10^{-6} + 541.7 \times 10^{-6}\\ &= -271 \times 10^{-6} \text{ in./in.}\end{aligned}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are

$$\begin{aligned}\delta_{AC} = \epsilon_{AC}(AC) &= (+271 \times 10^{-6})(12 \text{ in.})\\ &= +3.25 \times 10^{-3} \text{ in.}\\ \delta_{CB} = \epsilon_{CB}(CB) &= (-271 \times 10^{-6})(12 \text{ in.})\\ &= -3.25 \times 10^{-3} \text{ in.}\end{aligned}$$

Thus, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero.

Sample Problem 2.3



The $\frac{1}{2}$-in.-diameter rod *CE* and the $\frac{3}{4}$-in.-diameter rod *DF* are attached to the rigid bar *ABCD* as shown. Knowing that the rods are made of aluminum and using $E = 10.6 \times 10^6$ psi, determine (*a*) the force in each rod caused by the loading shown and (*b*) the corresponding deflection of point *A*.

STRATEGY: To solve this statically indeterminate problem, you must supplement static equilibrium with a relative deflection analysis of the two rods.

MODELING: Draw the free body diagram of the bar (Fig. 1)

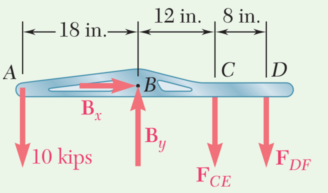


Fig. 1 Free-body diagram of rigid bar *ABCD*.

ANALYSIS:

Statics. Considering the free body of bar *ABCD* in Fig. 1, note that the reaction at *B* and the forces exerted by the rods are indeterminate. However, using statics,

$$+\circlearrowleft \Sigma M_B = 0: \quad (10 \text{ kips})(18 \text{ in.}) - F_{CE}(12 \text{ in.}) - F_{DF}(20 \text{ in.}) = 0$$

$$12F_{CE} + 20F_{DF} = 180 \qquad \textbf{(1)}$$

Geometry. After application of the 10-kip load, the position of the bar is $A'BC'D'$ (Fig. 2). From the similar triangles BAA', BCC', and BDD',

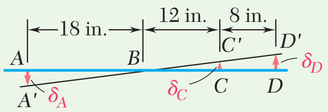


Fig. 2 Linearly proportional displacements along rigid bar *ABCD*.

$$\frac{\delta_C}{12 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_C = 0.6\delta_D \qquad \textbf{(2)}$$

$$\frac{\delta_A}{18 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_A = 0.9\delta_D \qquad \textbf{(3)}$$

Deformations. Using Eq. (2.9), and the data shown in Fig. 3, write

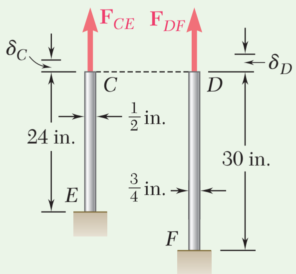


Fig. 3 Forces and deformations in *CE* and *DF*.

$$\delta_C = \frac{F_{CE}L_{CE}}{A_{CE}E} \qquad \delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E}$$

Substituting for δ_C and δ_D into Eq. (2), write

$$\delta_C = 0.6\delta_D \qquad \frac{F_{CE}L_{CE}}{A_{CE}E} = 0.6\frac{F_{DF}L_{DF}}{A_{DF}E}$$

$$F_{CE} = 0.6\frac{L_{DF}}{L_{CE}}\frac{A_{CE}}{A_{DF}}F_{DF} = 0.6\left(\frac{30 \text{ in.}}{24 \text{ in.}}\right)\left[\frac{\frac{1}{4}\pi(\frac{1}{2} \text{ in.})^2}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2}\right]F_{DF} \qquad F_{CE} = 0.333F_{DF}$$

Force in Each Rod. Substituting for F_{CE} into Eq. (1) and recalling that all forces have been expressed in kips,

$$12(0.333F_{DF}) + 20F_{DF} = 180 \qquad F_{DF} = 7.50 \text{ kips} \blacktriangleleft$$

$$F_{CE} = 0.333F_{DF} = 0.333(7.50 \text{ kips}) \qquad F_{CE} = 2.50 \text{ kips} \blacktriangleleft$$

(continued)

Deflections. The deflection of point D is

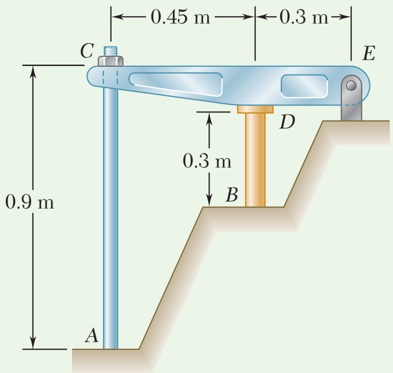
$$\delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E} = \frac{(7.50 \times 10^3 \text{ lb})(30 \text{ in.})}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2(10.6 \times 10^6 \text{ psi})} \qquad \delta_D = 48.0 \times 10^{-3} \text{ in.}$$

Using Eq. (3),

$$\delta_A = 0.9\delta_D = 0.9(48.0 \times 10^{-3} \text{ in.}) \qquad \delta_A = 43.2 \times 10^{-3} \text{ in.} \blacktriangleleft$$

REFLECT and THINK: You should note that as the rigid bar rotates about B, the deflections at C and D are proportional to their distance from the pivot point B, but *the forces exerted by the rods at these points are not*. Being statically indeterminate, these forces depend upon the deflection attributes of the rods as well as the equilibrium of the rigid bar.

Sample Problem 2.4



The rigid bar CDE is attached to a pin support at E and rests on the 30-mm-diameter brass cylinder BD. A 22-mm-diameter steel rod AC passes through a hole in the bar and is secured by a nut that is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C, while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod AC: Steel	Cylinder BD: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°C$	$\alpha = 20.9 \times 10^{-6}/°C$

STRATEGY: You can use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at B removed, the temperature rise of the cylinder causes point B to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1, equal to δ_T so that the final deflection of B will be zero (Fig. 2)

MODELING: Draw the free-body diagram of the entire assembly (Fig. 1).

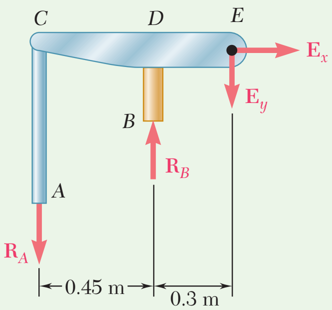


Fig. 1 Free-body diagram of bolt, cylinder and bar.

ANALYSIS:

Statics. Considering the free body of the entire assembly, write

$$+\circlearrowleft \Sigma M_E = 0: \qquad R_A(0.75 \text{ m}) - R_B(0.3 \text{ m}) = 0 \qquad R_A = 0.4R_B \qquad \textbf{(1)}$$

(continued)

Deflection δ_T. Because of a temperature rise of $50° - 20° = 30°C$, the length of the brass cylinder increases by δ_T. (Fig. 2*a*).

$$\delta_T = L(\Delta T)\alpha = (0.3 \text{ m})(30°C)(20.9 \times 10^{-6}/°C) = 188.1 \times 10^{-6} \text{ m} \downarrow$$

Deflection δ_1. From Fig. 2*b*, note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9 \text{ m})}{\frac{1}{4}\pi(0.022 \text{ m})^2(200 \text{ GPa})} = 11.84 \times 10^{-9} R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9} R_A) = 4.74 \times 10^{-9} R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3 \text{ m})}{\frac{1}{4}\pi(0.03 \text{ m})^2(105 \text{ GPa})} = 4.04 \times 10^{-9} R_B \uparrow$$

Recall from Eq. (1) that $R_A = 0.4R_B$, so

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9} R_B \uparrow$$

But $\delta_T = \delta_1$: $\quad 188.1 \times 10^{-6} \text{ m} = 5.94 \times 10^{-9} R_B \quad R_B = 31.7 \text{ kN}$

Stress in Cylinder: $\sigma_B = \dfrac{R_B}{A} = \dfrac{31.7 \text{ kN}}{\frac{1}{4}\pi(0.03 \text{ m})^2}$ $\quad \sigma_B = 44.8 \text{ MPa}$ ◀

REFLECT and THINK: This example illustrates the large stresses that can develop in statically indeterminate systems due to even modest temperature changes. Note that if this assembly was statically determinate (i.e., the steel rod was removed), no stress at all would develop in the cylinder due to the temperature change.

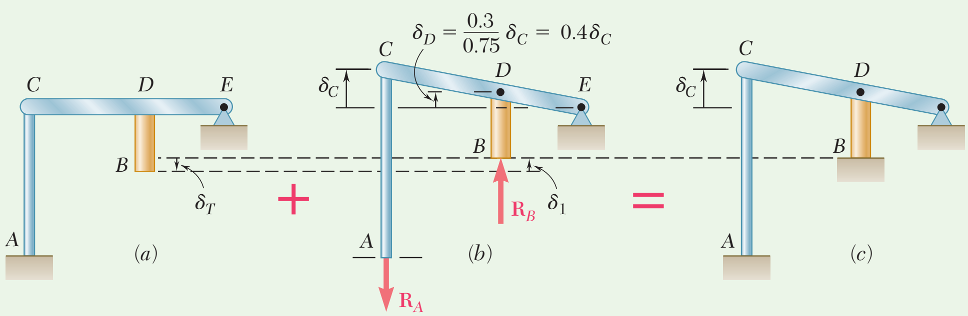


Fig. 2 Superposition of thermal and restraint force deformations (*a*) Support at *B* removed. (*b*) Reaction at *B* applied. (*c*) Final position.

Problems

2.33 An axial centric force of magnitude $P = 450$ kN is applied to the composite block shown by means of a rigid end plate. Knowing that $h = 10$ mm, determine the normal stress in (*a*) the brass core, (*b*) the aluminum plates.

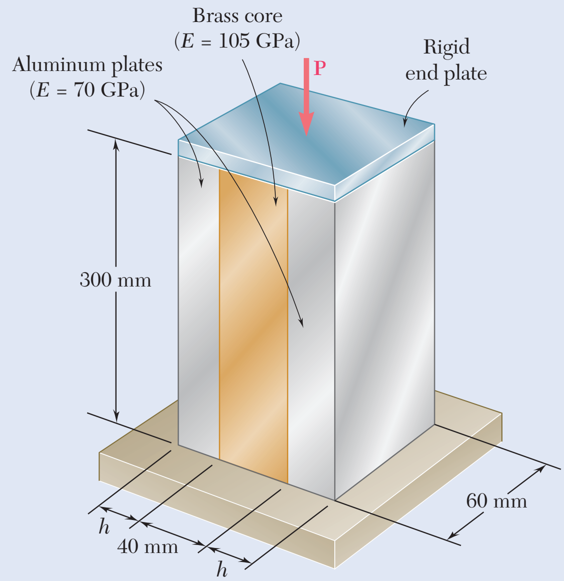


Fig. P2.33

2.34 For the composite block shown in Prob. 2.33, determine (*a*) the value of h if the portion of the load carried by the aluminum plates is half the portion of the load carried by the brass core, (*b*) the total load if the stress in the brass is 80 MPa.

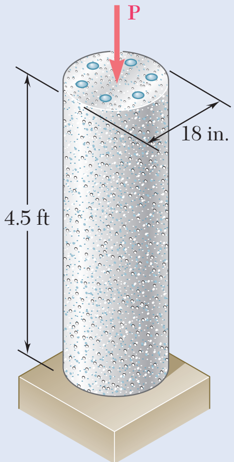


Fig. P2.35

2.35 The 4.5-ft concrete post is reinforced with six steel bars, each with a $1\frac{1}{8}$-in. diameter. Knowing that $E_s = 29 \times 10^6$ psi and $E_c = 4.2 \times 10^6$ psi, determine the normal stresses in the steel and in the concrete when a 350-kip axial centric force **P** is applied to the post.

2.36 For the post of Prob. 2.35, determine the maximum centric force that can be applied if the allowable normal stress is 20 ksi in the steel and 2.4 ksi in the concrete.

2.37 An axial force of 200 kN is applied to the assembly shown by means of rigid end plates. Determine (*a*) the normal stress in the aluminum shell, (*b*) the corresponding deformation of the assembly.

2.38 The length of the assembly shown decreases by 0.40 mm when an axial force is applied by means of rigid end plates. Determine (*a*) the magnitude of the applied force, (*b*) the corresponding stress in the brass core.

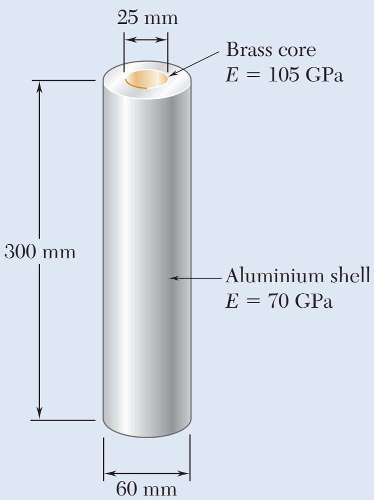


Fig. *P2.37* and *P2.38*

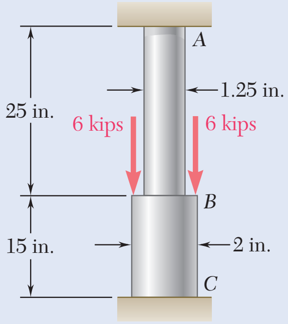


Fig. P2.39

2.39 A polystyrene rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends and supports two 6-kip loads as shown. Knowing that $E = 0.45 \times 10^6$ psi, determine (*a*) the reactions at *A* and *C*, (*b*) the normal stress in each portion of the rod.

2.40 Three steel rods ($E = 29 \times 10^6$ psi) support an 8.5-kip load **P**. Each of the rods *AB* and *CD* has a 0.32-in^2 cross-sectional area and rod *EF* has a 1-in^2 cross-sectional area. Neglecting the deformation of bar *BED*, determine (*a*) the change in length of rod *EF*, (*b*) the stress in each rod.

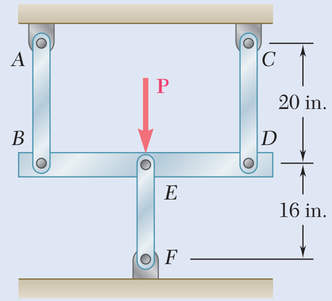


Fig. *P2.40*

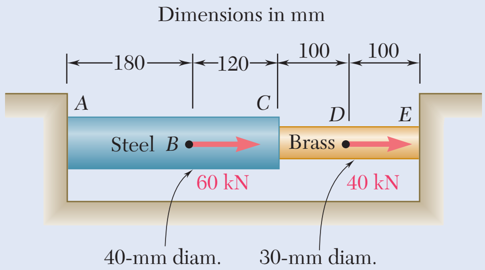


Fig. P2.41

2.41 Two cylindrical rods, one of steel and the other of brass, are joined at *C* and restrained by rigid supports at *A* and *E*. For the loading shown and knowing that $E_s = 200$ GPa and $E_b = 105$ GPa, determine (*a*) the reactions at *A* and *E*, (*b*) the deflection of point *C*.

2.42 Solve Prob. 2.41, assuming that rod *AC* is made of brass and rod *CE* is made of steel.

2.43 Each of the rods *BD* and *CE* is made of brass ($E = 105$ GPa) and has a cross-sectional area of 200 mm^2. Determine the deflection of end *A* of the rigid member *ABC* caused by the 2-kN load.

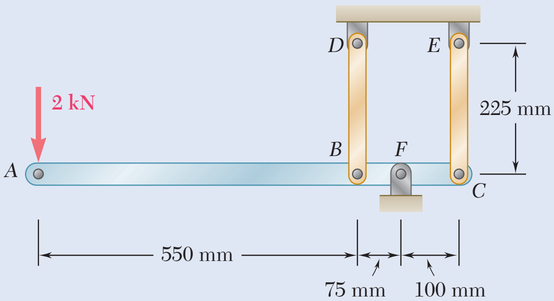


Fig. P2.43

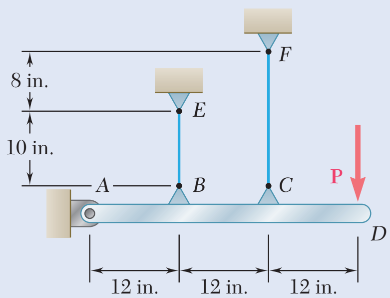


Fig. P2.44

2.44 The rigid bar *AD* is supported by two steel wires of $\frac{1}{16}$-in. diameter ($E = 29 \times 10^6$ psi) and a pin and bracket at *A*. Knowing that the wires were initially taut, determine (*a*) the additional tension in each wire when a 220-lb load **P** is applied at *D*, (*b*) the corresponding deflection of point *D*.

2.45 The rigid bar *ABC* is suspended from three wires of the same material. The cross-sectional area of the wire at *B* is equal to half of the cross-sectional area of the wires at *A* and *C*. Determine the tension in each wire caused by the load **P** shown.

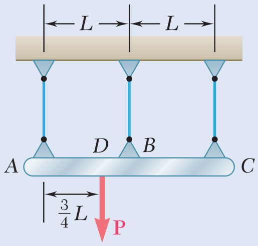


Fig. P2.45

2.46 The rigid bar *AD* is supported by two steel wires of $\frac{1}{16}$-in. diameter ($E = 29 \times 10^6$ psi) and a pin and bracket at *D*. Knowing that the wires were initially taut, determine (*a*) the additional tension in each wire when a 120-lb load **P** is applied at *B*, (*b*) the corresponding deflection of point *B*.

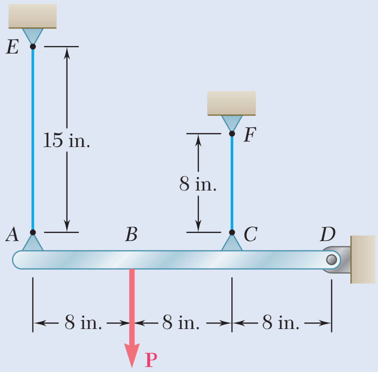


Fig. P2.46

2.47 The aluminum shell is fully bonded to the brass core and the assembly is unstressed at a temperature of 15°C. Considering only axial deformations, determine the stress in the aluminum when the temperature reaches 195°C.

2.48 Solve Prob. 2.47, assuming that the core is made of steel (E_s = 200 GPa, $\alpha_s = 11.7 \times 10^{-6}$/°C) instead of brass.

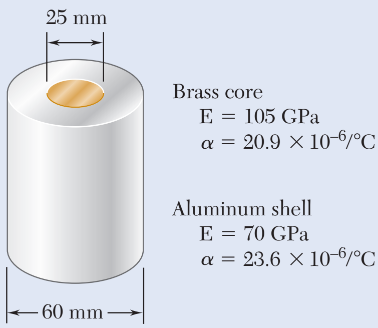


Fig. P2.47

2.49 The brass shell ($\alpha_b = 11.6 \times 10^{-6}$/°F) is fully bonded to the steel core ($\alpha_s = 6.5 \times 10^{-6}$/°F). Determine the largest allowable increase in temperature if the stress in the steel core is not to exceed 8 ksi.

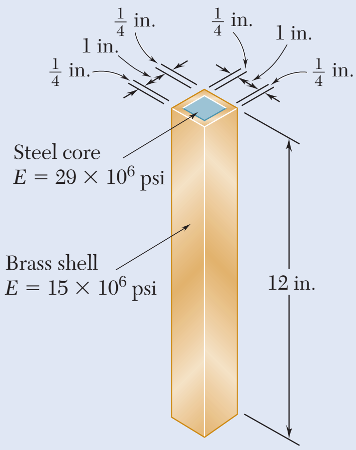


Fig. P2.49

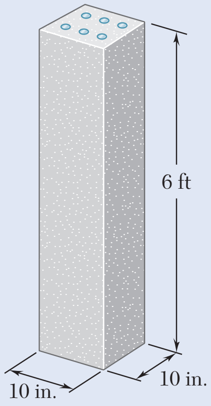


Fig. P2.50

2.50 The concrete post ($E_c = 3.6 \times 10^6$ psi and $\alpha_c = 5.5 \times 10^{-6}/°\text{F}$) is reinforced with six steel bars, each of $\frac{7}{8}$-in. diameter ($E_s = 29 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}/°\text{F}$). Determine the normal stresses induced in the steel and in the concrete by a temperature rise of 65°F.

2.51 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel ($E_s = 200$ GPa, $\alpha_s = 11.7 \times 10^{-6}/°\text{C}$) and portion *BC* is made of brass ($E_b = 105$ GPa, $\alpha_b = 20.9 \times 10^{-6}/°\text{C}$). Knowing that the rod is initially unstressed, determine the compressive force induced in *ABC* when there is a temperature rise of 50°C.

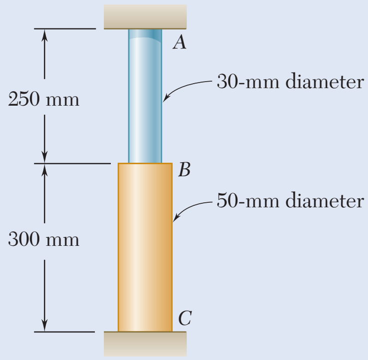


Fig. P2.51

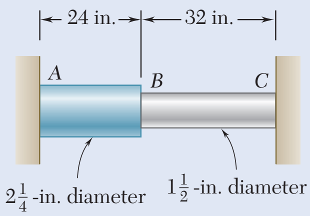


Fig. P2.52

2.52 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/°\text{F}$) and portion *BC* is made of aluminum ($E_a = 10.4 \times 10^6$ psi, $\alpha_a = 13.3 \times 10^{-6}/°\text{F}$). Knowing that the rod is initially unstressed, determine (*a*) the normal stresses induced in portions *AB* and *BC* by a temperature rise of 70°F, (*b*) the corresponding deflection of point *B*.

2.53 Solve Prob. 2.52, assuming that portion *AB* of the composite rod is made of aluminum and portion *BC* is made of steel.

2.54 The steel rails of a railroad track ($E_s = 200$ GPa, $\alpha_s = 11.7 \times 10^{-6}/°\text{C}$) were laid at a temperature of 6°C. Determine the normal stress in the rails when the temperature reaches 48°C, assuming that the rails (*a*) are welded to form a continuous track, (*b*) are 10 m long with 3-mm gaps between them.

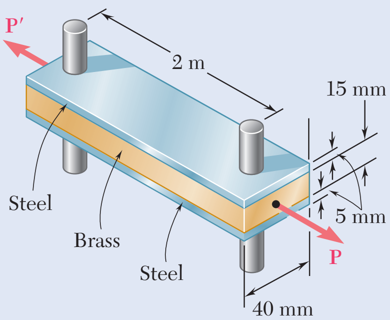


Fig. P2.55

2.55 Two steel bars ($E_s = 200$ GPa and $\alpha_s = 11.7 \times 10^{-6}/°\text{C}$) are used to reinforce a brass bar ($E_b = 105$ GPa, $\alpha_b = 20.9 \times 10^{-6}/°\text{C}$) that is subjected to a load $P = 25$ kN. When the steel bars were fabricated, the distance between the centers of the holes that were to fit on the pins was made 0.5 mm smaller than the 2 m needed. The steel bars were then placed in an oven to increase their length so that they would just fit on the pins. Following fabrication, the temperature in the steel bars dropped back to room temperature. Determine (*a*) the increase in temperature that was required to fit the steel bars on the pins, (*b*) the stress in the brass bar after the load is applied to it.

2.56 Determine the maximum load P that can be applied to the brass bar of Prob. 2.55 if the allowable stress in the steel bars is 30 MPa and the allowable stress in the brass bar is 25 MPa.

2.57 An aluminum rod (E_a = 70 GPa, α_a = 23.6 × 10^{-6}/°C) and a steel link (E_s = 200 GPa, α_s = 11.7 × 10^{-6}/°C) have the dimensions shown at a temperature of 20°C. The steel link is heated until the aluminum rod can be fitted freely into the link. The temperature of the whole assembly is then raised to 150°C. Determine the final normal stress (*a*) in the rod, (*b*) in the link.

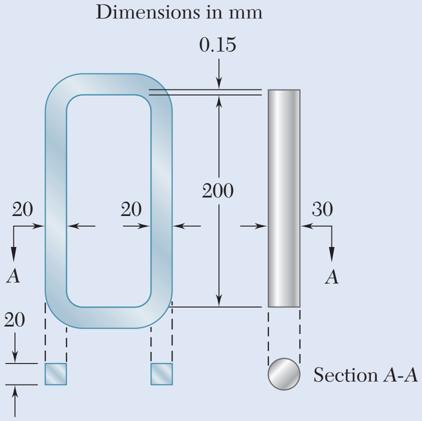


Fig. *P2.57*

2.58 Knowing that a 0.02-in. gap exists when the temperature is 75°F, determine (*a*) the temperature at which the normal stress in the aluminum bar will be equal to −11 ksi, (*b*) the corresponding exact length of the aluminum bar.

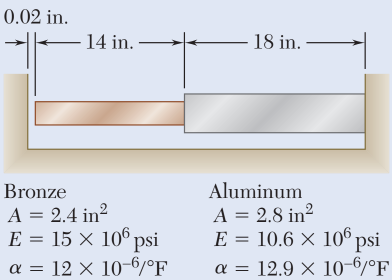


Fig. P2.58 and P2.59

2.59 Determine (*a*) the compressive force in the bars shown after a temperature rise of 180°F, (*b*) the corresponding change in length of the bronze bar.

2.60 At room temperature (20°C) a 0.5-mm gap exists between the ends of the rods shown. At a later time when the temperature has reached 140°C, determine (*a*) the normal stress in the aluminum rod, (*b*) the change in length of the aluminum rod.

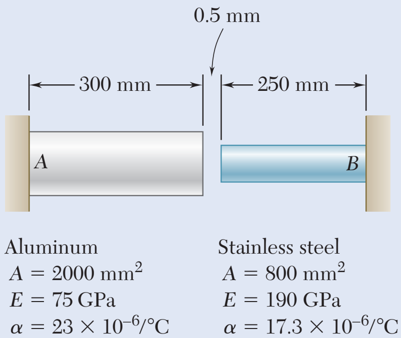


Fig. *P2.60*

2.4 POISSON'S RATIO

When a homogeneous slender bar is axially loaded, the resulting stress and strain satisfy Hooke's law, as long as the elastic limit of the material is not exceeded. Assuming that the load **P** is directed along the x axis (Fig. 2.29*a*), $\sigma_x = P/A$, where A is the cross-sectional area of the bar, and from Hooke's law,

$$\epsilon_x = \sigma_x/E \tag{2.16}$$

where E is the modulus of elasticity of the material.

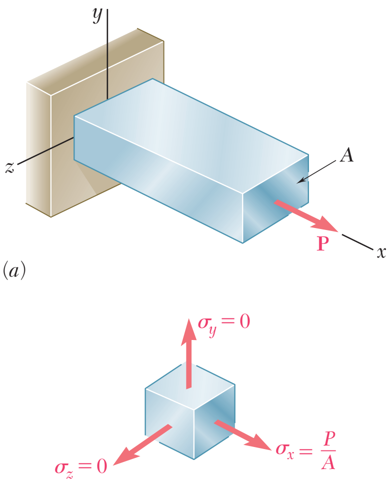


Fig. 2.29 A bar in uniaxial tension and a representative stress element.

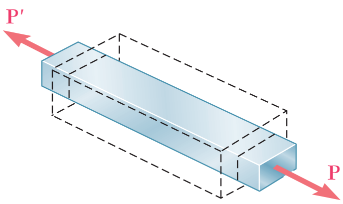


Fig. 2.30 Materials undergo transverse contraction when elongated under axial load.

Also, the normal stresses on faces perpendicular to the y and z axes are zero: $\sigma_y = \sigma_z = 0$ (Fig. 2.29*b*). It would be tempting to conclude that the corresponding strains ϵ_y and ϵ_z are also zero. This *is not the case*. In all engineering materials, the elongation produced by an axial tensile force **P** in the direction of the force is accompanied by a contraction in any transverse direction (Fig. 2.30).[†] In this section and the following sections, all materials are assumed to be both *homogeneous* and *isotropic* (i.e., their mechanical properties are independent of both *position* and *direction*). It follows that the strain must have the same value for any transverse direction. Therefore, the loading shown in Fig. 2.29 must have $\epsilon_y = \epsilon_z$. This common value is the *lateral strain*. An important constant for a given material is its *Poisson's ratio*, named after the French mathematician Siméon Denis Poisson (1781–1840) and denoted by the Greek letter ν (nu).

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.17}$$

or

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.18}$$

for the loading condition represented in Fig. 2.29. Note the use of a minus sign in these equations to obtain a positive value for ν, as the axial and lateral strains have opposite signs for all engineering materials.[‡] Solving Eq. (2.18) for ϵ_y and ϵ_z, and recalling Eq. (2.16), write the following relationships, which fully describe the condition of strain under an axial load applied in a direction parallel to the x axis:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.19}$$

[†] It also would be tempting, but equally wrong, to assume that the volume of the rod remains unchanged as a result of the combined effect of the axial elongation and transverse contraction (see Sec. 2.6).

[‡] However, some experimental materials, such as polymer foams, expand laterally when stretched. Since the axial and lateral strains have then the same sign, Poisson's ratio of these materials is negative. (*See* Roderic Lakes, "Foam Structures with a Negative Poisson's Ratio," *Science*, 27 February 1987, Volume 235, pp. 1038–1040.)

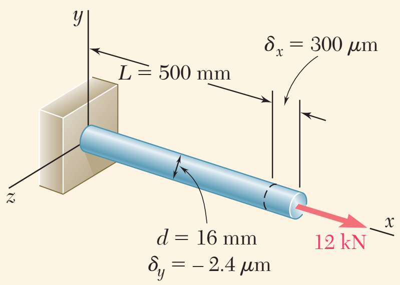


Fig. 2.31 Axially loaded rod.

Concept Application 2.7

A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

The cross-sectional area of the rod is

$$A = \pi r^2 = \pi(8 \times 10^{-3}\,\text{m})^2 = 201 \times 10^{-6}\,\text{m}^2$$

Choosing the x axis along the axis of the rod (Fig. 2.31), write

$$\sigma_x = \frac{P}{A} = \frac{12 \times 10^3\,\text{N}}{201 \times 10^{-6}\,\text{m}^2} = 59.7\,\text{MPa}$$

$$\epsilon_x = \frac{\delta_x}{L} = \frac{300\,\mu\text{m}}{500\,\text{mm}} = 600 \times 10^{-6}$$

$$\epsilon_y = \frac{\delta_y}{d} = \frac{-2.4\,\mu\text{m}}{16\,\text{mm}} = -150 \times 10^{-6}$$

From Hooke's law, $\sigma_x = E\epsilon_x$,

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{59.7\,\text{MPa}}{600 \times 10^{-6}} = 99.5\,\text{GPa}$$

and from Eq. (2.18),

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{-150 \times 10^{-6}}{600 \times 10^{-6}} = 0.25$$

2.5 MULTIAXIAL LOADING: GENERALIZED HOOKE'S LAW

All the examples considered so far in this chapter have dealt with slender members subjected to axial loads, i.e., to forces directed along a single axis. Consider now structural elements subjected to loads acting in the directions of the three coordinate axes and producing normal stresses σ_x, σ_y, and σ_z that are all different from zero (Fig. 2.32). This condition is a

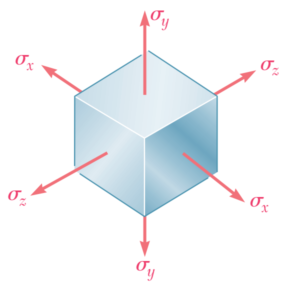


Fig. 2.32 State of stress for multiaxial loading.

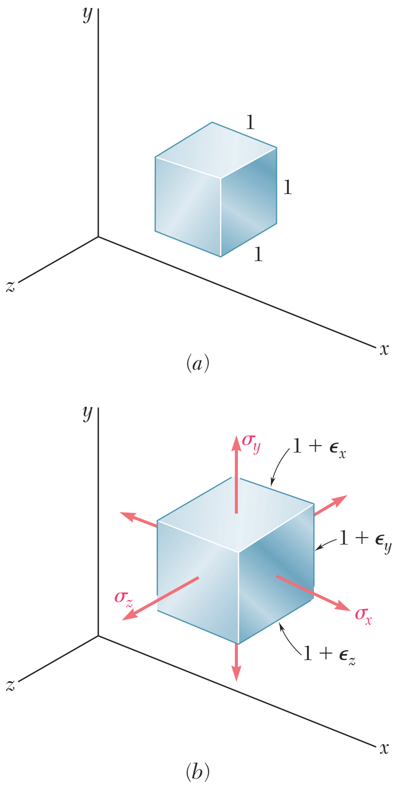


Fig. 2.33 Deformation of unit cube under multiaxial loading: (*a*) unloaded; (*b*) deformed.

multiaxial loading. Note that this is not the general stress condition described in Sec. 1.3, since no shearing stresses are included among the stresses shown in Fig. 2.32.

Consider an element of an isotropic material in the shape of a cube (Fig. 2.33*a*). Assume the side of the cube to be equal to unity, since it is always possible to select the side of the cube as a unit of length. Under the given multiaxial loading, the element will deform into a *rectangular parallelepiped* of sides equal to $1 + \epsilon_x$, $1 + \epsilon_y$, and $1 + \epsilon_z$, where ϵ_x, ϵ_y, and ϵ_z denote the values of the normal strain in the directions of the three coordinate axes (Fig. 2.33*b*). Note that, as a result of the deformations of the other elements of the material, the element under consideration could also undergo a translation, but the concern here is with the *actual deformation* of the element, not with any possible superimposed rigid-body displacement.

In order to express the strain components ϵ_x, ϵ_y, ϵ_z in terms of the stress components σ_x, σ_y, σ_z, consider the effect of each stress component and combine the results. This approach will be used repeatedly in this text, and is based on the *principle of superposition*. This principle states that the effect of a given combined loading on a structure can be obtained by *determining the effects of the various loads separately and combining the results*, provided that the following conditions are satisfied:

1. Each effect is linearly related to the load that produces it.
2. The deformation resulting from any given load is small and does not affect the conditions of application of the other loads.

For multiaxial loading, the first condition is satisfied if the stresses do not exceed the proportional limit of the material, and the second condition is also satisfied if the stress on any given face does not cause deformations of the other faces that are large enough to affect the computation of the stresses on those faces.

Considering the effect of the stress component σ_x, recall from Sec. 2.4 that σ_x causes a strain equal to σ_x/E in the x direction and strains equal to $-\nu\sigma_x/E$ in each of the y and z directions. Similarly, the stress component σ_y, if applied separately, will cause a strain σ_y/E in the y direction and strains $-\nu\sigma_y/E$ in the other two directions. Finally, the stress component σ_z causes a strain σ_z/E in the z direction and strains $-\nu\sigma_z/E$ in the x and y directions. Combining the results, the components of strain corresponding to the given multiaxial loading are

$$
\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned}
\tag{2.20}
$$

Equations (2.20) are the *generalized Hooke's law for the multiaxial loading of a homogeneous isotropic material*. As indicated earlier, these results are valid only as long as the stresses do not exceed the proportional limit and the deformations involved remain small. Also, a positive value for a stress component signifies tension and a negative value compression. Similarly, a positive value for a strain component indicates expansion in the corresponding direction and a negative value contraction.

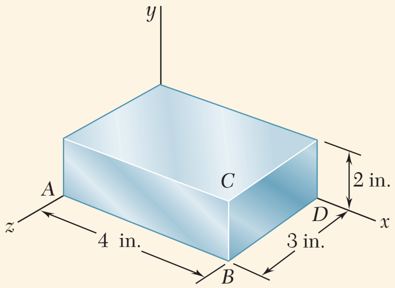


Fig. 2.34 Steel block under uniform pressure *p*.

Concept Application 2.8

The steel block shown (Fig. 2.34) is subjected to a uniform pressure on all its faces. Knowing that the change in length of edge *AB* is -1.2×10^{-3} in., determine (*a*) the change in length of the other two edges and (*b*) the pressure *p* applied to the faces of the block. Assume $E = 29 \times 10^6$ psi and $\nu = 0.29$.

a. Change in Length of Other Edges. Substituting $\sigma_x = \sigma_y = \sigma_z = -p$ into Eqs. (2.20), the three strain components have the common value

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{p}{E}(1 - 2\nu) \quad \textbf{(1)}$$

Since

$$\epsilon_x = \delta_x/AB = (-1.2 \times 10^{-3} \text{ in.})/(4 \text{ in.})$$
$$= -300 \times 10^{-6} \text{ in./in.}$$

obtain

$$\epsilon_y = \epsilon_z = \epsilon_x = -300 \times 10^{-6} \text{ in./in.}$$

from which

$$\delta_y = \epsilon_y(BC) = (-300 \times 10^{-6})(2 \text{ in.}) = -600 \times 10^{-6} \text{ in.}$$
$$\delta_z = \epsilon_z(BD) = (-300 \times 10^{-6})(3 \text{ in.}) = -900 \times 10^{-6} \text{ in.}$$

b. Pressure. Solving Eq. (1) for *p*,

$$p = -\frac{E\epsilon_x}{1 - 2\nu} = -\frac{(29 \times 10^6 \text{ psi})(-300 \times 10^{-6})}{1 - 0.58}$$
$$p = 20.7 \text{ ksi}$$

*2.6 DILATATION AND BULK MODULUS

This section examines the effect of the normal stresses σ_x, σ_y, and σ_z on the volume of an element of isotropic material. Consider the element shown in Fig. 2.33. In its unstressed state, it is in the shape of a cube of unit volume. Under the stresses σ_x, σ_y, σ_z, it deforms into a rectangular parallelepiped of volume

$$v = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)$$

Since the strains ϵ_x, ϵ_y, ϵ_z are much smaller than unity, their products can be omitted in the expansion of the product. Therefore,

$$v = 1 + \epsilon_x + \epsilon_y + \epsilon_z$$

The change in volume *e* of the element is

$$e = v - 1 = 1 + \epsilon_x + \epsilon_y + \epsilon_z - 1$$

or

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{2.21}$$

Since the element originally had a unit volume, e represents *the change in volume per unit volume* and is called the *dilatation* of the material. Substituting for ϵ_x, ϵ_y, and ϵ_z from Eqs. (2.20) into (2.21), the change is

$$e = \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{2\nu(\sigma_x + \sigma_y + \sigma_z)}{E}$$

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{2.22}$$ [†]

When a body is subjected to a uniform hydrostatic pressure p, each of the stress components is equal to $-p$ and Eq. (2.22) yields

$$e = -\frac{3(1 - 2\nu)}{E}p \tag{2.23}$$

Introducing the constant

$$k = \frac{E}{3(1 - 2\nu)} \tag{2.24}$$

Eq. (2.23) is given in the form

$$e = -\frac{p}{k} \tag{2.25}$$

The constant k is known as the *bulk modulus* or *modulus of compression* of the material. It is expressed in pascals or in psi.

Because a stable material subjected to a hydrostatic pressure can only *decrease* in volume, the dilatation e in Eq. (2.25) is negative, and the bulk modulus k is a positive quantity. Referring to Eq. (2.24), $1 - 2\nu > 0$ or $\nu < \frac{1}{2}$. Recall from Sec. 2.4 that ν is positive for all engineering materials. Thus, for any engineering material,

$$0 < \nu < \tfrac{1}{2} \tag{2.26}$$

Note that an ideal material having ν equal to zero can be stretched in one direction without any lateral contraction. On the other hand, an ideal material for which $\nu = \frac{1}{2}$ and $k = \infty$ is perfectly incompressible ($e = 0$). Referring to Eq. (2.22) and noting that since $\nu < \frac{1}{2}$ in the elastic range, stretching an engineering material in one direction, for example in the x direction ($\sigma_x > 0$, $\sigma_y = \sigma_z = 0$), results in an increase of its volume ($e > 0$).[‡]

[†]Since the dilatation e represents a change in volume, it must be independent of the orientation of the element considered. It then follows from Eqs. (2.21) and (2.22) that the quantities $\epsilon_x + \epsilon_y + \epsilon_z$ and $\sigma_x + \sigma_y + \sigma_z$ are also independent of the orientation of the element. This property will be verified in Chap. 7.

[‡]However, in the plastic range, the volume of the material remains nearly constant.

Concept Application 2.9

Determine the change in volume ΔV of the steel block shown in Fig. 2.34, when it is subjected to the hydrostatic pressure $p = 180$ MPa. Use $E = 200$ GPa and $\nu = 0.29$.

From Eq. (2.24), the bulk modulus of steel is

$$k = \frac{E}{3(1 - 2\nu)} = \frac{200 \text{ GPa}}{3(1 - 0.58)} = 158.7 \text{ GPa}$$

and from Eq. (2.25), the dilatation is

$$e = -\frac{p}{k} = -\frac{180 \text{ MPa}}{158.7 \text{ GPa}} = -1.134 \times 10^{-3}$$

Since the volume V of the block in its unstressed state is

$$V = (80 \text{ mm})(40 \text{ mm})(60 \text{ mm}) = 192 \times 10^3 \text{ mm}^3$$

and e represents the change in volume per unit volume, $e = \Delta V/V$,

$$\Delta V = eV = (-1.134 \times 10^{-3})(192 \times 10^3 \text{ mm}^3)$$

$$\Delta V = -218 \text{ mm}^3$$

2.7 SHEARING STRAIN

When we derived in Sec. 2.5 the relations (2.20) between normal stresses and normal strains in a homogeneous isotropic material, we assumed that no shearing stresses were involved. In the more general stress situation represented in Fig. 2.35, shearing stresses τ_{xy}, τ_{yz}, and τ_{zx} are present (as well as the corresponding shearing stresses τ_{yx}, τ_{zy}, and τ_{xz}). These stresses have no direct effect on the normal strains and, as long as all the deformations involved remain small, they will not affect the derivation nor the validity of Eqs. (2.20). The shearing stresses, however, tend to deform a cubic element of material into an *oblique* parallelepiped.

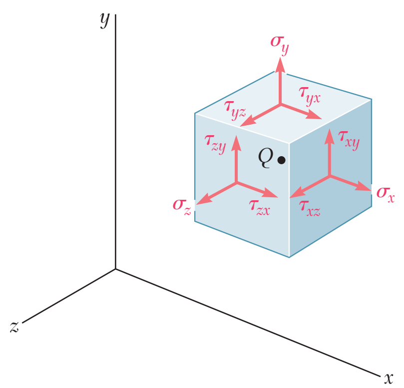


Fig. 2.35 Positive stress components at point *Q* for a general state of stress.

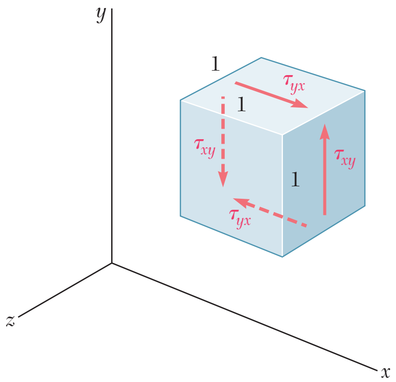


Fig. 2.36 Unit cubic element subjected to shearing stress.

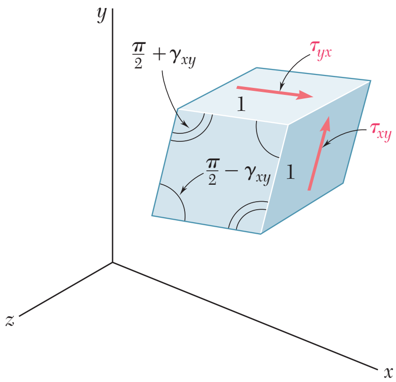


Fig. 2.37 Deformation of unit cubic element due to shearing stress.

Consider a cubic element (Fig. 2.36) subjected to only the shearing stresses τ_{xy} and τ_{yx} applied to faces of the element respectively perpendicular to the x and y axes. (Recall from Sec. 1.4 that $\tau_{xy} = \tau_{yx}$.) The cube is observed to deform into a rhomboid of sides equal to one (Fig. 2.37). Two of the angles formed by the four faces under stress are reduced from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \gamma_{xy}$, while the other two are increased from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \gamma_{xy}$. The small angle γ_{xy} (expressed in radians) defines the *shearing strain* corresponding to the x and y directions. When the deformation involves a *reduction* of the angle formed by the two faces oriented toward the positive x and y axes (as shown in Fig. 2.37), the shearing strain γ_{xy} is *positive;* otherwise, it is negative.

As a result of the deformations of the other elements of the material, the element under consideration also undergoes an overall rotation. The concern here is with the *actual deformation* of the element, not with any possible superimposed rigid-body displacement.[†]

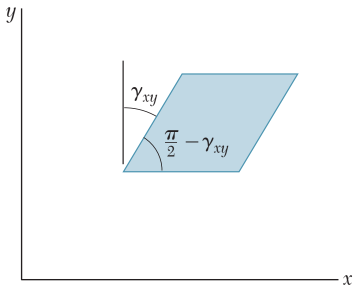


Fig. 2.38 Cubic element as viewed in *xy*-plane after rigid rotation.

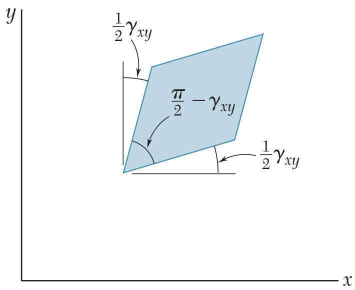


Fig. 2.39 Cubic element as viewed in *xy*-plane with equal rotation of *x* and *y* faces.

Plotting successive values of τ_{xy} against the corresponding values of γ_{xy}, the shearing stress-strain diagram is obtained for the material. (This can be accomplished by carrying out a torsion test, as you will see in Chap. 3.) This diagram is similar to the normal stress-strain diagram from the tensile test described earlier; however, the values for the yield strength, ultimate strength, etc., are about half as large in shear as they are in tension. As for normal stresses and strains, the initial portion of the shearing stress-strain diagram is a straight line. For values of the shearing stress that do not exceed the proportional limit in shear, it can be written for any homogeneous isotropic material that

$$\tau_{xy} = G\gamma_{xy} \tag{2.27}$$

This relationship is *Hooke's law for shearing stress and strain*, and the constant G is called the *modulus of rigidity* or *shear modulus* of the material.

[†] In defining the strain γ_{xy}, some authors arbitrarily assume that the actual deformation of the element is accompanied by a rigid-body rotation where the horizontal faces of the element do not rotate. The strain γ_{xy} is then represented by the angle through which the other two faces have rotated (Fig. 2.38). Others assume a rigid-body rotates where the horizontal faces rotate through $\frac{1}{2}\gamma_{xy}$ counterclockwise and the vertical faces through $\frac{1}{2}\gamma_{xy}$ clockwise (Fig. 2.39). Since both assumptions are unnecessary and may lead to confusion, in this text you will associate the shearing strain γ_{xy} with the *change in the angle* formed by the two faces, rather than with the *rotation of a given face* under restrictive conditions.

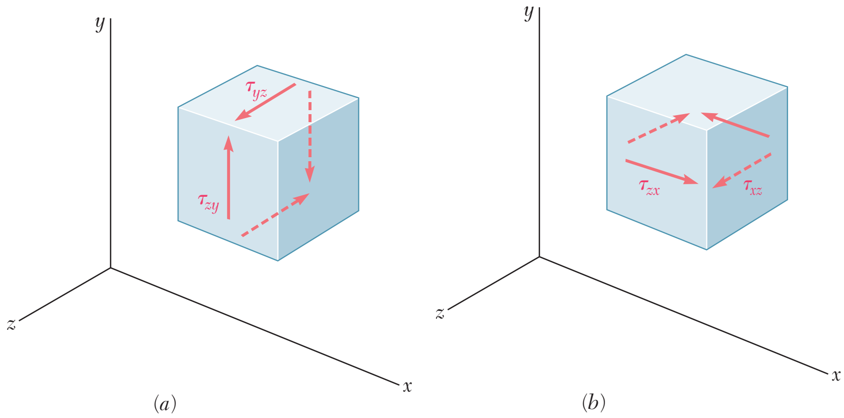


Fig. 2.40 States of pure shear in: (*a*) *yz*-plane; (*b*) *xz*-plane.

Since the strain γ_{xy} is defined as an angle in radians, it is dimensionless, and the modulus G is expressed in the same units as τ_{xy} in pascals or in psi. The modulus of rigidity G of any given material is less than one-half, but more than one-third of the modulus of elasticity E of that material.†

Now consider a small element of material subjected to shearing stresses τ_{yz} and τ_{zy} (Fig. 2.40*a*), where the shearing strain γ_{yz} is the change in the angle formed by the faces under stress. The shearing strain γ_{zx} is found in a similar way by considering an element subjected to shearing stresses τ_{zx} and τ_{xz} (Fig. 2.40*b*). For values of the stress that do not exceed the proportional limit, you can write two additional relationships:

$$\tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{2.28}$$

where the constant G is the same as in Eq. (2.27).

For the general stress condition represented in Fig. 2.35, and as long as none of the stresses involved exceeds the corresponding proportional limit, you can apply the principle of superposition and combine the results. The generalized Hooke's law for a homogeneous isotropic material under the most general stress condition is

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}
\end{aligned} \tag{2.29}$$

An examination of Eqs. (2.29) leads us to three distinct constants, E, ν, and G, which are used to predict the deformations caused in a given material by an arbitrary combination of stresses. Only two of these constants need be determined experimentally for any given material. The next section explains that the third constant can be obtained through a very simple computation.

†See Prob. 2.90.

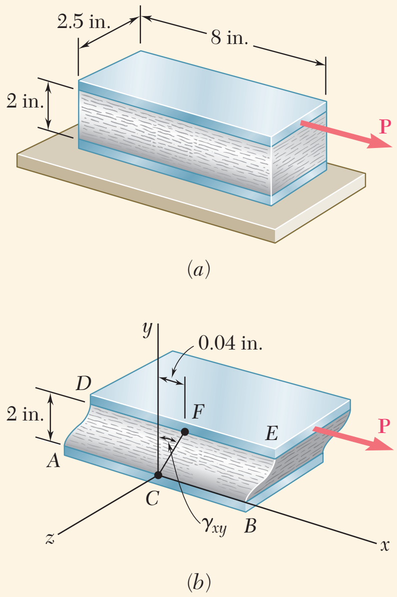


Fig. 2.41 (*a*) Rectangular block loaded in shear. (*b*) Deformed block showing the shearing strain.

Concept Application 2.10

A rectangular block of a material with a modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P** (Fig. 2.41*a*). Knowing that the upper plate moves through 0.04 in. under the action of the force, determine (*a*) the average shearing strain in the material and (*b*) the force **P** exerted on the upper plate.

a. Shearing Strain. The coordinate axes are centered at the midpoint *C* of edge *AB* and directed as shown (Fig. 2.41*b*). The shearing strain γ_{xy} is equal to the angle formed by the vertical and the line *CF* joining the midpoints of edges *AB* and *DE*. Noting that this is a very small angle and recalling that it should be expressed in radians, write

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04 \text{ in.}}{2 \text{ in.}} \qquad \gamma_{xy} = 0.020 \text{ rad}$$

b. Force Exerted on Upper Plate. Determine the shearing stress τ_{xy} in the material. Using Hooke's law for shearing stress and strain,

$$\tau_{xy} = G\gamma_{xy} = (90 \times 10^3 \text{ psi})(0.020 \text{ rad}) = 1800 \text{ psi}$$

The force exerted on the upper plate is

$$P = \tau_{xy} A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36.0 \times 10^3 \text{ lb}$$

$$P = 36.0 \text{ kips}$$

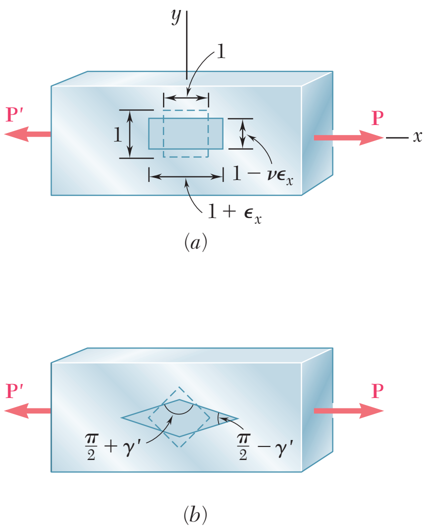


Fig. 2.42 Representations of strain in an axially-loaded bar: (*a*) cubic strain element faces aligned with coordinate axes; (*b*) cubic strain element faces rotated 45° about *z*-axis.

2.8 DEFORMATIONS UNDER AXIAL LOADING—RELATION BETWEEN E, ν, AND G

Section 2.4 showed that a slender bar subjected to an axial tensile load **P** directed along the *x* axis will elongate in the *x* direction and contract in both of the transverse *y* and *z* directions. If ϵ_x denotes the axial strain, the lateral strain is expressed as $\epsilon_y = \epsilon_z = -\nu\epsilon_x$, where ν is Poisson's ratio. Thus, an element in the shape of a cube of side equal to one and oriented as shown in Fig. 2.42*a* will deform into a rectangular parallelepiped of sides $1 + \epsilon_x$, $1 - \nu\epsilon_x$, and $1 - \nu\epsilon_x$. (Note that only one face of the element is shown in the figure.) On the other hand, if the element is oriented at 45° to the axis of the load (Fig. 2.42*b*), the face shown deforms into a rhombus. Therefore, the axial load **P** causes a shearing strain γ' equal to the amount by which each of the angles shown in Fig. 2.42*b* increases or decreases.†

The fact that shearing strains, as well as normal strains, result from an axial loading is not a surprise, since it was observed at the end of Sec. 1.4 that an axial load **P** causes normal and shearing stresses of equal magnitude on four of the faces of an element oriented at 45° to the axis of the member. This was illustrated in Fig. 1.38, which has been repeated

†Note that the load **P** also produces normal strains in the element shown in Fig. 2.42*b* (see Prob. 2.72).

here. It was also shown in Sec. 1.3 that the shearing stress is maximum on a plane forming an angle of 45° with the axis of the load. It follows from Hooke's law for shearing stress and strain that the shearing strain γ' associated with the element of Fig. 2.42*b* is also maximum: $\gamma' = \gamma_m$.

While a more detailed study of the transformations of strain is covered in Chap. 7, this section provides a relationship between the maximum shearing strain $\gamma' = \gamma_m$ associated with the element of Fig. 2.42*b* and the normal strain ϵ_x in the direction of the load. Consider the prismatic element obtained by intersecting the cubic element of Fig. 2.42*a* by a diagonal plane (Fig. 2.43*a* and *b*). Referring to Fig. 2.42*a*, this new element will deform into that shown in Fig. 2.43*c*, which has horizontal and vertical sides equal to $1 + \epsilon_x$ and $1 - \nu\epsilon_x$. But the angle formed by the oblique and horizontal faces of Fig. 2.43*b* is precisely half of one of the right angles of the cubic element in Fig. 2.42*b*. The angle β into which this angle deforms must be equal to half of $\pi/2 - \gamma_m$. Therefore,

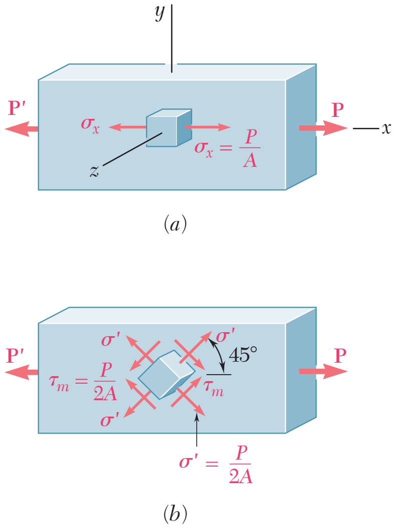


Fig. 1.38 (repeated)

$$\beta = \frac{\pi}{4} - \frac{\gamma_m}{2}$$

Applying the formula for the tangent of the difference of two angles,

$$\tan\beta = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma_m}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma_m}{2}} = \frac{1 - \tan\frac{\gamma_m}{2}}{1 + \tan\frac{\gamma_m}{2}}$$

or since $\gamma_m/2$ is a very small angle,

$$\tan\beta = \frac{1 - \frac{\gamma_m}{2}}{1 + \frac{\gamma_m}{2}} \qquad \textbf{(2.30)}$$

From Fig. 2.43*c*, observe that

$$\tan\beta = \frac{1 - \nu\epsilon_x}{1 + \epsilon_x} \qquad \textbf{(2.31)}$$

Equating the right-hand members of Eqs. (2.30) and (2.31) and solving for γ_m, results in

$$\gamma_m = \frac{(1 + \nu)\epsilon_x}{1 + \frac{1 - \nu}{2}\epsilon_x}$$

Since $\epsilon_x \ll 1$, the denominator in the expression obtained can be assumed equal to one. Therefore,

$$\gamma_m = (1 + \nu)\epsilon_x \qquad \textbf{(2.32)}$$

which is the desired relation between the maximum shearing strain γ_m and the axial strain ϵ_x.

To obtain a relation among the constants E, ν, and G, we recall that, by Hooke's law, $\gamma_m = \tau_m/G$, and for an axial loading, $\epsilon_x = \sigma_x/E$. Equation (2.32) can be written as

$$\frac{\tau_m}{G} = (1 + \nu)\frac{\sigma_x}{E}$$

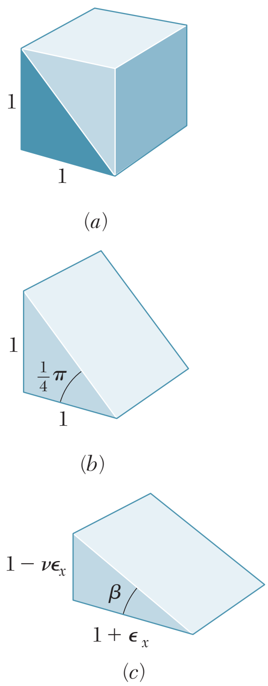


Fig. 2.43 (*a*) Cubic strain unit element, to be sectioned on a diagonal plane. (*b*) Undeformed section of unit element. (*c*) Deformed section of unit element.

or

$$\frac{E}{G} = (1 + \nu)\frac{\sigma_x}{\tau_m} \tag{2.33}$$

Recall from Fig. 1.38 that $\sigma_x = P/A$ and $\tau_m = P/2A$, where A is the cross-sectional area of the member. Thus, $\sigma_x/\tau_m = 2$. Substituting this value into Eq. (2.33) and dividing both members by 2, the relationship is

$$\frac{E}{2G} = 1 + \nu \tag{2.34}$$

which can be used to determine one of the constants E, ν, or G from the other two. For example, solving Eq. (2.34) for G,

$$G = \frac{E}{2(1 + \nu)} \tag{2.35}$$

*2.9 STRESS-STRAIN RELATIONSHIPS FOR FIBER-REINFORCED COMPOSITE MATERIALS

Fiber-reinforced composite materials are fabricated by embedding fibers of a strong, stiff material into a weaker, softer material called a *matrix*. The relationship between the normal stress and the corresponding normal strain created in a lamina or layer of a composite material depends upon the direction in which the load is applied. Different moduli of elasticity, E_x, E_y, and E_z, are required to describe the relationship between normal stress and normal strain, according to whether the load is applied parallel to the fibers, perpendicular to the layer, or in a transverse direction.

Consider again the layer of composite material discussed in Sec. 2.1D and subject it to a uniaxial tensile load parallel to its fibers (Fig. 2.44*a*). It is assumed that the properties of the fibers and of the matrix have been combined or "smeared" into a fictitious, equivalent homogeneous material possessing these combined properties. In a small element of that layer of smeared material (Fig. 2.44*b*), the corresponding normal stress is σ_x and $\sigma_y = \sigma_z = 0$. As indicated in Sec. 2.1D, the corresponding normal strain in the x direction is $\epsilon_x = \sigma_x/E_x$, where E_x is the modulus of elasticity of the composite material in the x direction. As for isotropic materials, the elongation of the material in the x direction is accompanied by contractions in the y and z directions. These contractions depend upon the placement of the fibers in the matrix and generally will be different. Therefore, the lateral strains ϵ_y and ϵ_z also will be different, and the corresponding Poisson's ratios are

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} \quad \text{and} \quad \nu_{xz} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.36}$$

Note that the first subscript in each of the Poisson's ratios ν_{xy} and ν_{xz} in Eqs. (2.36) refers to the direction of the load and the second to the direction of the contraction.

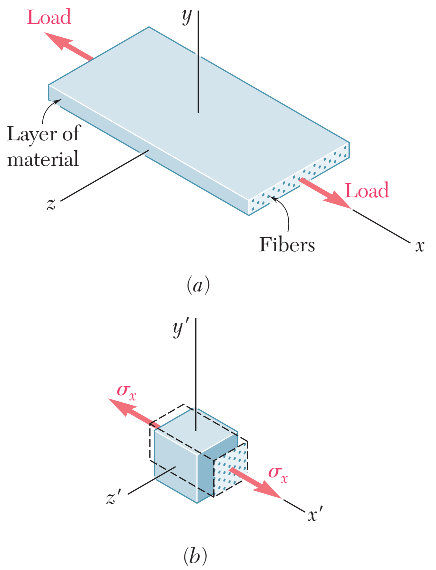


Fig. 2.44 Orthotropic fiber-reinforced composite material under uniaxial tensile load.

In the case of the *multiaxial loading* of a layer of a composite material, equations similar to Eqs. (2.20) of Sec. 2.5 can be used to describe the stress-strain relationship. In this case, three different values of the modulus of elasticity and six different values of Poisson's ratio are involved. We write

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} - \frac{\nu_{zx}\sigma_z}{E_z} \\
\epsilon_y &= -\frac{\nu_{xy}\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}\sigma_z}{E_z} \\
\epsilon_z &= -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} + \frac{\sigma_z}{E_z}
\end{aligned} \tag{2.37}$$

Equations (2.37) can be considered as defining the transformation of stress into strain for the given layer. It follows from a general property of such transformations that the coefficients of the stress components are symmetric:

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \qquad \frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z} \qquad \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x} \tag{2.38}$$

While different, these equations show that Poisson's ratios ν_{xy} and ν_{yx} are not independent; either of them can be obtained from the other if the corresponding values of the modulus of elasticity are known. The same is true of ν_{yz} and ν_{zy}, and of ν_{zx} and ν_{xz}.

Consider now the effect of shearing stresses on the faces of a small element of smeared layer. As discussed in Sec. 2.7 for isotropic materials, these stresses come in pairs of equal and opposite vectors applied to opposite sides of the given element and have no effect on the normal strains. Thus, Eqs. (2.37) remain valid. The shearing stresses, however, create shearing strains that are defined by equations similar to the last three of Eqs. (2.29) of Sec. 2.7, except that three different values of the modulus of rigidity, G_{xy}, G_{yz}, and G_{zx}, must be used:

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}} \tag{2.39}$$

The fact that the three components of strain ϵ_x, ϵ_y, and ϵ_z can be expressed in terms of the normal stresses only and do not depend upon any shearing stresses characterizes *orthotropic materials* and distinguishes them from other anisotropic materials.

As in Sec. 2.1D, a flat *laminate* is obtained by superposing a number of layers or laminas. If the fibers in all layers are given the same orientation to withstand an axial tensile load, the laminate itself will be orthotropic. If the lateral stability of the laminate is increased by positioning some of its layers so that their fibers are at a right angle to the fibers of the other layers, the resulting laminate also will be orthotropic. On the other hand, if any of the layers of a laminate are positioned so that their fibers are neither parallel nor perpendicular to the fibers of other layers, the lamina generally will not be orthotropic.[†]

[†]For more information on fiber-reinforced composite materials, see Hyer, M. W., *Stress Analysis of Fiber-Reinforced Composite Materials*, DEStech Publications, Inc., Lancaster, PA, 2009.

Concept Application 2.11

A 60-mm cube is made from layers of graphite epoxy with fibers aligned in the x direction. The cube is subjected to a compressive load of 140 kN in the x direction. The properties of the composite material are: $E_x = 155.0$ GPa, $E_y = 12.10$ GPa, $E_z = 12.10$ GPa, $\nu_{xy} = 0.248$, $\nu_{xz} = 0.248$, and $\nu_{yz} = 0.458$. Determine the changes in the cube dimensions, knowing that (*a*) the cube is free to expand in the y and z directions (Fig. 2.45*a*); (*b*) the cube is free to expand in the z direction, but is restrained from expanding in the y direction by two fixed frictionless plates (Fig. 2.45*b*).

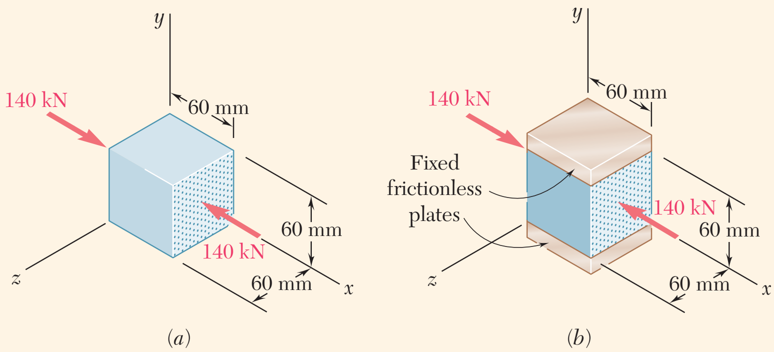


Fig. 2.45 Graphite-epoxy cube undergoing compression loading along the fiber direction; (*a*) unrestrained cube; (*b*) cube restrained in *y* direction.

a. Free in *y* and *z* Directions. Determine the stress σ_x in the direction of loading.

$$\sigma_x = \frac{P}{A} = \frac{-140 \times 10^3 \text{ N}}{(0.060 \text{ m})(0.060 \text{ m})} = -38.89 \text{ MPa}$$

Since the cube is not loaded or restrained in the y and z directions, we have $\sigma_y = \sigma_z = 0$. Thus, the right-hand members of Eqs. (2.37) reduce to their first terms. Substituting the given data into these equations,

$$\epsilon_x = \frac{\sigma_x}{E_x} = \frac{-38.89 \text{ MPa}}{155.0 \text{ GPa}} = -250.9 \times 10^{-6}$$

$$\epsilon_y = -\frac{\nu_{xy}\sigma_x}{E_x} = -\frac{(0.248)(-38.89 \text{ MPa})}{155.0 \text{ GPa}} = +62.22 \times 10^{-6}$$

$$\epsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} = -\frac{(0.248)(-38.69 \text{ MPa})}{155.0 \text{ GPa}} = +62.22 \times 10^{-6}$$

The changes in the cube dimensions are obtained by multiplying the corresponding strains by the length $L = 0.060$ m of the side of the cube:

$$\delta_x = \epsilon_x L = (-250.9 \times 10^{-6})(0.060 \text{ m}) = -15.05 \ \mu\text{m}$$

$$\delta_y = \epsilon_y L = (+62.2 \times 10^{-6})(0.060 \text{ m}) = +3.73 \ \mu\text{m}$$

$$\delta_z = \epsilon_z L = (+62.2 \times 10^{-6})(0.060 \text{ m}) = +3.73 \ \mu\text{m}$$

(continued)

b. Free in *z* Direction, Restrained in *y* Direction. The stress in the x direction is the same as in part a, namely, $\sigma_x = 38.89$ MPa. Since the cube is free to expand in the z direction as in part a, $\sigma_z = 0$. But since the cube is now restrained in the y direction, the stress σ_y is not zero. On the other hand, since the cube cannot expand in the y direction, $\delta_y = 0$. Thus, $\epsilon_y = \delta_y/L = 0$. Set $\sigma_z = 0$ and $\epsilon_y = 0$ in the second of Eqs. (2.37) and solve that equation for σ_y:

$$\sigma_y = \left(\frac{E_y}{E_x}\right)\nu_{xy}\sigma_x = \left(\frac{12.10}{155.0}\right)(0.248)(-38.89 \text{ MPa})$$
$$= -752.9 \text{ kPa}$$

Now that the three components of stress have been determined, use the first and last of Eqs. (2.37) to compute the strain components ϵ_x and ϵ_z. But the first of these equations contains Poisson's ratio ν_{yx}, and as you saw earlier this ratio *is not equal* to the ratio ν_{xy} that was among the given data. To find ν_{yx}, use the first of Eqs. (2.38) and write

$$\nu_{yx} = \left(\frac{E_y}{E_x}\right)\nu_{xy} = \left(\frac{12.10}{155.0}\right)(0.248) = 0.01936$$

Now set $\sigma_z = 0$ in the first and third of Eqs. (2.37) and substitute the given values of E_x, E_y, ν_{xz}, and ν_{yz}, as well as the values obtained for σ_x, σ_y, and ν_{yx}, resulting in

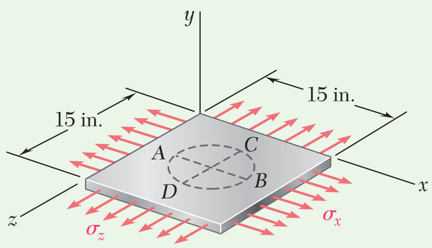
$$\epsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} = \frac{-38.89 \text{ MPa}}{155.0 \text{ GPa}} - \frac{(0.01936)(-752.9 \text{ kPa})}{12.10 \text{ GPa}}$$
$$= -249.7 \times 10^{-6}$$

$$\epsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} = -\frac{(0.248)(-38.89 \text{ MPa})}{155.0 \text{ GPa}} - \frac{(0.458)(-752.9 \text{ kPa})}{12.10 \text{ GPa}}$$
$$= +90.72 \times 10^{-6}$$

The changes in the cube dimensions are obtained by multiplying the corresponding strains by the length $L = 0.060$ m of the side of the cube:

$$\delta_x = \epsilon_x L = (-249.7 \times 10^{-6})(0.060 \text{ m}) = -14.98\ \mu\text{m}$$
$$\delta_y = \epsilon_y L = (0)(0.060 \text{ m}) = 0$$
$$\delta_z = \epsilon_z L = (+90.72 \times 10^{-6})(0.060 \text{ m}) = +5.44\ \mu\text{m}$$

Comparing the results of parts a and b, note that the difference between the values for the deformation δ_x in the direction of the fibers is negligible. However, the difference between the values for the lateral deformation δ_z is not negligible when the cube is restrained from deforming in the y direction.



Sample Problem 2.5

A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = \frac{3}{4}$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi. For $E = 10 \times 10^6$ psi and $\nu = \frac{1}{3}$, determine the change in (*a*) the length of diameter *AB*, (*b*) the length of diameter *CD*, (*c*) the thickness of the plate, and (*d*) the volume of the plate.

STRATEGY: You can use the generalized Hooke's Law to determine the components of strain. These strains can then be used to evaluate the various dimensional changes to the plate, and through the dilatation, also assess the volume change.

ANALYSIS:

Hooke's Law. Note that $\sigma_y = 0$. Using Eqs. (2.20), find the strain in each of the coordinate directions.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[(12 \text{ ksi}) - 0 - \frac{1}{3}(20 \text{ ksi})\right] = +0.533 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) + 0 - \frac{1}{3}(20 \text{ ksi})\right] = -1.067 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) - 0 + (20 \text{ ksi})\right] = +1.600 \times 10^{-3} \text{ in./in.}$$

a. Diameter *AB*. The change in length is $\delta_{B/A} = \epsilon_x d$.

$$\delta_{B/A} = \epsilon_x d = (+0.533 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{B/A} = +4.8 \times 10^{-3} \text{ in.} \quad ◀$$

b. Diameter *CD*.

$$\delta_{C/D} = \epsilon_z d = (+1.600 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{C/D} = +14.4 \times 10^{-3} \text{ in.} \quad ◀$$

c. Thickness. Recalling that $t = \frac{3}{4}$ in.,

$$\delta_t = \epsilon_y t = (-1.067 \times 10^{-3} \text{ in./in.})(\tfrac{3}{4} \text{ in.})$$

$$\delta_t = -0.800 \times 10^{-3} \text{ in.} \quad ◀$$

d. Volume of the Plate. Using Eq. (2.21),

$$e = \epsilon_x + \epsilon_y + \epsilon_z = (+0.533 - 1.067 + 1.600)10^{-3} = +1.067 \times 10^{-3}$$

$$\Delta V = eV = +1.067 \times 10^{-3}[(15 \text{ in.})(15 \text{ in.})(\tfrac{3}{4} \text{ in.})]$$

$$\Delta V = +0.180 \text{ in}^3 \quad ◀$$

Problems

2.61 A standard tension test is used to determine the properties of an experimental plastic. The test specimen is a $\frac{5}{8}$-in.-diameter rod and it is subjected to an 800-lb tensile force. Knowing that an elongation of 0.45 in. and a decrease in diameter of 0.025 in. are observed in a 5-in. gage length, determine the modulus of elasticity, the modulus of rigidity, and Poisson's ratio for the material.

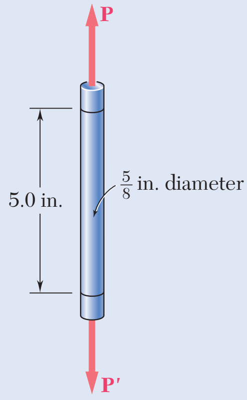


Fig. P2.61

2.62 A 2-m length of an aluminum pipe of 240-mm outer diameter and 10-mm wall thickness is used as a short column to carry a 640-kN centric axial load. Knowing that $E = 73$ GPa and $\nu = 0.33$, determine (*a*) the change in length of the pipe, (*b*) the change in its outer diameter, (*c*) the change in its wall thickness.

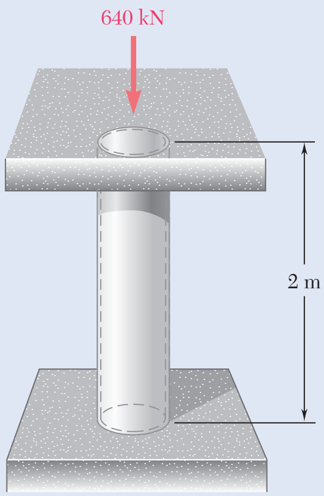


Fig. *P2.62*

2.63 A line of slope 4:10 has been scribed on a cold-rolled yellow-brass plate, 150 mm wide and 6 mm thick. Knowing that $E = 105$ GPa and $\nu = 0.34$, determine the slope of the line when the plate is subjected to a 200-kN centric axial load as shown.

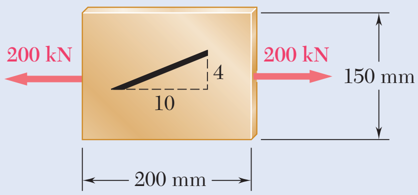


Fig. P2.63

2.64 A 2.75-kN tensile load is applied to a test coupon made from 1.6-mm flat steel plate ($E = 200$ GPa, $\nu = 0.30$). Determine the resulting change (*a*) in the 50-mm gage length, (*b*) in the width of portion *AB* of the test coupon, (*c*) in the thickness of portion *AB*, (*d*) in the cross-sectional area of portion *AB*.

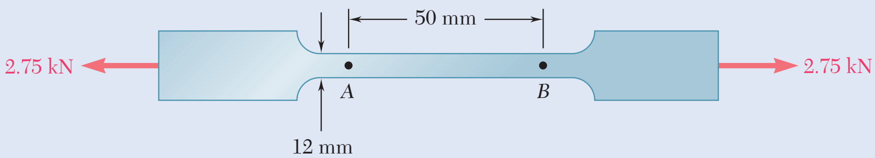


Fig. P2.64

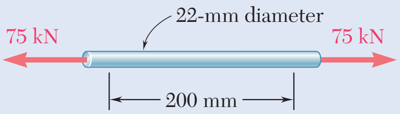


Fig. *P2.65*

2.65 In a standard tensile test a steel rod of 22-mm diameter is subjected to a tension force of 75 kN. Knowing that $\nu = 0.30$ and $E = 200$ GPa, determine (*a*) the elongation of the rod in a 200-mm gage length, (*b*) the change in diameter of the rod.

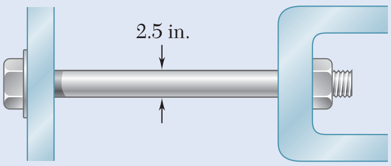


Fig. P2.66

2.66 The change in diameter of a large steel bolt is carefully measured as the nut is tightened. Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine the internal force in the bolt if the diameter is observed to decrease by 0.5×10^{-3} in.

2.67 The brass rod *AD* is fitted with a jacket that is used to apply a hydrostatic pressure of 48 MPa to the 240-mm portion *BC* of the rod. Knowing that $E = 105$ GPa and $\nu = 0.33$, determine (*a*) the change in the total length *AD*, (*b*) the change in diameter at the middle of the rod.

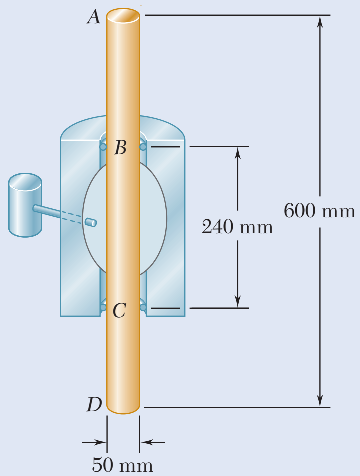


Fig. P2.67

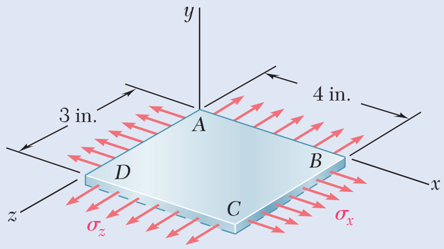


Fig. P2.68

2.68 A fabric used in air-inflated structures is subjected to a biaxial loading that results in normal stresses $\sigma_x = 18$ ksi and $\sigma_z = 24$ ksi. Knowing that the properties of the fabric can be approximated as $E = 12.6 \times 10^6$ psi and $\nu = 0.34$, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

2.69 A 1-in. square was scribed on the side of a large steel pressure vessel. After pressurization the biaxial stress condition at the square is as shown. Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

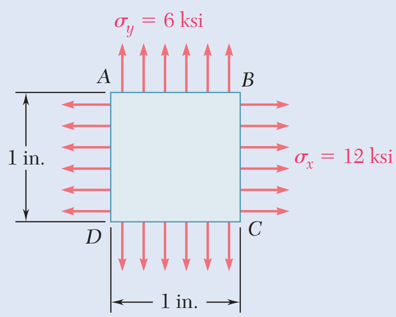


Fig. P2.69

2.70 The block shown is made of a magnesium alloy for which $E = 45$ GPa and $\nu = 0.35$. Knowing that $\sigma_x = -180$ MPa, determine (*a*) the magnitude of σ_y for which the change in the height of the block will be zero, (*b*) the corresponding change in the area of the face *ABCD*, (*c*) the corresponding change in the volume of the block.

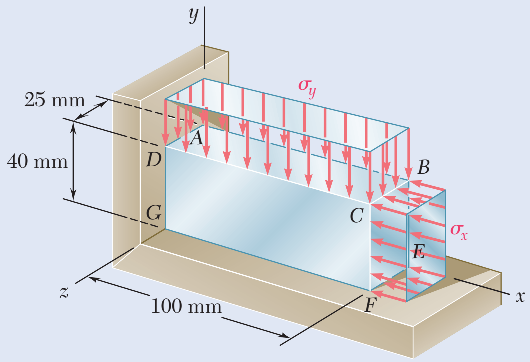


Fig. P2.70

2.71 The homogeneous plate *ABCD* is subjected to a biaxial loading as shown. It is known that $\sigma_z = \sigma_0$ and that the change in length of the plate in the x direction must be zero, that is, $\epsilon_x = 0$. Denoting by E the modulus of elasticity and by ν Poisson's ratio, determine (*a*) the required magnitude of σ_x, (*b*) the ratio σ_0/ϵ_z.

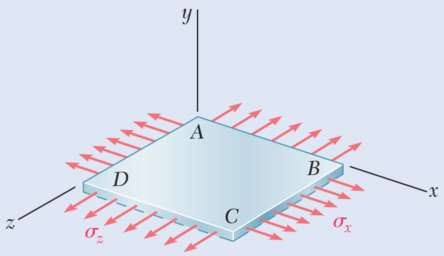


Fig. P2.71

2.72 For a member under axial loading, express the normal strain ϵ' in a direction forming an angle of 45° with the axis of the load in terms of the axial strain ϵ_x by (*a*) comparing the hypotenuses of the triangles shown in Fig. 2.43, which represent respectively an element before and after deformation, (*b*) using the values of the corresponding stresses σ' and σ_x shown in Fig. 1.38, and the generalized Hooke's law.

2.73 In many situations it is known that the normal stress in a given direction is zero. For example, $\sigma_z = 0$ in the case of the thin plate shown. For this case, which is known as *plane stress*, show that if the strains ϵ_x and ϵ_y have been determined experimentally, we can express σ_x, σ_y, and ϵ_z as follows:

$$\sigma_x = E\frac{\epsilon_x + \nu\epsilon_y}{1 - \nu^2}$$

$$\sigma_y = E\frac{\epsilon_y + \nu\epsilon_x}{1 - \nu^2}$$

$$\epsilon_z = -\frac{\nu}{1 - \nu}(\epsilon_x + \epsilon_y)$$

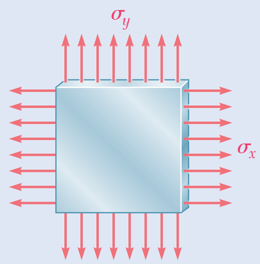


Fig. P2.73

2.74 In many situations physical constraints prevent strain from occurring in a given direction. For example, $\epsilon_z = 0$ in the case shown, where longitudinal movement of the long prism is prevented at every point. Plane sections perpendicular to the longitudinal axis remain plane and the same distance apart. Show that for this situation, which is known as *plane strain*, we can express σ_z, ϵ_x, and ϵ_y as follows:

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$
$$\epsilon_x = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$
$$\epsilon_y = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

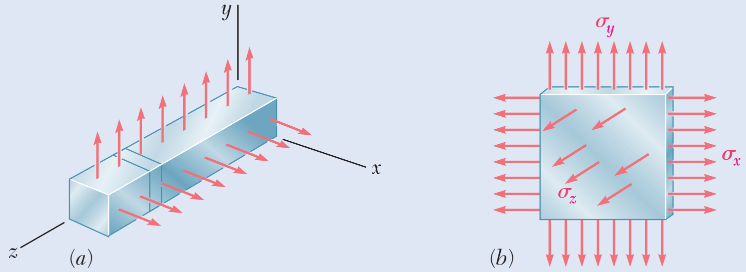


Fig. P2.74

2.75 The plastic block shown is bonded to a rigid support and to a vertical plate to which a 55-kip load **P** is applied. Knowing that for the plastic used $G = 150$ ksi, determine the deflection of the plate.

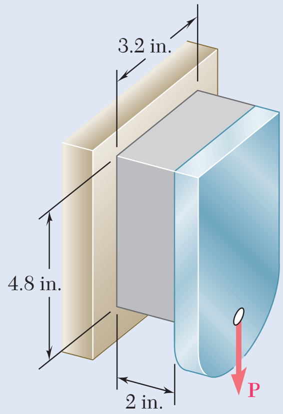


Fig. *P2.75*

2.76 What load **P** should be applied to the plate of Prob. 2.75 to produce a $\frac{1}{16}$-in. deflection?

2.77 Two blocks of rubber with a modulus of rigidity $G = 12$ MPa are bonded to rigid supports and to a plate AB. Knowing that $c = 100$ mm and $P = 45$ kN, determine the smallest allowable dimensions a and b of the blocks if the shearing stress in the rubber is not to exceed 1.4 MPa and the deflection of the plate is to be at least 5 mm.

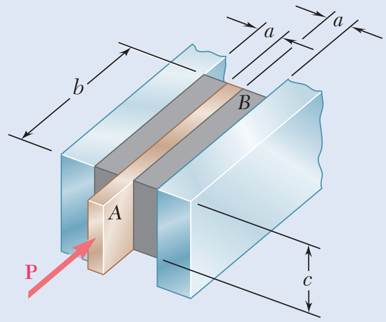


Fig. P2.77 and P2.78

2.78 Two blocks of rubber with a modulus of rigidity $G = 10$ MPa are bonded to rigid supports and to a plate AB. Knowing that $b = 200$ mm and $c = 125$ mm, determine the largest allowable load P and the smallest allowable thickness a of the blocks if the shearing stress in the rubber is not to exceed 1.5 MPa and the deflection of the plate is to be at least 6 mm.

2.79 An elastomeric bearing ($G = 130$ psi) is used to support a bridge girder as shown to provide flexibility during earthquakes. The beam must not displace more than $\frac{3}{8}$ in. when a 5-kip lateral load is applied as shown. Knowing that the maximum allowable shearing stress is 60 psi, determine (*a*) the smallest allowable dimension *b*, (*b*) the smallest required thickness *a*.

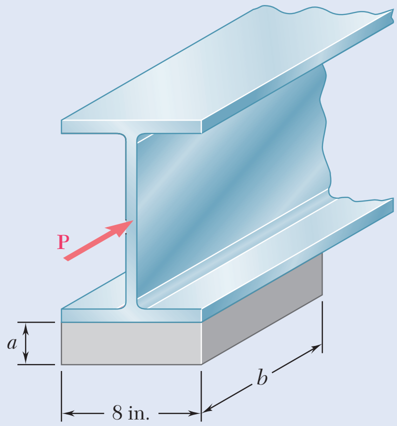


Fig. P2.79

2.80 For the elastomeric bearing in Prob. 2.79 with $b = 10$ in. and $a = 1$ in., determine the shearing modulus *G* and the shear stress τ for a maximum lateral load $P = 5$ kips and a maximum displacement $\delta = 0.4$ in.

2.81 A vibration isolation unit consists of two blocks of hard rubber bonded to a plate *AB* and to rigid supports as shown. Knowing that a force of magnitude $P = 25$ kN causes a deflection $\delta = 1.5$ mm of plate *AB*, determine the modulus of rigidity of the rubber used.

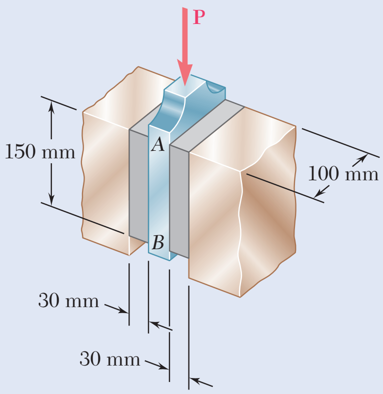


Fig. P2.81 and P2.82

2.82 A vibration isolation unit consists of two blocks of hard rubber with a modulus of rigidity $G = 19$ MPa bonded to a plate *AB* and to rigid supports as shown. Denoting by *P* the magnitude of the force applied to the plate and by δ the corresponding deflection, determine the effective spring constant, $k = P/\delta$, of the system.

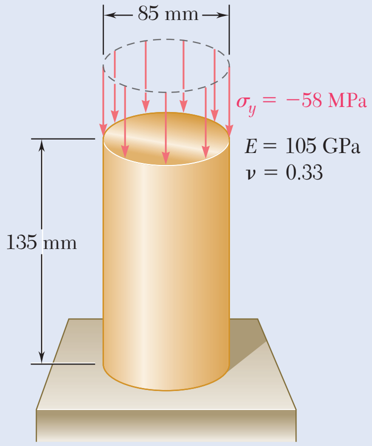


Fig. P2.84

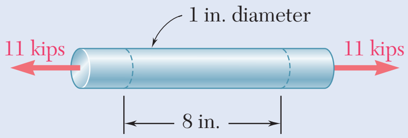


Fig. P2.85

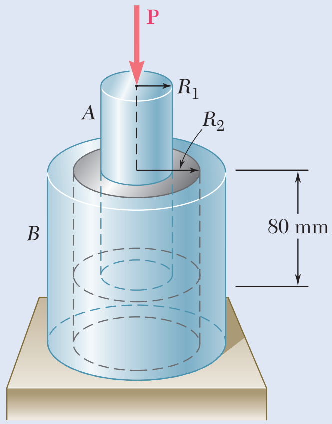


Fig. *P2.87* and P2.88

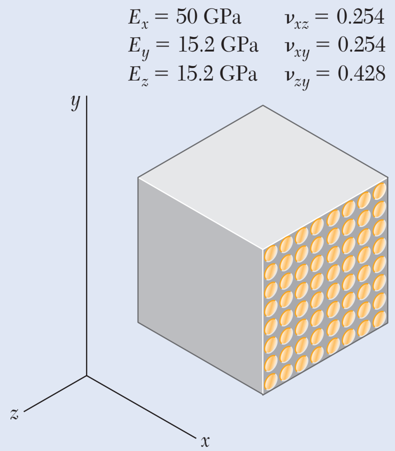


Fig. P2.91

***2.83** A 6-in.-diameter solid steel sphere is lowered into the ocean to a point where the pressure is 7.1 ksi (about 3 miles below the surface). Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine (*a*) the decrease in diameter of the sphere, (*b*) the decrease in volume of the sphere, (*c*) the percent increase in the density of the sphere.

***2.84** (*a*) For the axial loading shown, determine the change in height and the change in volume of the brass cylinder shown. (*b*) Solve part *a*, assuming that the loading is hydrostatic with $\sigma_x = \sigma_y = \sigma_z = -70$ MPa.

***2.85** Determine the dilatation *e* and the change in volume of the 8-in. length of the rod shown if (*a*) the rod is made of steel with $E = 29 \times 10^6$ psi and $\nu = 0.30$, (*b*) the rod is made of aluminum with $E = 10.6 \times 10^6$ psi and $\nu = 0.35$.

******2.86*** Determine the change in volume of the 50-mm gage length segment *AB* in Prob. 2.64 (*a*) by computing the dilatation of the material, (*b*) by subtracting the original volume of portion *AB* from its final volume.

******2.87*** A vibration isolation support consists of a rod *A* of radius $R_1 = 10$ mm and a tube *B* of inner radius $R_2 = 25$ mm bonded to an 80-mm-long hollow rubber cylinder with a modulus of rigidity $G = 12$ MPa. Determine the largest allowable force **P** that can be applied to rod *A* if its deflection is not to exceed 2.50 mm.

***2.88** A vibration isolation support consists of a rod *A* of radius R_1 and a tube *B* of inner radius R_2 bonded to an 80-mm-long hollow rubber cylinder with a modulus of rigidity $G = 10.93$ MPa. Determine the required value of the ratio R_2/R_1 if a 10-kN force **P** is to cause a 2-mm deflection of rod *A*.

***2.89** The material constants *E*, *G*, *k*, and ν are related by Eqs. (2.24) and (2.34). Show that any one of the constants may be expressed in terms of any other two constants. For example, show that (*a*) $k = GE/(9G - 3E)$ and (*b*) $\nu = (3k - 2G)/(6k + 2G)$.

***2.90** Show that for any given material, the ratio *G/E* of the modulus of rigidity over the modulus of elasticity is always less than $\frac{1}{2}$ but more than $\frac{1}{3}$. [*Hint:* Refer to Eq. (2.34) and to Sec. 2.1*e*.]

***2.91** A composite cube with 40-mm sides and the properties shown is made with glass polymer fibers aligned in the *x* direction. The cube is constrained against deformations in the *y* and *z* directions and is subjected to a tensile load of 65 kN in the *x* direction. Determine (*a*) the change in the length of the cube in the *x* direction and (*b*) the stresses σ_x, σ_y, and σ_z.

***2.92** The composite cube of Prob. 2.91 is constrained against deformation in the *z* direction and elongated in the *x* direction by 0.035 mm due to a tensile load in the *x* direction. Determine (*a*) the stresses σ_x, σ_y, and σ_z and (*b*) the change in the dimension in the *y* direction.

2.10 STRESS AND STRAIN DISTRIBUTION UNDER AXIAL LOADING: SAINT-VENANT'S PRINCIPLE

We have assumed so far that, in an axially loaded member, the normal stresses are uniformly distributed in any section perpendicular to the axis of the member. As we saw in Sec. 1.2A, such an assumption may be quite in error in the immediate vicinity of the points of application of the loads. However, the determination of the actual stresses in a given section of the member requires the solution of a statically indeterminate problem.

In Sec. 2.2, you saw that statically indeterminate problems involving the determination of *forces* can be solved by considering the *deformations* caused by these forces. It is thus reasonable to conclude that the determination of the *stresses* in a member requires the analysis of the strains produced by the stresses in the member. This is essentially the approach found in advanced textbooks, where the mathematical theory of elasticity is used to determine the distribution of stresses corresponding to various modes of application of the loads at the ends of the member. Given the more limited mathematical means at our disposal, our analysis of stresses will be restricted to the particular case when two rigid plates are used to transmit the loads to a member made of a homogeneous isotropic material (Fig. 2.46).

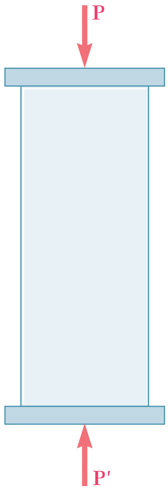


Fig. 2.46 Axial load applied by rigid plates.

If the loads are applied at the center of each plate,[†] the plates will move toward each other without rotating, causing the member to get shorter, while increasing in width and thickness. It is assumed that the member will remain straight, plane sections will remain plane, and all elements of the member will deform in the same way, since this assumption is compatible with the given end conditions. Figure 2.47 shows a rubber model before and after loading.[‡] Now, if all elements deform in the same

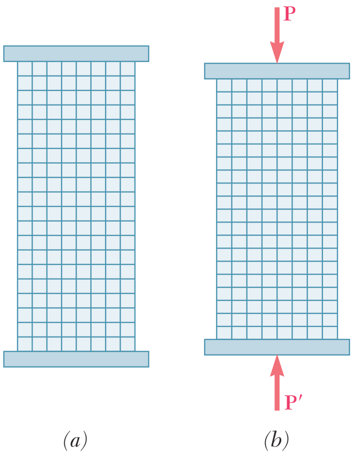


Fig. 2.47 Axial load applied by rigid plates to rubber model.

[†]More precisely, the common line of action of the loads should pass through the centroid of the cross section (cf. Sec. 1.2A).

[‡]Note that for long, slender members, another configuration is possible and will prevail if the load is sufficiently large; the member *buckles* and assumes a curved shape. This will be discussed in Chap. 10.

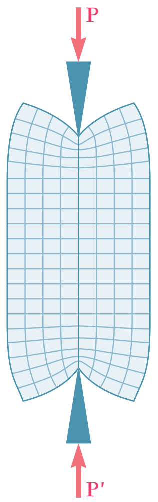


Fig. 2.48 Concentrated axial load applied to rubber model.

way, the distribution of strains throughout the member must be uniform. In other words, the axial strain ϵ_y and the lateral strain $\epsilon_x = -\nu\epsilon_y$ are constant. But, if the stresses do not exceed the proportional limit, Hooke's law applies, and $\sigma_y = E\epsilon_y$, so the normal stress σ_y is also constant. Thus, the distribution of stresses is uniform throughout the member, and at any point,

$$\sigma_y = (\sigma_y)_{\text{ave}} = \frac{P}{A}$$

If the loads are concentrated, as in Fig. 2.48, the elements in the immediate vicinity of the points of application of the loads are subjected to very large stresses, while other elements near the ends of the member are unaffected by the loading. This results in large deformations, strains, and stresses near the points of application of the loads, while no deformation takes place at the corners. Considering elements farther and farther from the ends, a progressive equalization of the deformations and a more uniform distribution of the strains and stresses are seen across a section of the member. Using the mathematical theory of elasticity found in advanced textbooks, Fig. 2.49 shows the resulting distribution of stresses across various sections of a thin rectangular plate subjected to concentrated loads. Note

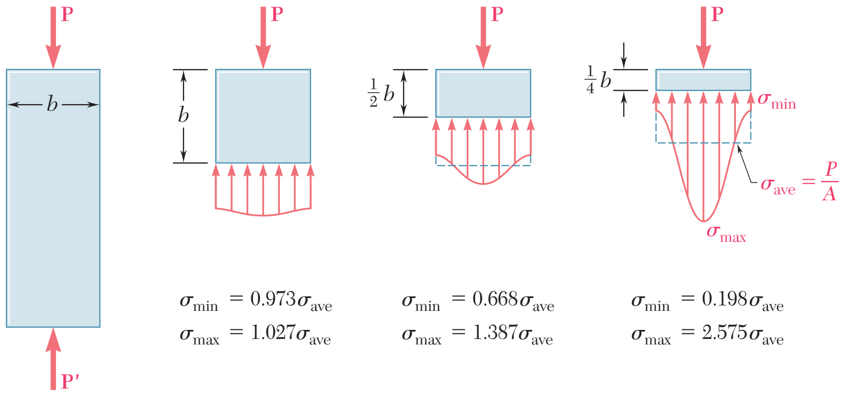


Fig. 2.49 Stress distributions in a plate under concentrated axial loads.

that at a distance b from either end, where b is the width of the plate, the stress distribution is nearly uniform across the section, and the value of the stress σ_y at any point of that section can be assumed to be equal to the average value P/A. Thus, at a distance equal to or greater than the width of the member, the distribution of stresses across a section is the same, whether the member is loaded as shown in Fig. 2.46 or Fig. 2.48. In other words, except in the immediate vicinity of the points of application of the loads, the stress distribution is assumed independent of the actual mode of application of the loads. This statement, which applies to axial loadings and to practically any type of load, is known as *Saint-Venant's principle*, after the French mathematician and engineer Adhémar Barré de Saint-Venant (1797–1886).

While Saint-Venant's principle makes it possible to replace a given loading by a simpler one to compute the stresses in a structural member, keep in mind two important points when applying this principle:

1. The actual loading and the loading used to compute the stresses must be *statically equivalent*.

2. Stresses cannot be computed in this manner in the immediate vicinity of the points of application of the loads. Advanced theoretical or experimental methods must be used to determine the distribution of stresses in these areas.

You should also observe that the plates used to obtain a uniform stress distribution in the member of Fig. 2.47 must allow the member to freely expand laterally. Thus, the plates cannot be rigidly attached to the member; assume them to be just in contact with the member and smooth enough not to impede lateral expansion. While such end conditions can be achieved for a member in compression, they cannot be physically realized in the case of a member in tension. It does not matter, whether or not an actual fixture can be realized and used to load a member so that the distribution of stresses in the member is uniform. The important thing is to *imagine a model* that will allow such a distribution of stresses and to keep this model in mind so that it can be compared with the actual loading conditions.

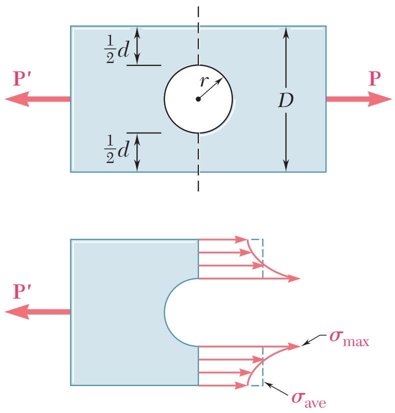


Fig. 2.50 Stress distribution near circular hole in flat bar under axial loading.

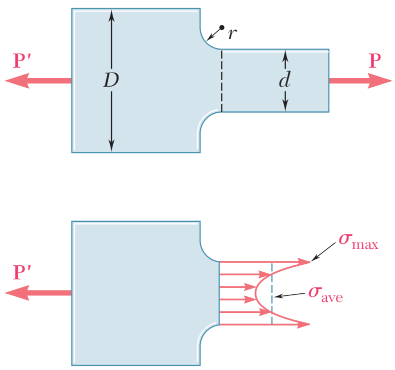


Fig. 2.51 Stress distribution near fillets in flat bar under axial loading.

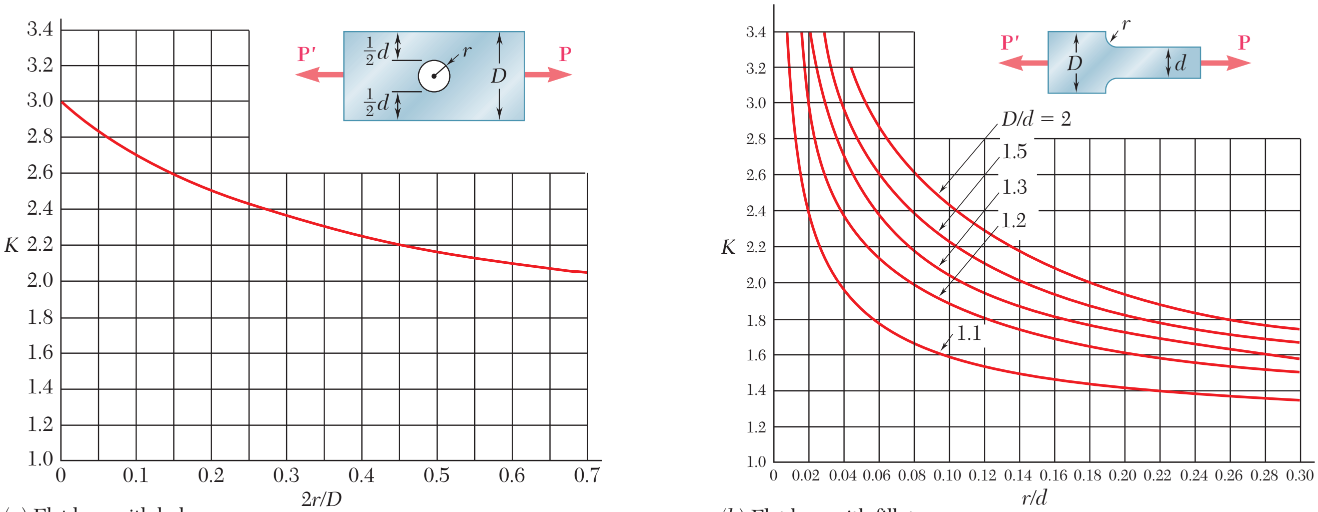
2.11 STRESS CONCENTRATIONS

As you saw in the preceding section, the stresses near the points of application of concentrated loads can reach values much larger than the average value of the stress in the member. When a structural member contains a discontinuity, such as a hole or a sudden change in cross section, high localized stresses can occur. Figures 2.50 and 2.51 show the distribution of stresses in critical sections corresponding to two situations. Figure 2.50 shows a flat bar with a *circular hole* and shows the stress distribution in a section passing through the center of the hole. Figure 2.51 shows a flat bar consisting of two portions of different widths connected by *fillets;* here the stress distribution is in the narrowest part of the connection, where the highest stresses occur.

These results were obtained experimentally through the use of a photoelastic method. Fortunately for the engineer, these results are independent of the size of the member and of the material used; they depend only upon the ratios of the geometric parameters involved (i.e., the ratio $2r/D$ for a circular hole, and the ratios r/d and D/d for fillets). Furthermore, the designer is more interested in the *maximum value* of the stress in a given section than the actual distribution of stresses. The main concern is to determine *whether* the allowable stress will be exceeded under a given loading, not *where* this value will be exceeded. Thus, the ratio

$$K = \frac{\sigma_{max}}{\sigma_{ave}} \tag{2.40}$$

is computed in the critical (narrowest) section of the discontinuity. This ratio is the *stress-concentration factor* of the discontinuity. Stress-concentration factors can be computed in terms of the ratios of the geometric parameters involved, and the results can be expressed in tables or graphs, as shown in Fig. 2.52. To determine the maximum stress occurring near a discontinuity in a given member subjected to a given axial load P, the designer needs to compute the average stress $\sigma_{ave} = P/A$ in the critical section and multiply the result obtained by the appropriate value of the stress-concentration factor K. Note that this procedure is valid only as long as σ_{max} does not exceed the proportional limit of the material, since the values of K plotted in Fig. 2.52 were obtained by assuming a linear relation between stress and strain.



(*a*) Flat bars with holes (*b*) Flat bars with fillets

Fig. 2.52 Stress concentration factors for flat bars under axial loading. Note that the average stress must be computed across the narrowest section: $\sigma_{ave} = P/td$, where t is the thickness of the bar. (Source: W. D. Pilkey and D.F. Pilkey, *Peterson's Stress Concentration Factors*, 3rd ed., John Wiley & Sons, New York, 2008.)

Concept Application 2.12

Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick and, respectively, 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

First compute the ratios

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \qquad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20$$

Using the curve in Fig. 2.52*b* corresponding to $D/d = 1.50$, the value of the stress-concentration factor corresponding to $r/d = 0.20$ is

$$K = 1.82$$

Then carrying this value into Eq. (2.40) and solving for σ_{ave},

$$\sigma_{ave} = \frac{\sigma_{max}}{1.82}$$

But σ_{max} cannot exceed the allowable stress $\sigma_{all} = 165$ MPa. Substituting this value for σ_{max}, the average stress in the narrower portion ($d = 40$ mm) of the bar should not exceed the value

$$\sigma_{ave} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}$$

Recalling that $\sigma_{ave} = P/A$,

$$P = A\sigma_{ave} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa}) = 36.3 \times 10^3 \text{ N}$$

$$P = 36.3 \text{ kN}$$

2.12 PLASTIC DEFORMATIONS

The results in the preceding sections were based on the assumption of a linear stress-strain relationship, where the proportional limit of the material was never exceeded. This is a reasonable assumption in the case of brittle materials, which rupture without yielding. For ductile materials, however, this implies that the yield strength of the material is not exceeded. The deformations will remain within the elastic range and the structural member will regain its original shape after all loads have been removed. However, if the stresses in any part of the member exceed the yield strength of the material, plastic deformations occur, and most of the results obtained in earlier sections cease to be valid. Then a more involved analysis, based on a nonlinear stress-strain relationship, must be carried out.

While an analysis taking into account the actual stress-strain relationship is beyond the scope of this text, we gain considerable insight into plastic behavior by considering an idealized *elastoplastic material* for which the stress-strain diagram consists of the two straight-line segments shown in Fig. 2.53. Note that the stress-strain diagram for mild steel in the elastic and plastic ranges is similar to this idealization. As long as the stress σ is less than the yield strength σ_Y, the material behaves elastically and obeys Hooke's law, $\sigma = E\epsilon$. When σ reaches the value σ_Y, the material starts yielding and keeps deforming plastically under a constant load. If the load is removed, unloading takes place along a straight-line segment CD parallel to the initial portion AY of the loading curve. The segment AD of the horizontal axis represents the strain corresponding to the permanent set or plastic deformation resulting from the loading and unloading of the specimen. While no actual material behaves exactly as shown in Fig. 2.53, this stress-strain diagram will prove useful in discussing the plastic deformations of ductile materials such as mild steel.

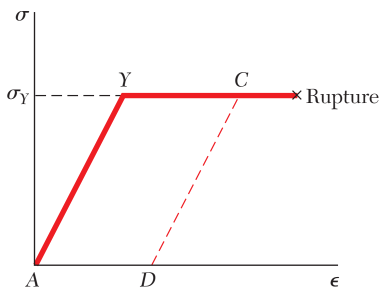


Fig. 2.53 Stress-strain diagram for an idealized elastoplastic material.

Concept Application 2.13

A rod of length $L = 500$ mm and cross-sectional area $A = 60$ mm^2 is made of an elastoplastic material having a modulus of elasticity $E = 200$ GPa in its elastic range and a yield point $\sigma_Y = 300$ MPa. The rod is subjected to an axial load until it is stretched 7 mm and the load is then removed. What is the resulting permanent set?

Referring to the diagram of Fig. 2.53, the maximum strain represented by the abscissa of point C is

$$\epsilon_C = \frac{\delta_C}{L} = \frac{7 \text{ mm}}{500 \text{ mm}} = 14 \times 10^{-3}$$

However, the yield strain, represented by the abscissa of point Y, is

$$\epsilon_Y = \frac{\sigma_Y}{E} = \frac{300 \times 10^6 \text{ Pa}}{200 \times 10^9 \text{ Pa}} = 1.5 \times 10^{-3}$$

The strain after unloading is represented by the abscissa ϵ_D of point D. Note from Fig. 2.53 that

$$\begin{aligned}\epsilon_D &= AD = YC = \epsilon_C - \epsilon_Y \\ &= 14 \times 10^{-3} - 1.5 \times 10^{-3} = 12.5 \times 10^{-3}\end{aligned}$$

The permanent set is the deformation δ_D corresponding to the strain ϵ_D.

$$\delta_D = \epsilon_D L = (12.5 \times 10^{-3})(500 \text{ mm}) = 6.25 \text{ mm}$$

Concept Application 2.14

A 30-in.-long cylindrical rod of cross-sectional area $A_r = 0.075\text{ in}^2$ is placed inside a tube of the same length and of cross-sectional area $A_t = 0.100\text{ in}^2$. The ends of the rod and tube are attached to a rigid support on one side, and to a rigid plate on the other, as shown in the longitudinal section of Fig. 2.54*a*. The rod and tube are both assumed to be elastoplastic, with moduli of elasticity $E_r = 30 \times 10^6$ psi and $E_t = 15 \times 10^6$ psi, and yield strengths $(\sigma_r)_Y = 36$ ksi and $(\sigma_t)_Y = 45$ ksi. Draw the load-deflection diagram of the rod-tube assembly when a load P is applied to the plate as shown.

Determine the internal force and the elongation of the rod as it begins to yield

$$(P_r)_Y = (\sigma_r)_Y A_r = (36\text{ ksi})(0.075\text{ in}^2) = 2.7\text{ kips}$$

$$(\delta_r)_Y = (\epsilon_r)_Y L = \frac{(\sigma_r)_Y}{E_r}L = \frac{36 \times 10^3\text{ psi}}{30 \times 10^6\text{ psi}}(30\text{ in.})$$

$$= 36 \times 10^{-3}\text{ in.}$$

Since the material is elastoplastic, the force-elongation diagram *of the rod alone* consists of oblique and horizontal straight lines, as shown in Fig. 2.54*b*. Following the same procedure for the tube,

$$(P_t)_Y = (\sigma_t)_Y A_t = (45\text{ ksi})(0.100\text{ in}^2) = 4.5\text{ kips}$$

$$(\delta_t)_Y = (\epsilon_t)_Y L = \frac{(\sigma_t)_Y}{E_t}L = \frac{45 \times 10^3\text{ psi}}{15 \times 10^6\text{ psi}}(30\text{ in.})$$

$$= 90 \times 10^{-3}\text{ in.}$$

The load-deflection diagram of *the tube alone* is shown in Fig. 2.54*c*. Observing that the load and deflection of the rod-tube combination are

$$P = P_r + P_t \qquad \delta = \delta_r = \delta_t$$

we draw the required load-deflection diagram by adding the ordinates of the diagrams obtained for both the rod and the tube (Fig. 2.54*d*). Points Y_r and Y_t correspond to the onset of yield.

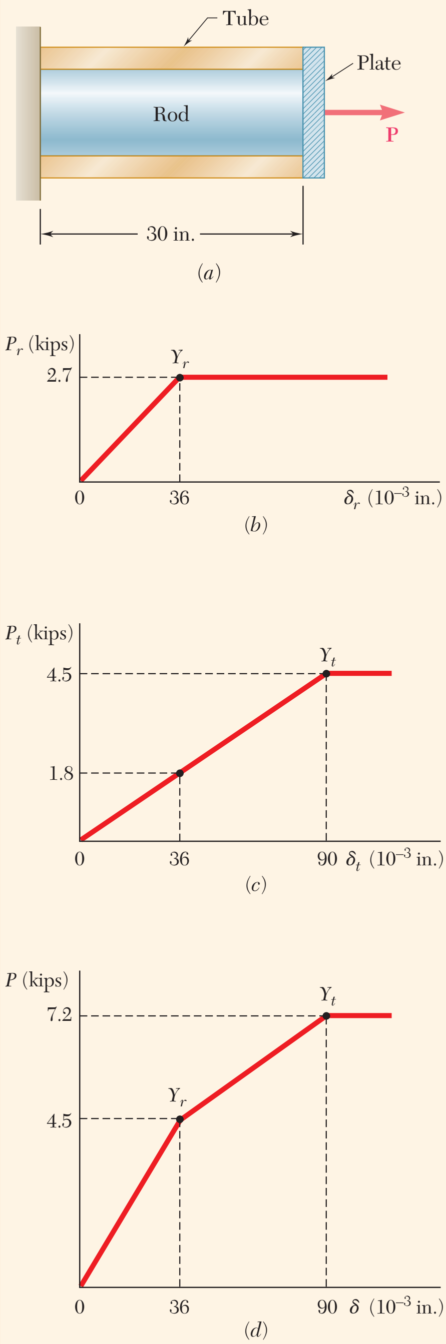


Fig. 2.54 (*a*) Concentric rod-tube assembly axially loaded by rigid plate. (*b*) Load-deflection response of the rod. (*c*) Load-deflection response of the tube. (*d*) Combined load-deflection response of the rod-tube assembly.

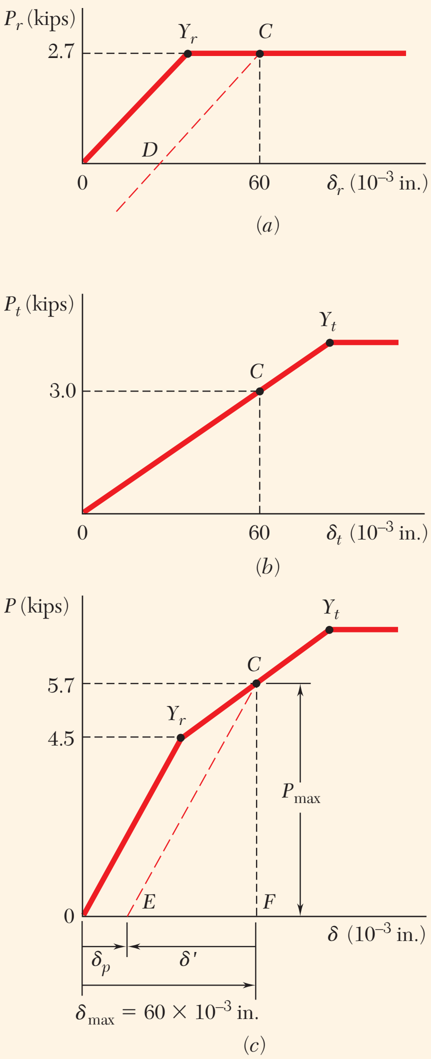


Fig. 2.55 (*a*) Rod load-deflection response with elastic unloading (red dashed line). (*b*) Tube load-deflection response; note that the given loading does not yield the tube, so unloading is along the original elastic loading line. (*c*) Combined rod-tube assembly load-deflection response with elastic unloading (red dashed line).

Concept Application 2.15

If the load **P** applied to the rod-tube assembly of Concept Application 2.14 is increased from zero to 5.7 kips and decreased back to zero, determine (*a*) the maximum elongation of the assembly and (*b*) the permanent set after the load has been removed.

a. Maximum Elongation. Referring to Fig. 2.54*d*, the load $P_{max} = 5.7$ kips corresponds to a point located on the segment Y_rY_t of the load-deflection diagram of the assembly. Thus, the rod has reached the plastic range with $P_r = (P_r)_Y = 2.7$ kips and $\sigma_r = (\sigma_r)_Y = 36$ ksi. However the tube is still in the elastic range with

$$P_t = P - P_r = 5.7 \text{ kips} - 2.7 \text{ kips} = 3.0 \text{ kips}$$

$$\sigma_t = \frac{P_t}{A_t} = \frac{3.0 \text{ kips}}{0.1 \text{ in}^2} = 30 \text{ ksi}$$

$$\delta_t = \epsilon_t L = \frac{\sigma_t}{E_t} L = \frac{30 \times 10^3 \text{ psi}}{15 \times 10^6 \text{ psi}}(30 \text{ in.}) = 60 \times 10^{-3} \text{ in.}$$

The maximum elongation of the assembly is

$$\delta_{max} = \delta_t = 60 \times 10^{-3} \text{ in.}$$

b. Permanent Set. As the load **P** decreases from 5.7 kips to zero, the internal forces P_r and P_t both decrease along a straight line, as shown in Fig. 2.55*a* and *b*. The force P_r decreases along line *CD* parallel to the initial portion of the loading curve, while the force P_t decreases along the original loading curve, since the yield stress was not exceeded in the tube. Their sum *P* will decrease along a line *CE* parallel to the portion $0Y_r$ of the load-deflection curve of the assembly (Fig. 2.55*c*). Referring to Fig. 2.55*c*, the slope of $0Y_r$ (and thus of *CE*) is

$$m = \frac{4.5 \text{ kips}}{36 \times 10^{-3} \text{ in.}} = 125 \text{ kips/in.}$$

The segment of line *FE* in Fig. 2.55*c* represents the deformation δ' of the assembly during the unloading phase, and the segment 0*E* is the permanent set δ_p after the load **P** has been removed. From triangle *CEF*,

$$\delta' = -\frac{P_{max}}{m} = -\frac{5.7 \text{ kips}}{125 \text{ kips/in.}} = -45.6 \times 10^{-3} \text{ in.}$$

The permanent set is

$$\delta_P = \delta_{max} + \delta' = 60 \times 10^{-3} - 45.6 \times 10^{-3}$$
$$= 14.4 \times 10^{-3} \text{ in.}$$

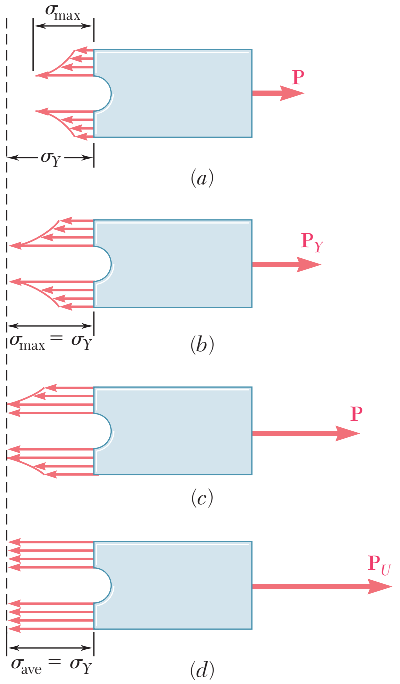


Fig. 2.56 Distribution of stresses in elastic-perfectly plastic material under increasing load.

Stress Concentrations. Recall that the discussion of stress concentrations of Sec. 2.11 was carried out under the assumption of a linear stress-strain relationship. The stress distributions shown in Figs. 2.50 and 2.51, and the stress-concentration factors plotted in Fig. 2.52 cannot be used when plastic deformations take place, i.e., when σ_{max} exceeds the yield strength σ_Y.

Consider again the flat bar with a circular hole of Fig. 2.50, and let us assume that the material is elastoplastic, i.e., that its stress-strain diagram is as shown in Fig. 2.53. As long as no plastic deformation takes place, the distribution of stresses is as indicated in Sec. 2.11 (Fig. 2.50*a*). The area under the stress-distribution curve represents the integral $\int \sigma \, dA$, which is equal to the load P. Thus this area and the value of σ_{max} must increase as the load P increases. As long as $\sigma_{max} \leq \sigma_Y$, all of the stress distributions obtained as P increases will have the shape shown in Fig. 2.50 and repeated in Fig. 2.56*a*. However, as P is increased beyond P_Y corresponding to $\sigma_{max} = \sigma_Y$ (Fig. 2.56*b*), the stress-distribution curve must flatten in the vicinity of the hole (Fig. 2.56*c*), since the stress cannot exceed the value σ_Y. This indicates that the material is yielding in the vicinity of the hole. As the load P is increased, the plastic zone where yield takes place keeps expanding until it reaches the edges of the plate (Fig. 2.56*d*). At that point, the distribution of stresses across the plate is uniform, $\sigma = \sigma_Y$, and the corresponding value $P = P_U$ of the load is the largest that can be applied to the bar without causing rupture.

It is interesting to compare the maximum value P_Y of the load that can be applied if no permanent deformation is to be produced in the bar with the value P_U that will cause rupture. Recalling the average stress, $\sigma_{ave} = P/A$, where A is the net cross-sectional area and the stress concentration factor, $K = \sigma_{max}/\sigma_{ave}$, write

$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K} \tag{2.41}$$

for any value of σ_{max} that does not exceed σ_Y. When $\sigma_{max} = \sigma_Y$ (Fig. 2.56*b*), $P = P_Y$, and Eq. (2.40) yields

$$P_Y = \frac{\sigma_Y A}{K} \tag{2.42}$$

On the other hand, when $P = P_U$ (Fig. 2.56*d*), $\sigma_{ave} = \sigma_Y$ and

$$P_U = \sigma_Y A \tag{2.43}$$

Comparing Eqs. (2.42) and (2.43),

$$P_Y = \frac{P_U}{K} \tag{2.44}$$

*2.13 RESIDUAL STRESSES

In Concept Application 2.13 of the preceding section, we considered a rod that was stretched beyond the yield point. As the load was removed, the rod did not regain its original length; it had been permanently deformed.

However, after the load was removed, all stresses disappeared. You should not assume that this will always be the case. Indeed, when only some of the parts of an indeterminate structure undergo plastic deformations, as in Concept Application 2.15, or when different parts of the structure undergo different plastic deformations, the stresses in the various parts of the structure will not return to zero after the load has been removed. Stresses called *residual stresses* will remain in various parts of the structure.

While computation of residual stresses in an actual structure can be quite involved, the following concept application provides a general understanding of the method to be used for their determination.

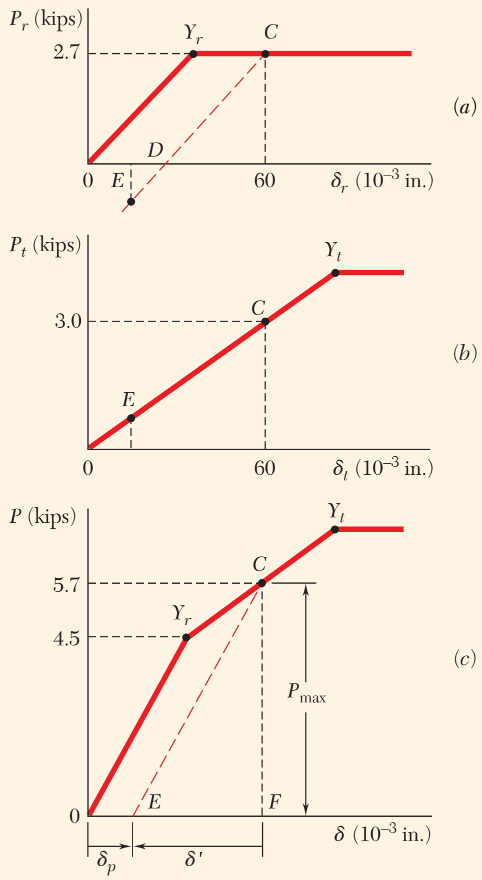


Fig. 2.57 (*a*) Rod load-deflection response with elastic unloading (red dashed line). (*b*) Tube load-deflection response; the given loading does not yield the tube, so unloading is along elastic loading line with residual tensile stress. (*c*) Combined rod-tube assembly load-deflection response with elastic unloading (red dashed line).

Concept Application 2.16

Determine the residual stresses in the rod and tube of Fig. 2.54*a* after the load **P** has been increased from zero to 5.7 kips and decreased back to zero.

Observe from the diagrams of Fig. 2.57 (similar to those in the previous concept application) that, after the load **P** has returned to zero, the internal forces P_r and P_t are *not* equal to zero. Their values have been indicated by point E in parts *a* and *b*. The corresponding stresses are not equal to zero either after the assembly has been unloaded. To determine these residual stresses, first determine the reverse stresses σ'_r and σ'_t caused by the unloading and add them to the maximum stresses $\sigma_r = 36$ ksi and $\sigma_t = 30$ ksi found in part *a* of Concept Application 2.15.

The strain caused by the unloading is the same in both the rod and the tube. It is equal to δ'/L, where δ' is the deformation of the assembly during unloading found in Concept Application 2.15:

$$\epsilon' = \frac{\delta'}{L} = \frac{-45.6 \times 10^{-3}\text{ in.}}{30\text{ in.}} = -1.52 \times 10^{-3}\text{ in./in.}$$

The corresponding reverse stresses in the rod and tube are

$$\sigma'_r = \epsilon' E_r = (-1.52 \times 10^{-3})(30 \times 10^6\text{ psi}) = -45.6\text{ ksi}$$

$$\sigma'_t = \epsilon' E_t = (-1.52 \times 10^{-3})(15 \times 10^6\text{ psi}) = -22.8\text{ ksi}$$

Then the residual stresses are found by superposing the stresses due to loading and the reverse stresses due to unloading.

$$(\sigma_r)_{\text{res}} = \sigma_r + \sigma'_r = 36\text{ ksi} - 45.6\text{ ksi} = -9.6\text{ ksi}$$

$$(\sigma_t)_{\text{res}} = \sigma_t + \sigma'_t = 30\text{ ksi} - 22.8\text{ ksi} = +7.2\text{ ksi}$$

Temperature Changes. Plastic deformations caused by temperature changes can also result in residual stresses. For example, consider a small plug that is to be welded to a large plate (Fig. 2.58). The plug can be considered a small rod AB to be welded across a small hole in the plate. During the welding process, the temperature of the rod will be raised to over 1000°C, at which point its modulus of elasticity, stiffness, and stress will be almost zero. Since the plate is large, its temperature will not be increased significantly above room temperature (20°C). Thus, when the welding is completed, rod AB is at $T = 1000°C$ with no stress and is attached to the plate, which is at 20°C.

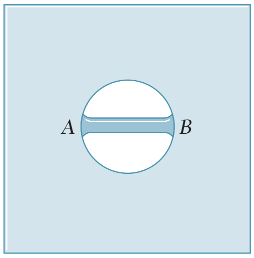


Fig. 2.58 Small rod welded to a large plate.

As the rod cools, its modulus of elasticity increases. At about 500°C, it will approach its normal value of about 200 GPa. As the temperature of the rod decreases further, a situation similar to that considered in Sec. 2.3 and illustrated in Fig. 2.26 develops. Solving Eq. (2.15) for ΔT, making σ equal to the yield strength, assuming $\sigma_Y = 300$ MPa for the steel used, and $\alpha = 12 \times 10^{-6}/°C$, the temperature change that causes the rod to yield is

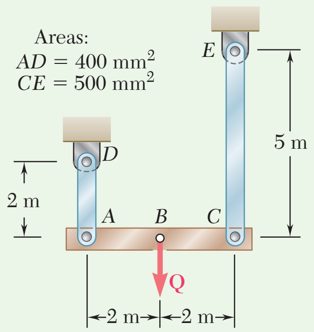
$$\Delta T = -\frac{\sigma}{E\alpha} = -\frac{300 \text{ MPa}}{(200 \text{ GPa})(12 \times 10^{-6}/°\text{C})} = -125°\text{C}$$

So the rod starts yielding at about 375°C and keeps yielding at a fairly constant stress level as it cools down to room temperature. As a result of welding, a residual stress (approximately equal to the yield strength of the steel used) is created in the plug and in the weld.

Residual stresses also occur as a result of the cooling of metals that have been cast or hot rolled. In these cases, the outer layers cool more rapidly than the inner core. This causes the outer layers to reacquire their stiffness (E returns to its normal value) faster than the inner core. When the entire specimen has returned to room temperature, the inner core will contract more than the outer layers. The result is residual longitudinal tensile stresses in the inner core and residual compressive stresses in the outer layers.

Residual stresses due to welding, casting, and hot rolling can be quite large (of the order of magnitude of the yield strength). These stresses can be removed by reheating the entire specimen to about 600°C and then allowing it to cool slowly over a period of 12 to 24 hours.

Sample Problem 2.6



The rigid beam *ABC* is suspended from two steel rods as shown and is initially horizontal. The midpoint *B* of the beam is deflected 10 mm downward by the slow application of the force **Q**, after which the force is slowly removed. Knowing that the steel used for the rods is elastoplastic with $E = 200$ GPa and $\sigma_Y = 300$ MPa, determine (*a*) the required maximum value of *Q* and the corresponding position of the beam and (*b*) the final position of the beam.

STRATEGY: You can assume that plastic deformation would occur first in rod AD (which is a good assumption—*why?*), and then check this assumption.

MODELING AND ANALYSIS:

Statics. Since **Q** is applied at the midpoint of the beam (Fig. 1),

$$P_{AD} = P_{CE} \qquad \text{and} \qquad Q = 2P_{AD}$$

Elastic Action (Fig. 2). The maximum value of *Q* and the maximum elastic deflection of point *A* occur when $\sigma = \sigma_Y$ in rod *AD*.

$$(P_{AD})_{\max} = \sigma_Y A = (300 \text{ MPa})(400 \text{ mm}^2) = 120 \text{ kN}$$

$$Q_{\max} = 2(P_{AD})_{\max} = 2(120 \text{ kN}) \qquad Q_{\max} = 240 \text{ kN} \blacktriangleleft$$

$$\delta_{A_1} = \epsilon L = \frac{\sigma_Y}{E} L = \left(\frac{300 \text{ MPa}}{200 \text{ GPa}}\right)(2 \text{ m}) = 3 \text{ mm}$$

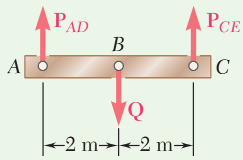


Fig. 1 Free-body diagram of rigid beam.

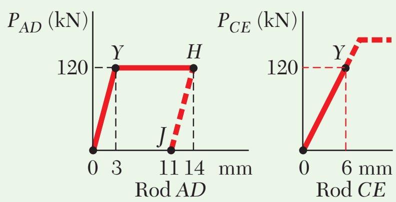


Fig. 2 Load-deflection diagrams for steel rods.

Since $P_{CE} = P_{AD} = 120$ kN, the stress in rod CE is

$$\sigma_{CE} = \frac{P_{CE}}{A} = \frac{120 \text{ kN}}{500 \text{ mm}^2} = 240 \text{ MPa}$$

The corresponding deflection of point C is

$$\delta_{C_1} = \epsilon L = \frac{\sigma_{CE}}{E} L = \left(\frac{240 \text{ MPa}}{200 \text{ GPa}}\right)(5 \text{ m}) = 6 \text{ mm}$$

The corresponding deflection of point B is

$$\delta_{B_1} = \tfrac{1}{2}(\delta_{A_1} + \delta_{C_1}) = \tfrac{1}{2}(3 \text{ mm} + 6 \text{ mm}) = 4.5 \text{ mm}$$

Since $\delta_B = 10$ mm, plastic deformation will occur.

Plastic Deformation. For $Q = 240$ kN, plastic deformation occurs in rod AD, where $\sigma_{AD} = \sigma_Y = 300$ MPa. Since the stress in rod CE is within the elastic range, δ_C remains equal to 6 mm. From Fig. 3, the deflection δ_A for which $\delta_B = 10$ mm is obtained by writing

$$\delta_{B_2} = 10 \text{ mm} = \tfrac{1}{2}(\delta_{A_2} + 6 \text{ mm}) \qquad \delta_{A_2} = 14 \text{ mm}$$

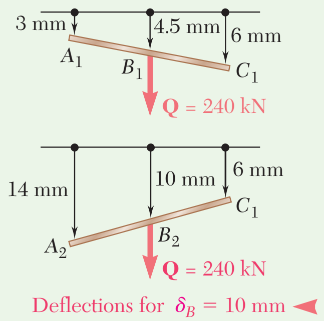


Fig. 3 Deflection of fully-loaded beam.

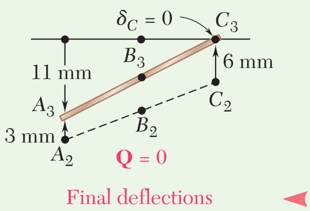


Fig. 4 Beam's final deflections with load removed.

Unloading. As force **Q** is slowly removed, the force P_{AD} decreases along line HJ parallel to the initial portion of the load-deflection diagram of rod AD. The final deflection of point A is

$$\delta_{A_3} = 14 \text{ mm} - 3 \text{ mm} = 11 \text{ mm}$$

Since the stress in rod CE remained within the elastic range, note that the final deflection of point C is zero. Fig. 4 illustrates the final position of the beam.

REFLECT and THINK: Due to symmetry in this determinate problem, the axial forces in the rods are equal. Given that the rods have identical material properties and that the cross-sectional area of rod AD is smaller than rod CE, you would therefore expect that rod AD would reach yield first (as assumed in the STRATEGY step).

Problems

2.93 Knowing that, for the plate shown, the allowable stress is 125 MPa, determine the maximum allowable value of P when (*a*) $r = 12$ mm, (*b*) $r = 18$ mm.

2.94 Knowing that $P = 38$ kN, determine the maximum stress when (*a*) $r = 10$ mm, (*b*) $r = 16$ mm, (*c*) $r = 18$ mm.

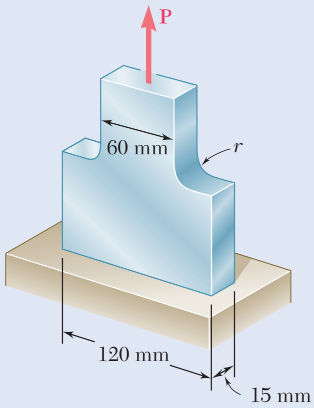


Fig. P2.93 and P2.94

2.95 A hole is to be drilled in the plate at A. The diameters of the bits available to drill the hole range from $\frac{1}{2}$ to $1\frac{1}{2}$ in. in $\frac{1}{4}$-in. increments. If the allowable stress in the plate is 21 ksi, determine (*a*) the diameter d of the largest bit that can be used if the allowable load **P** at the hole is to exceed that at the fillets, (*b*) the corresponding allowable load **P**.

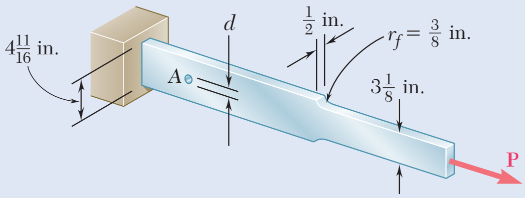


Fig. *P2.95* and *P2.96*

2.96 (*a*) For $P = 13$ kips and $d = \frac{1}{2}$ in., determine the maximum stress in the plate shown. (*b*) Solve part *a*, assuming that the hole at A is not drilled.

2.97 Knowing that the hole has a diameter of 9 mm, determine (*a*) the radius r_f of the fillets for which the same maximum stress occurs at the hole A and at the fillets, (*b*) the corresponding maximum allowable load **P** if the allowable stress is 100 MPa.

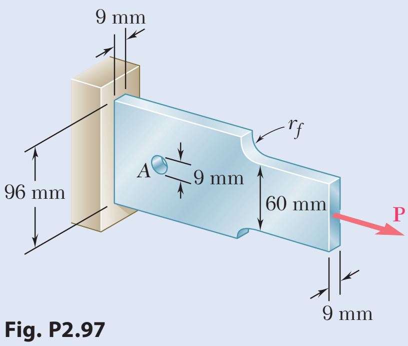


Fig. P2.97

2.98 For $P = 100$ kN, determine the minimum plate thickness t required if the allowable stress is 125 MPa.

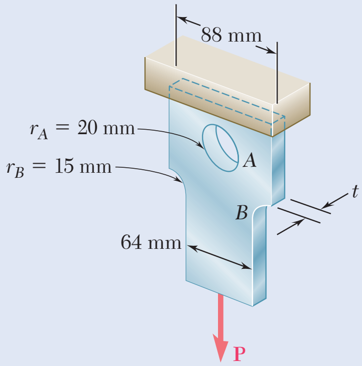


Fig. P2.98

2.99 (a) Knowing that the allowable stress is 20 ksi, determine the maximum allowable magnitude of the centric load **P**. (b) Determine the percent change in the maximum allowable magnitude of **P** if the raised portions are removed at the ends of the specimen.

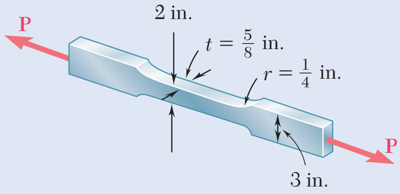


Fig. P2.99

2.100 A centric axial force is applied to the steel bar shown. Knowing that $\sigma_{all} = 20$ ksi, determine the maximum allowable load **P**.

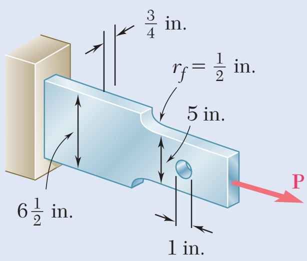


Fig. P2.100

2.101 The cylindrical rod AB has a length $L = 5$ ft and a 0.75-in. diameter; it is made of a mild steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi. A force **P** is applied to the bar and then removed to give it a permanent set δ_P. Determine the maximum value of the force **P** and the maximum amount δ_m by which the bar should be stretched if the desired value of δ_P is (a) 0.1 in., (b) 0.2 in.

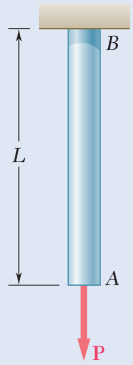


Fig. P2.101 and P2.102

2.102 The cylindrical rod AB has a length $L = 6$ ft and a 1.25-in. diameter; it is made of a mild steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi. A force **P** is applied to the bar until end A has moved down by an amount δ_m. Determine the maximum value of the force **P** and the permanent set of the bar after the force has been removed, knowing (a) $\delta_m = 0.125$ in., (b) $\delta_m = 0.250$ in.

2.103 Rod *AB* is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 345$ MPa. After the rod has been attached to the rigid lever *CD*, it is found that end *C* is 6 mm too high. A vertical force **Q** is then applied at *C* until this point has moved to position *C*′. Determine the required magnitude of **Q** and the deflection δ_1 if the lever is to *snap* back to a horizontal position after **Q** is removed.

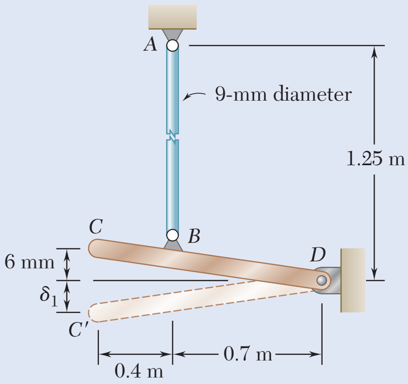


Fig. *P2.103*

2.104 Solve Prob. 2.103, assuming that the yield point of the mild steel is 250 MPa.

2.105 Rod *ABC* consists of two cylindrical portions *AB* and *BC;* it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. A force **P** is applied to the rod and then removed to give it a permanent set $\delta_P = 2$ mm. Determine the maximum value of the force **P** and the maximum amount δ_m by which the rod should be stretched to give it the desired permanent set.

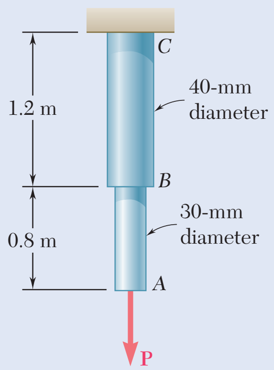


Fig. P2.105 and P2.106

2.106 Rod *ABC* consists of two cylindrical portions *AB* and *BC;* it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. A force **P** is applied to the rod until its end *A* has moved down by an amount $\delta_m = 5$ mm. Determine the maximum value of the force **P** and the permanent set of the rod after the force has been removed.

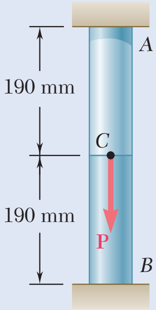


Fig. P2.107

2.107 Rod *AB* consists of two cylindrical portions *AC* and *BC*, each with a cross-sectional area of 1750 mm^2. Portion *AC* is made of a mild steel with $E = 200$ GPa and $\sigma_Y = 250$ MPa, and portion *BC* is made of a high-strength steel with $E = 200$ GPa and $\sigma_Y = 345$ MPa. A load **P** is applied at *C* as shown. Assuming both steels to be elastoplastic, determine (*a*) the maximum deflection of *C* if *P* is gradually increased from zero to 975 kN and then reduced back to zero, (*b*) the maximum stress in each portion of the rod, (*c*) the permanent deflection of *C*.

2.108 For the composite rod of Prob. 2.107, if *P* is gradually increased from zero until the deflection of point *C* reaches a maximum value of $\delta_m = 0.3$ mm and then decreased back to zero, determine, (*a*) the maximum value of *P*, (*b*) the maximum stress in each portion of the rod, (*c*) the permanent deflection of *C* after the load is removed.

2.109 Each cable has a cross-sectional area of 100 mm^2 and is made of an elastoplastic material for which $\sigma_Y = 345$ MPa and $E = 200$ GPa. A force **Q** is applied at *C* to the rigid bar *ABC* and is gradually increased from 0 to 50 kN and then reduced to zero. Knowing that the cables were initially taut, determine (*a*) the maximum stress that occurs in cable *BD*, (*b*) the maximum deflection of point *C*, (*c*) the final displacement of point *C*. (*Hint:* In part *c*, cable *CE* is not taut.)

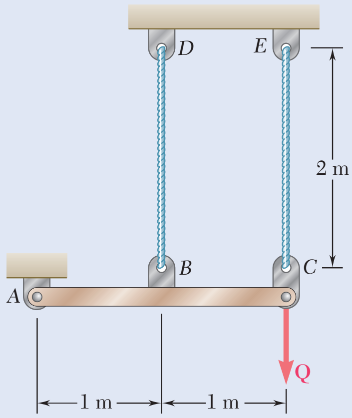


Fig. *P2.109*

2.110 Solve Prob. 2.109, assuming that the cables are replaced by rods of the same cross-sectional area and material. Further assume that the rods are braced so that they can carry compressive forces.

2.111 Two tempered-steel bars, each $\frac{3}{16}$ in. thick, are bonded to a $\frac{1}{2}$-in. mild-steel bar. This composite bar is subjected as shown to a centric axial load of magnitude *P*. Both steels are elastoplastic with $E = 29 \times 10^6$ psi and with yield strengths equal to 100 ksi and 50 ksi, respectively, for the tempered and mild steel. The load *P* is gradually increased from zero until the deformation of the bar reaches a maximum value $\delta_m = 0.04$ in. and then decreased back to zero. Determine (*a*) the maximum value of *P*, (*b*) the maximum stress in the tempered-steel bars, (*c*) the permanent set after the load is removed.

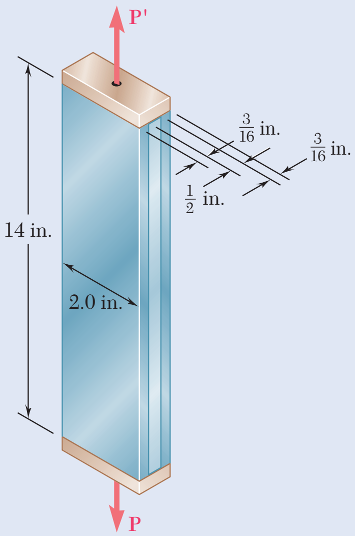


Fig. P2.111

2.112 For the composite bar of Prob. 2.111, if P is gradually increased from zero to 98 kips and then decreased back to zero, determine (*a*) the maximum deformation of the bar, (*b*) the maximum stress in the tempered-steel bars, (*c*) the permanent set after the load is removed.

2.113 The rigid bar ABC is supported by two links, AD and BE, of uniform 37.5 × 6-mm rectangular cross section and made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. The magnitude of the force **Q** applied at B is gradually increased from zero to 260 kN. Knowing that $a = 0.640$ m, determine (*a*) the value of the normal stress in each link, (*b*) the maximum deflection of point B.

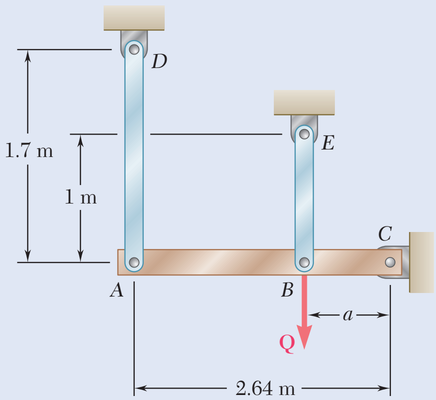


Fig. P2.113

2.114 Solve Prob. 2.113, knowing that $a = 1.76$ m and that the magnitude of the force **Q** applied at B is gradually increased from zero to 135 kN.

***2.115** Solve Prob. 2.113, assuming that the magnitude of the force **Q** applied at B is gradually increased from zero to 260 kN and then decreased back to zero. Knowing that $a = 0.640$ m, determine (*a*) the residual stress in each link, (*b*) the final deflection of point B. Assume that the links are braced so that they can carry compressive forces without buckling.

2.116 A uniform steel rod of cross-sectional area A is attached to rigid supports and is unstressed at a temperature of 45°F. The steel is assumed to be elastoplastic with $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Knowing that $\alpha = 6.5 \times 10^{-6}$/°F, determine the stress in the bar (*a*) when the temperature is raised to 320°F, (*b*) after the temperature has returned to 45°F.

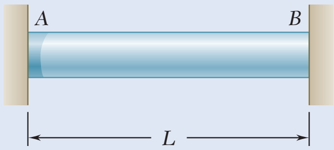


Fig. P2.116

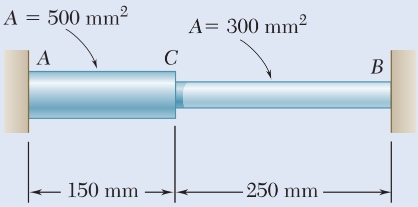


Fig. P2.117

2.117 The steel rod *ABC* is attached to rigid supports and is unstressed at a temperature of 25°C. The steel is assumed elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. The temperature of both portions of the rod is then raised to 150°C. Knowing that $\alpha = 11.7 \times 10^{-6}$/°C, determine (*a*) the stress in both portions of the rod, (*b*) the deflection of point *C*.

***2.118** Solve Prob. 2.117, assuming that the temperature of the rod is raised to 150°C and then returned to 25°C.

***2.119** For the composite bar of Prob. 2.111, determine the residual stresses in the tempered-steel bars if *P* is gradually increased from zero to 98 kips and then decreased back to zero.

***2.120** For the composite bar in Prob. 2.111, determine the residual stresses in the tempered-steel bars if *P* is gradually increased from zero until the deformation of the bar reaches a maximum value $\delta_m = 0.04$ in. and is then decreased back to zero.

***2.121** Narrow bars of aluminum are bonded to the two sides of a thick steel plate as shown. Initially, at $T_1 = 70$°F, all stresses are zero. Knowing that the temperature will be slowly raised to T_2 and then reduced to T_1, determine (*a*) the highest temperature T_2 that does *not* result in residual stresses, (*b*) the temperature T_2 that will result in a residual stress in the aluminum equal to 58 ksi. Assume $\alpha_a = 12.8 \times 10^{-6}$/°F for the aluminum and $\alpha_s = 6.5 \times 10^{-6}$/°F for the steel. Further assume that the aluminum is elastoplastic with $E = 10.9 \times 10^6$ psi and $\alpha_Y = 58$ ksi. (*Hint:* Neglect the small stresses in the plate.)

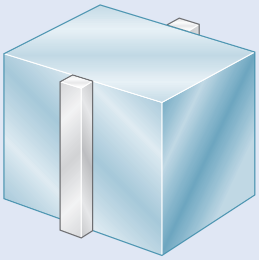


Fig. P2.121

***2.122** Bar *AB* has a cross-sectional area of 1200 mm^2 and is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. Knowing that the force **F** increases from 0 to 520 kN and then decreases to zero, determine (*a*) the permanent deflection of point *C*, (*b*) the residual stress in the bar.

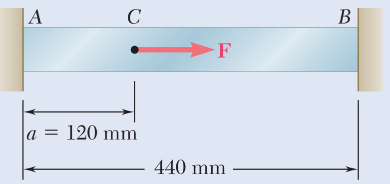


Fig. P2.122

***2.123** Solve Prob. 2.122, assuming that $a = 180$ mm.

Review and Summary

Normal Strain

Consider a rod of length L and uniform cross section, and its deformation δ under an axial load **P** (Fig. 2.59). The *normal strain* ϵ in the rod is defined as the *deformation per unit length*:

$$\epsilon = \frac{\delta}{L} \quad \textbf{(2.1)}$$

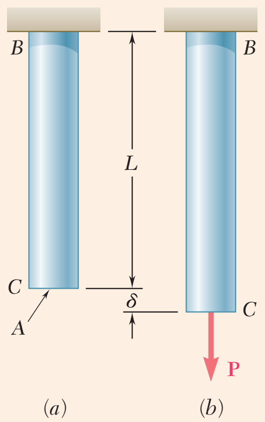


Fig. 2.59 Undeformed and deformed axially-loaded rod.

In the case of a rod of variable cross section, the normal strain at any given point Q is found by considering a small element of rod at Q:

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \quad \textbf{(2.2)}$$

Stress-Strain Diagram

A *stress-strain diagram* is obtained by plotting the stress σ versus the strain ϵ as the load increases. These diagrams can be used to distinguish between *brittle* and *ductile* materials. A brittle material ruptures without any noticeable prior change in the rate of elongation (Fig. 2.60), while a ductile material

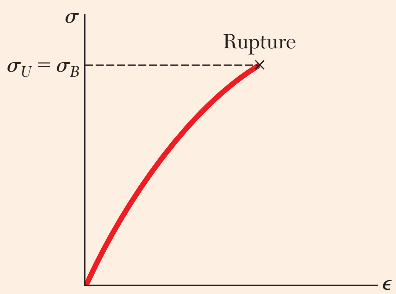


Fig. 2.60 Stress-strain diagram for a typical brittle material.

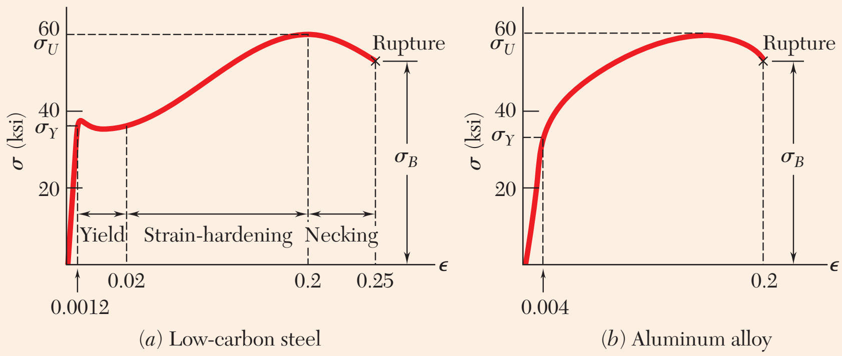


Fig. 2.61 Stress-strain diagrams of two typical ductile metal materials.

yields after a critical stress σ_Y (the *yield strength*) has been reached (Fig. 2.61). The specimen undergoes a large deformation before rupturing, with a relatively small increase in the applied load. An example of brittle material with different properties in tension and compression is *concrete*.

Hooke's Law and Modulus of Elasticity

The initial portion of the stress-strain diagram is a straight line. Thus, for small deformations, the stress is directly proportional to the strain:

$$\sigma = E\epsilon \tag{2.6}$$

This relationship is *Hooke's law*, and the coefficient *E* is the *modulus of elasticity* of the material. The *proportional limit* is the largest stress for which Eq. (2.4) applies.

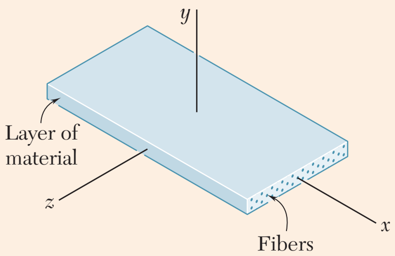


Fig. 2.62 Layer of fiber-reinforced composite material.

Properties of *isotropic* materials are independent of direction, while properties of *anisotropic* materials depend upon direction. *Fiber-reinforced composite materials* are made of fibers of a strong, stiff material embedded in layers of a weaker, softer material (Fig. 2.62).

Elastic Limit and Plastic Deformation

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically*. The largest stress for which this occurs is called the *elastic limit* of the material. If the elastic limit is exceeded, the stress and strain decrease in a linear fashion when the load is removed, and the strain does not return to zero (Fig. 2.63), indicating that a *permanent set* or *plastic deformation* of the material has taken place.

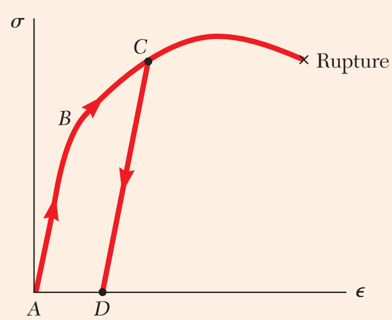


Fig. 2.63 Stress-strain response of ductile material loaded beyond yield and unloaded.

Fatigue and Endurance Limit

Fatigue causes the failure of structural or machine components after a very large number of repeated loadings, even though the stresses remain in the elastic range. A standard fatigue test determines the number n of successive loading-and-unloading cycles required to cause the failure of a specimen for any given maximum stress level σ and plots the resulting σ-n curve. The value of σ for which failure does not occur, even for an indefinitely large number of cycles, is known as the *endurance limit*.

Elastic Deformation Under Axial Loading

If a rod of length L and uniform cross section of area A is subjected at its end to a centric axial load **P** (Fig. 2.64), the corresponding deformation is

$$\delta = \frac{PL}{AE} \qquad \textbf{(2.9)}$$

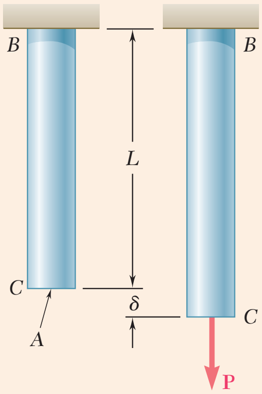


Fig. 2.64 Undeformed and deformed axially-loaded rod.

If the rod is loaded at several points or consists of several parts of various cross sections and possibly of different materials, the deformation δ of the rod must be expressed as the sum of the deformations of its component parts:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \qquad \textbf{(2.10)}$$

Statically Indeterminate Problems

Statically indeterminate problems are those in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration were complemented by relations involving deformations and obtained from the geometry of the problem. The forces in the rod and in the tube of Fig. 2.65, for instance, were determined by observing that their sum is equal to P, and that they cause equal deformations in the rod and in the tube. Similarly, the reactions at the supports of the bar of

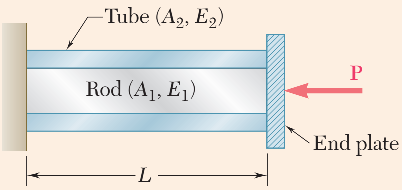


Fig. 2.65 Statically indeterminate problem where concentric rod and tube have same strain but different stresses.

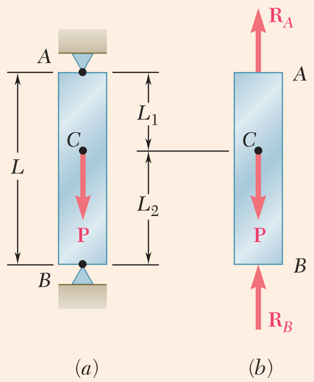


Fig. 2.66 (*a*) Axially-loaded statically-indeterminate member. (*b*) Free-body diagram.

Fig. 2.66 could not be obtained from the free-body diagram of the bar alone, but they could be determined by expressing that the total elongation of the bar must be equal to zero.

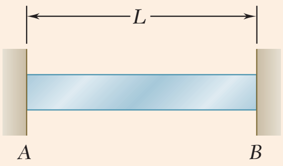


Fig. 2.67 Fully restrained bar of length *L*.

Problems with Temperature Changes

When the temperature of an *unrestrained rod AB* of length L is increased by ΔT, its elongation is

$$\delta_T = \alpha(\Delta T)L \qquad \textbf{(2.13)}$$

where α is the *coefficient of thermal expansion* of the material. The corresponding strain, called *thermal strain*, is

$$\epsilon_T = \alpha \Delta T \qquad \textbf{(2.14)}$$

and *no stress* is associated with this strain. However, if rod *AB* is *restrained* by fixed supports (Fig. 2.67), stresses develop in the rod as the temperature increases, because of the reactions at the supports. To determine the magnitude *P* of the reactions, the rod is first detached from its support at *B* (Fig. 2.68*a*).

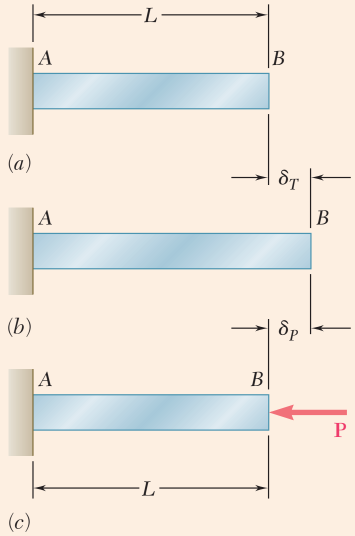


Fig. 2.68 Determination of reactions for bar of Fig. 2.67 subject to a temperature increase. (*a*) Support at *B* removed. (*b*) Thermal expansion. (*c*) Application of support reaction to counter thermal expansion.

The deformation δ_T of the rod occurs as it expands due to of the temperature change (Fig. 2.68*b*). The deformation δ_P caused by the force **P** is required to bring it back to its original length, so that it may be reattached to the support at *B* (Fig. 2.68*c*).

Lateral Strain and Poisson's Ratio

When an axial load **P** is applied to a homogeneous, slender bar (Fig. 2.69), it causes a strain, not only along the axis of the bar but in any transverse direction. This strain is the *lateral strain*, and the ratio of the lateral strain over the axial strain is called *Poisson's ratio*:

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.17}$$

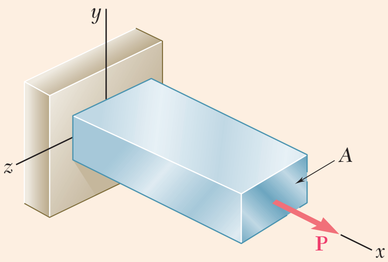


Fig. 2.69 A bar in uniaxial tension.

Multiaxial Loading

The condition of strain under an axial loading in the *x* direction is

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.19}$$

A *multiaxial loading* causes the state of stress shown in Fig. 2.70. The resulting strain condition was described by the *generalized Hooke's law* for a multiaxial loading.

$$\begin{aligned} \epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned} \tag{2.20}$$

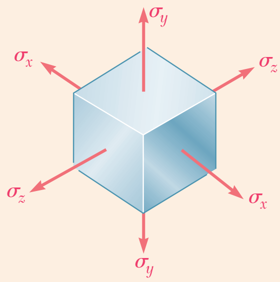


Fig. 2.70 State of stress for multiaxial loading.

Dilatation

If an element of material is subjected to the stresses σ_x, σ_y, σ_z, it will deform and a certain change of volume will result. The *change in volume per unit volume* is the *dilatation* of the material:

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{2.22}$$

Bulk Modulus

When a material is subjected to a hydrostatic pressure *p*,

$$e = -\frac{p}{k} \tag{2.25}$$

where *k* is the *bulk modulus* of the material:

$$k = \frac{E}{3(1 - 2\nu)} \tag{2.24}$$

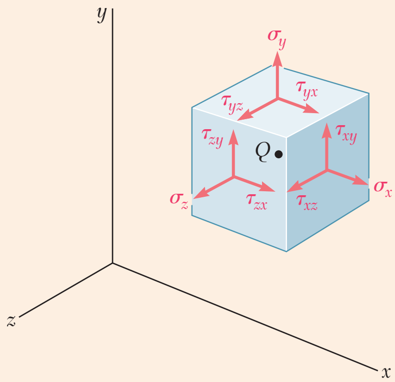


Fig. 2.71 Positive stress components at point Q for a general state of stress.

Shearing Strain: Modulus of Rigidity

The state of stress in a material under the most general loading condition involves shearing stresses, as well as normal stresses (Fig. 2.71). The shearing stresses tend to deform a cubic element of material into an oblique parallelepiped. The stresses τ_{xy} and τ_{yx} shown in Fig. 2.72 cause the angles formed by the faces on which they act to either increase or decrease by a small angle γ_{xy}. This angle defines the *shearing strain* corresponding to the x and y directions. Defining in a similar way the shearing strains γ_{yz} and γ_{zx}, the following relations were written:

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{2.27, 28}$$

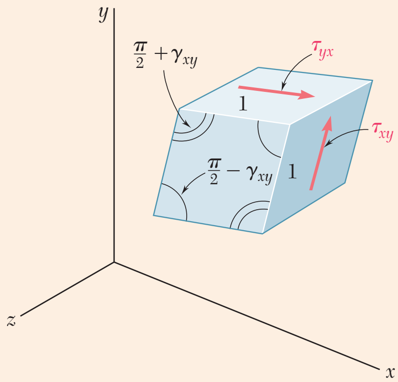


Fig. 2.72 Deformation of unit cubic element due to shearing stress.

which are valid for any homogeneous isotropic material within its proportional limit in shear. The constant G is the *modulus of rigidity* of the material, and the relationships obtained express *Hooke's law for shearing stress and strain*. Together with Eqs. (2.20), they form a group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

While an axial load exerted on a slender bar produces only normal strains—both axial and transverse—on an element of material oriented along the axis of the bar, it will produce both normal and shearing strains on an element rotated through 45° (Fig. 2.73). The three constants E, ν, and G are not independent. They satisfy the relation

$$\frac{E}{2G} = 1 + \nu \qquad \textbf{(2.34)}$$

This equation can be used to determine any of the three constants in terms of the other two.

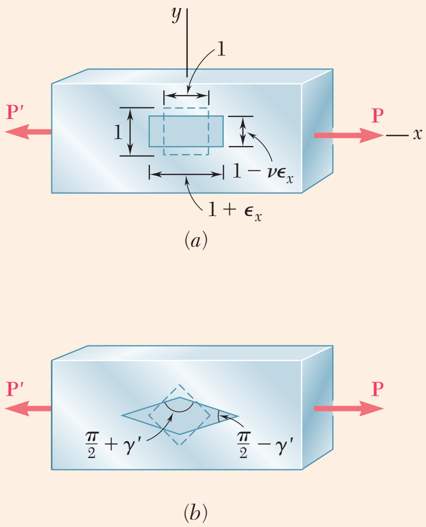


Fig. 2.73 Representations of strain in an axially-loaded bar: (*a*) cubic strain element with faces aligned with coordinate axes; (*b*) cubic strain element with faces rotated 45° about *z*-axis.

Saint-Venant's Principle

Saint-Venant's principle states that except in the immediate vicinity of the points of application of the loads, the distribution of stresses in a given member is independent of the actual mode of application of the loads. This principle makes it possible to assume a uniform distribution of stresses in a member subjected to concentrated axial loads, except close to the points of application of the loads, where stress concentrations will occur.

Stress Concentrations

Stress concentrations will also occur in structural members near a discontinuity, such as a hole or a sudden change in cross section. The ratio of the maximum value of the stress occurring near the discontinuity over the average stress computed in the critical section is referred to as the *stress-concentration factor* of the discontinuity:

$$K = \frac{\sigma_{\max}}{\sigma_{\text{ave}}} \qquad \textbf{(2.40)}$$

Plastic Deformations

Plastic deformations occur in structural members made of a ductile material when the stresses in some part of the member exceed the yield strength of the material. An idealized *elastoplastic material* is characterized by the stress-strain diagram shown in Fig. 2.74. When an indeterminate structure undergoes plastic deformations, the stresses do not, in general, return to zero after the load has been removed. The stresses remaining in the various parts of the structure are called *residual stresses* and can be determined by adding the maximum stresses reached during the loading phase and the reverse stresses corresponding to the unloading phase.

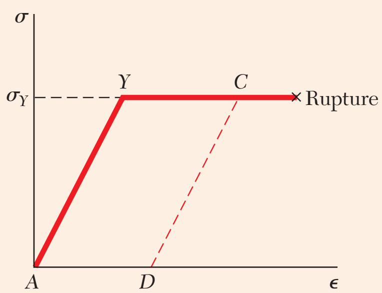


Fig. 2.74 Stress-strain diagram for an idealized elastoplastic material.

Review Problems

2.124 The uniform wire ABC, of unstretched length $2l$, is attached to the supports shown and a vertical load **P** is applied at the midpoint B. Denoting by A the cross-sectional area of the wire and by E the modulus of elasticity, show that, for $\delta << l$, the deflection at the midpoint B is

$$\delta = l\sqrt[3]{\frac{P}{AE}}$$

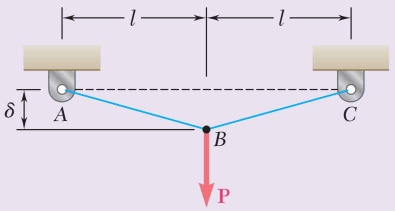


Fig. P2.124

2.125 The aluminum rod ABC ($E = 10.1 \times 10^6$ psi), which consists of two cylindrical portions AB and BC, is to be replaced with a cylindrical steel rod DE ($E = 29 \times 10^6$ psi) of the same overall length. Determine the minimum required diameter d of the steel rod if its vertical deformation is not to exceed the deformation of the aluminum rod under the same load and if the allowable stress in the steel rod is not to exceed 24 ksi.

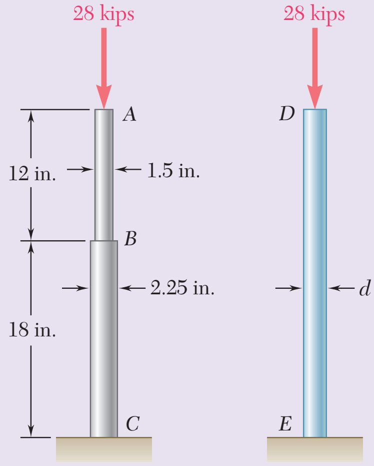


Fig. P2.125

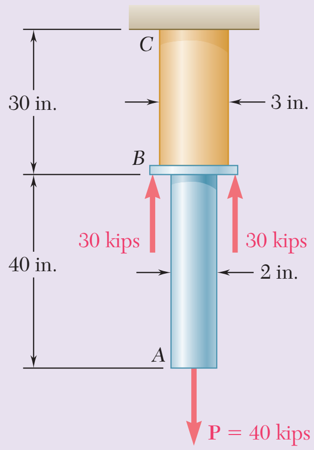


Fig. *P2.126*

2.126 Two solid cylindrical rods are joined at B and loaded as shown. Rod AB is made of steel ($E = 29 \times 10^6$ psi), and rod BC of brass ($E = 15 \times 10^6$ psi). Determine (*a*) the total deformation of the composite rod ABC, (*b*) the deflection of point B.

2.127 The brass strip AB has been attached to a fixed support at A and rests on a rough support at B. Knowing that the coefficient of friction is 0.60 between the strip and the support at B, determine the decrease in temperature for which slipping will impend.

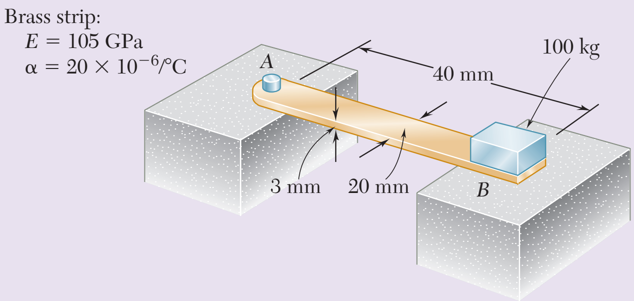


Fig. P2.127

2.128 The specimen shown is made from a 1-in.-diameter cylindrical steel rod with two 1.5-in.-outer-diameter sleeves bonded to the rod as shown. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the load **P** so that the total deformation is 0.002 in., (*b*) the corresponding deformation of the central portion BC.

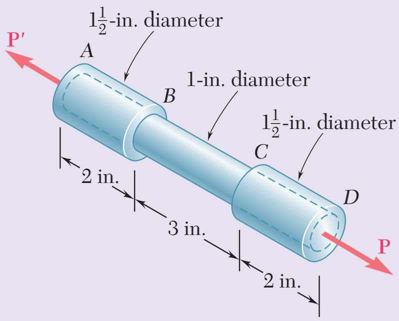


Fig. P2.128

2.129 Each of the four vertical links connecting the two rigid horizontal members is made of aluminum (E = 70 GPa) and has a uniform rectangular cross section of 10 × 40 mm. For the loading shown, determine the deflection of (*a*) point E, (*b*) point F, (*c*) point G.

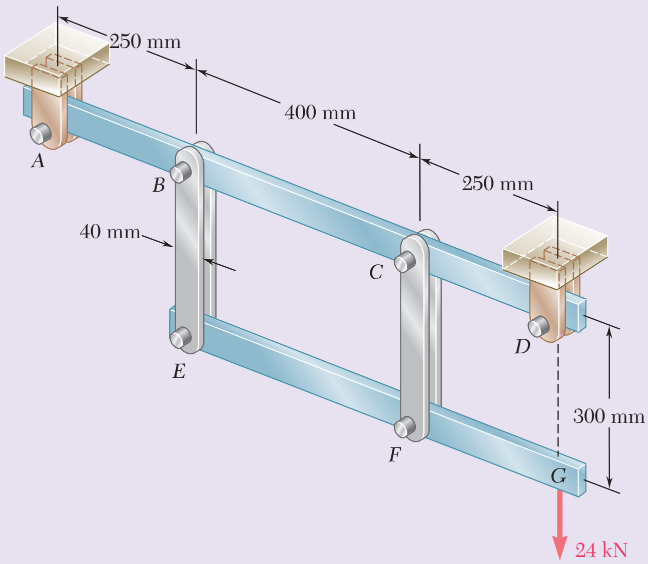


Fig. ***P2.129***

2.130 A 4-ft concrete post is reinforced with four steel bars, each with a $\frac{3}{4}$-in. diameter. Knowing that $E_s = 29 \times 10^6$ psi and $E_c = 3.6 \times 10^6$ psi, determine the normal stresses in the steel and in the concrete when a 150-kip axial centric force **P** is applied to the post.

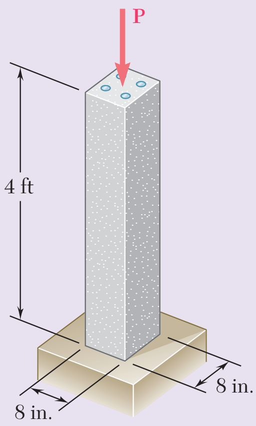


Fig. P2.130

2.131 The steel rods *BE* and *CD* each have a 16-mm diameter ($E = 200$ GPa); the ends of the rods are single-threaded with a pitch of 2.5 mm. Knowing that after being snugly fitted, the nut at *C* is tightened one full turn, determine (*a*) the tension in rod *CD*, (*b*) the deflection of point *C* of the rigid member *ABC*.

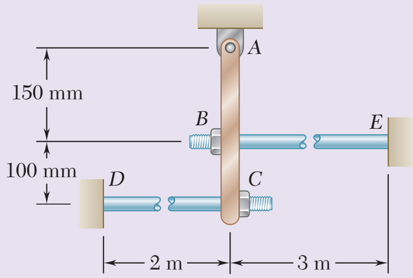


Fig. P2.131

2.132 The assembly shown consists of an aluminum shell ($E_a = 10.6 \times 10^6$ psi, $\alpha_a = 12.9 \times 10^{-6}/°F$) fully bonded to a steel core ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/°F$) and is unstressed. Determine (*a*) the largest allowable change in temperature if the stress in the aluminum shell is not to exceed 6 ksi, (*b*) the corresponding change in length of the assembly.

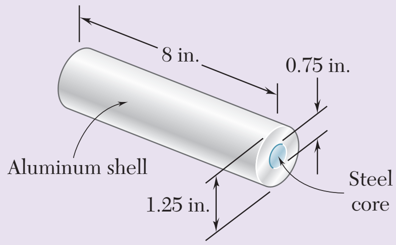


Fig. *P2.132*

2.133 The plastic block shown is bonded to a fixed base and to a horizontal rigid plate to which a force **P** is applied. Knowing that for the plastic used $G = 55$ ksi, determine the deflection of the plate when $P = 9$ kips.

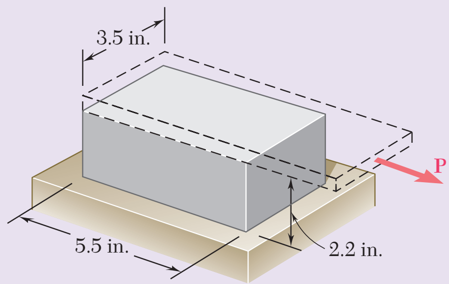


Fig. P2.133

2.134 The aluminum test specimen shown is subjected to two equal and opposite centric axial forces of magnitude P. (*a*) Knowing that $E = 70$ GPa and $\sigma_{all} = 200$ MPa, determine the maximum allowable value of P and the corresponding total elongation of the specimen. (*b*) Solve part *a*, assuming that the specimen has been replaced by an aluminum bar of the same length and a uniform 60 × 15-mm rectangular cross section.

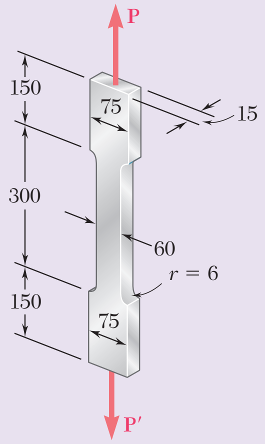


Fig. P2.134

2.135 The uniform rod BC has cross-sectional area A and is made of a mild steel that can be assumed to be elastoplastic with a modulus of elasticity E and a yield strength σ_Y. Using the block-and-spring system shown, it is desired to simulate the deflection of end C of the rod as the axial force **P** is gradually applied and removed, that is, the deflection of points C and C' should be the same for all values of P. Denoting by μ the coefficient of friction between the block and the horizontal surface, derive an expression for (*a*) the required mass m of the block, (*b*) the required constant k of the spring.

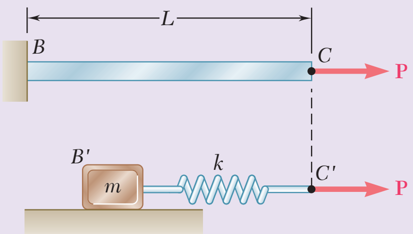


Fig. P2.135

Computer Problems

The following problems are designed to be solved with a computer. Write each program so that it can be used with either SI or U.S. customary units and in such a way that solid cylindrical elements may be defined by either their diameter or their cross-sectional area.

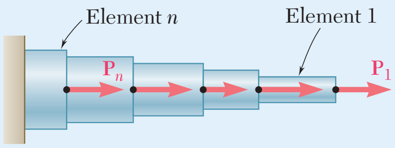


Fig. P2.C1

2.C1 A rod consisting of n elements, each of which is homogeneous and of uniform cross section, is subjected to the loading shown. The length of element i is denoted by L_i, its cross-sectional area by A_i, modulus of elasticity by E_i, and the load applied to its right end by $\mathbf{P}_i$, the magnitude P_i of this load being assumed to be positive if $\mathbf{P}_i$ is directed to the right and negative otherwise. (*a*) Write a computer program that can be used to determine the average normal stress in each element, the deformation of each element, and the total deformation of the rod. (*b*) Use this program to solve Probs. 2.20 and 2.126.

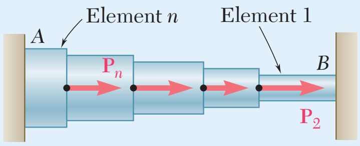


Fig. P2.C2

2.C2 Rod AB is horizontal with both ends fixed; it consists of n elements, each of which is homogeneous and of uniform cross section, and is subjected to the loading shown. The length of element i is denoted by L_i, its cross-sectional area by A_i, its modulus of elasticity by E_i, and the load applied to its right end by $\mathbf{P}_i$, the magnitude P_i of this load being assumed to be positive if $\mathbf{P}_i$ is directed to the right and negative otherwise. (Note that $P_1 = 0$.) (*a*) Write a computer program that can be used to determine the reactions at A and B, the average normal stress in each element, and the deformation of each element. (*b*) Use this program to solve Probs. 2.41 and 2.42.

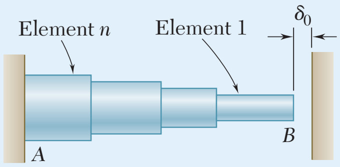


Fig. P2.C3

2.C3 Rod AB consists of n elements, each of which is homogeneous and of uniform cross section. End A is fixed, while initially there is a gap δ_0 between end B and the fixed vertical surface on the right. The length of element i is denoted by L_i, its cross-sectional area by A_i, its modulus of elasticity by E_i, and its coefficient of thermal expansion by α_i. After the temperature of the rod has been increased by ΔT, the gap at B is closed and the vertical surfaces exert equal and opposite forces on the rod. (*a*) Write a computer program that can be used to determine the magnitude of the reactions at A and B, the normal stress in each element, and the deformation of each element. (*b*) Use this program to solve Probs. 2.59 and 2.60.

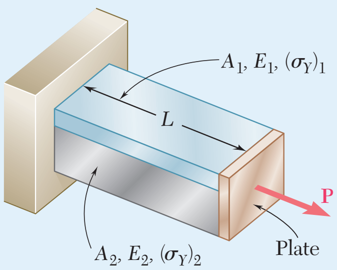


Fig. P2.C4

2.C4 Bar AB has a length L and is made of two different materials of given cross-sectional area, modulus of elasticity, and yield strength. The bar is subjected as shown to a load $\mathbf{P}$ that is gradually increased from zero until the deformation of the bar has reached a maximum value δ_m and then decreased back to zero. (*a*) Write a computer program that, for each of 25 values of δ_m equally spaced over a range extending from 0 to a value equal to 120% of the deformation causing both materials to yield, can be used to determine the maximum value P_m of the load, the maximum normal stress in each material, the permanent deformation δ_p of the bar, and the residual stress in each material. (*b*) Use this program to solve Probs. 2.111 and 2.112.

2.C5 The plate has a hole centered across the width. The stress concentration factor for a flat bar under axial loading with a centric hole is

$$K = 3.00 - 3.13\left(\frac{2r}{D}\right) + 3.66\left(\frac{2r}{D}\right)^2 - 1.53\left(\frac{2r}{D}\right)^3$$

where r is the radius of the hole and D is the width of the bar. Write a computer program to determine the allowable load **P** for the given values of r, D, the thickness t of the bar, and the allowable stress σ_{all} of the material. Knowing that $t = \frac{1}{4}$ in., $D = 3.0$ in. and $\sigma_{all} = 16$ ksi, determine the allowable load **P** for values of r from 0.125 in. to 0.75 in., using 0.125 in. increments.

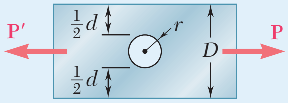


Fig. P2.C5

2.C6 A solid truncated cone is subjected to an axial force **P** as shown. The exact elongation is $(PL)/(2\pi c^2 E)$. By replacing the cone by n circular cylinders of equal thickness, write a computer program that can be used to calculate the elongation of the truncated cone. What is the percentage error in the answer obtained from the program using (*a*) $n = 6$, (*b*) $n = 12$, (*c*) $n = 60$?

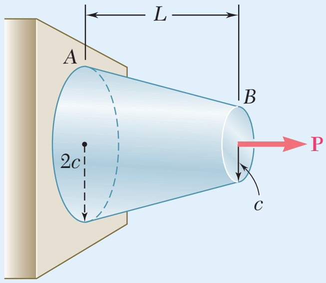


Fig. P2.C6