Review of Fundamental Mathematics

As explained in the Preface and in Chapter 1 of your textbook, managerial economics applies microeconomic theory to business decision making. The decision-making tools that you will learn can provide numerical answers to questions such as:

- What price should Harley-Davidson charge for its 2008 model Sportster?
- At Doctors Walk-In Clinic, what combination of doctors and nurses provides the expected level of medical services at the lowest possible total cost?
- What is the forecasted price of West Texas crude oil six months from now?
- How many PCs should Dell manufacture and sell each month in order to maximize profit?

In order to apply the decision-making rules developed in managerial economics, you need to understand some fundamental mathematics, most of which you learned in high school algebra. No calculus is used in the body of the textbook, and none is required to work the *Technical Problems* and *Applied Problems* at the end of each chapter in the textbook. Exercises are provided throughout this Review. Answers to the exercises can be found at the end of this Review.

Mathematical Functions

CONCEPT OF A FUNCTION

The relation between decision-making variables such as output, labor and capital employment, price, cost, and profit can be expressed using mathematical *functions*. A function shows mathematically the relation between a *dependent* variable and one or more *independent* variables. The dependent variable is often denoted by y, and the independent variable is denoted by x. The idea that "y is related to x" or that "y is a function of x" is expressed symbolically as

$$y = f(x)$$

This mathematical notation expresses the relation between y and x in the most *general* functional form. In contrast to this general form of expression, a *specific* functional form uses an equation to show the exact mathematical relation between y and x using an equation. For example,

$$y = 100 - 2x$$

gives a specific function relating y to values of x.

The equation showing the specific functional relation between y and x provides a "formula" for calculating the value of y for any specific value of x. In the specific function given above, when x equals 20, y equals 60 ($= 100 - 2 \times 20$). In economic analysis, it is frequently convenient to denote the dependent and independent variable(s) using notation that reminds us of the economic variables involved in the relation. Suppose, for example, that the quantity of golfing lessons (q) that a professional golf instructor gives each week depends on the price charged by the golf pro (p). The specific functional relation — which is called the golf pro's "demand" for lessons — can be expressed as

$$q = f(p) = 100 - 2p$$

Instead of using the "generic" mathematical names y and x, the dependent variable is denoted by q to suggest "quantity," and the independent variable is denoted by p to suggest "price." If the golf pro charges \$40 per lesson, then she will give 20 ($= 100 - 2 \times 40$) lessons per week.

INVERSE FUNCTIONS

Sometimes it is useful to "reverse" a functional relation so that x becomes the dependent variable and y becomes the independent variable:[1]

$$x = f^{-1}(y)$$

This function, known as an *inverse function*, gives the value of x for given values of y. The inverse of $y = f(x)$ can be derived algebraically by expressing x as a function of y.[2] Consider again the demand for golfing lessons. The inverse function gives the price that a golf pro can charge for a given quantity of lessons each week:

$$p = f(q) = 50 - 0.5q$$

If the golf pro wishes to sell 20 lessons per week, she charges a price of \$40 ($= 50 - 0.5 \times 20$) per lesson. Notice that all of the values of q and p satisfying the function $q = 100 - 2p$ also satisfy the inverse function $p = 50 - 0.5q$.

FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES

In many economic relations, the dependent variable is a function of, or depends on, the values of more than one independent variable. If y depends on both w and z, for example, the general functional relation is expressed as

$$y = f(w, z)$$

Suppose a firm employs two inputs, labor and capital, to produce its product. The price of labor is \$30 per hour, and the price of using capital is \$60 per hour. The total cost of production (C) can be expressed as a function of the amount of labor employed per hour (L) and the amount of capital used per hour (K):

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[1] For the function $y = f(x)$, mathematicians denote its inverse function as $x = f^{-1}(y)$. The "–1" is not an exponent here, but rather it denotes this function to be the inverse function of $y = f(x)$.

[2] Inverse functions do not exist for all functions. In this review, we will not investigate the conditions that ensure the existence of an inverse. In cases where inverse functions are required for managerial decision making, the required inverse functions generally do exist and can be rather easily derived.

$$C = f(L, K) = 30L + 60K$$

From this cost function, the manager calculates the cost of employing 10 workers and 5 units of capital to be \$600 ($= 30 \times 10 + 60 \times 5$).

Exercises (Answers to Exercises follow this Review on pages 17-18.)

1. In the following functions, find the value of the dependent variable when $x = 10$.
   a. $y = 300 - 20x$
   b. $y = 10 + 3x + x^2$
   c. $w = -20 + 3x$
   d. $s = 40x + 2$

2. Find the inverse functions.
   a. $y = 450 + 15x$
   b. $q = 1{,}000 - 25p$

3. In the functions below, find the value of the dependent variable for the values of the independent variables given.
   a. $y = f(x, w) = x^2 + w^2$, for $x = 25$ and $w = 5$
   b. $z = f(w, r) = 50w + 150r$, for $w = 200$ and $r = 32$

Linear Functions

Functions can be either *linear* or *nonlinear* in form. This section of the Review discusses the properties of linear functions, and then the next section examines nonlinear functions. The primary distinction between linear and nonlinear functions is the nature of their slopes: linear functions have constant slopes while curvilinear (i.e., nonlinear) functions have varying slopes.

DEFINITION OF LINEAR FUNCTIONS

A function, $y = f(x)$, is a linear function if a graph of all the combinations of x and y that satisfy the equation $y = f(x)$ lie on a straight line. Any linear relation between y and x can be expressed in the algebraic form

$$y = a + bx$$

where y is the dependent variable, x is an independent variable, a is the *intercept parameter*, and b is the *slope parameter*. The terms a and b are called *parameters*, rather than *variables*, because their values do not change along the graph of a specific linear function. The intercept parameter, a, gives the value of y when $x = 0$. For the line representing $y = f(x)$, when $x = 0$ the line crosses the y-axis. Hence the name "intercept" parameter.[3] Every line is characterized by a unique pair of values for a and b.

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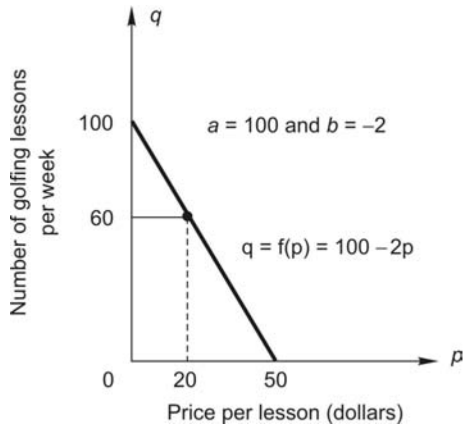
[3]Sometimes a is called the *y-intercept*. When the dependent variable is not named y, the term *y-intercept* can be confusing. It is best to think of the intercept parameter a as giving the point where the line for this function crosses the axis of the variable "on the other side of the equal sign." If, for example, $c = 30 - 5q$, the line passes through the c axis at the value $c = 30$ ($= 30 - 5 \times 0$). Also note that c need not be graphed on the vertical axis; 30 is where the line crosses the c-axis whether c is plotted on the vertical or on the horizontal axis.

Figure A.1 shows a graph of the golf pro's demand function which relates the number of lessons given each week (q) to the price charged for a lesson (p). The values of the parameters of this linear function are a = 100 and $b = -2$. Notice that the line passes through the q axis at 100 lessons per week. The slope of the line is -2. The slope of lines and curves are so important that we will now discuss in detail the meaning of the slope of a line and the slope of a curve.

FIGURE A.1: Demand function for golfing lessons



SLOPE OF A LINE

The slope of a line representing the function $y = f(x)$ is defined as the change in the dependent variable y divided by the change in the independent variable x:

$$slope\ of\ a\ line = b = \frac{\Delta y}{\Delta x}$$

where the Δ symbol indicates the *change* in a variable.[4] When y is plotted on the vertical axis and x is plotted on the horizontal axis, the change is y can be called *rise*, the change in x can be called *run*, so the slope of the line is frequently referred to as *rise over run*. In Figure A.1, moving from the top of the line where it intersects the q axis down to the bottom of the line where it intersects the p axis, the rise is $-100\,(=\Delta q)$. The run is +50 (Δp), and so the slope is -2.

The slope of a line can be either positive, negative, or zero. If x and y move in the same direction, Δx and Δy have the same algebraic sign. When an increase (decrease) in x causes an increase (decrease) in y, x and y are said to be *directly* or *positively* related. When x and y are directly or positively related, the slope of $y = f(x)$ is positive (i.e., $b > 0$). The graph of a line is upward sloping when x and y are directly or positively related. See Panel *A* in Figure A.2 for an example of a direct or positive relation.

Alternatively, x and y may move in opposite directions: an increase (decrease) in x causes a decrease (increase) in y. In this situation, x and y are said to be *inversely* or *negatively* related, and the slope of $y = f(x)$ is negative (i.e., $b < 0$). The graph of a line is downward sloping when variables are inversely or negatively related. See Panel *B* in Figure *A*.2 for an example of an inverse or negative relation.

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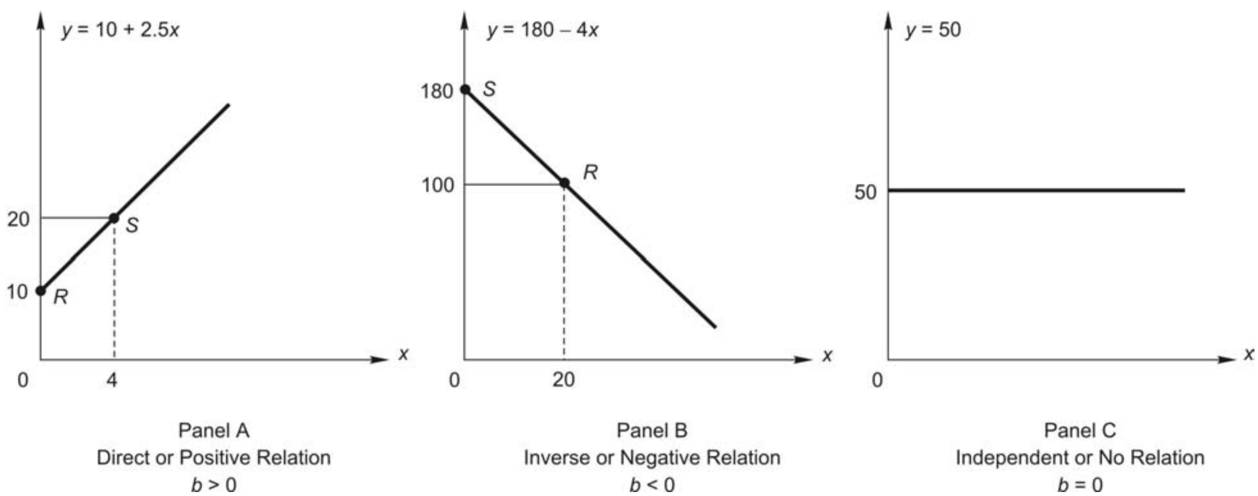
[4]The change in a variable is calculated by taking the final value of the variable (x_1) and subtracting the initial value (x_0): $\Delta x = x_1 - x_0$. When a variable increases (decreases) in value, the final value is greater (less) than the initial value of the variable, and the change in the variable is positive (negative).

In some situations, changes in x do not cause y to increase or to decrease. If changes in x have no effect on y, x and y are unrelated or independent. Suppose, for example, $y = 50$ for all values of x. The parameter values for this situation are $a = 50$ and $b = 0$. The graph of this linear function is a horizontal line at $y = 50$ (see Panel *C* of Figure *A*.2). In general, when x and y are independent, the slope of the line is zero.

FIGURE A.2: The slope of a line



CALCULATING THE SLOPE OF A LINE

Calculating the slope of a line is straightforward. First, locate any two points on the line. Then calculate Δy and Δx between the two points. Finally, divide Δy by Δx. Consider, for example, points *R* and *S* in Panel *A* of Figure *A*.2. Moving from *R* to *S*, $\Delta y = 20 - 10 = +10$ and $\Delta x = 4 - 0 = +4$. Now divide: $\Delta y / \Delta x = +10/+4 = +2.5$. Since the slope is positive, the line is upward sloping and the variables y and x are directly (or positively) related.

As discussed above, when variables are inversely (negatively) related, the slope of the line is negative. In Panel *B* of Figure *A*.2, consider moving from *R* to *S*. Note that $\Delta y = 180 - 100 = +80$ because the movement from *R* to *S* is *upward* (by 80 units). Since moving from *R* to *S* is a leftward movement, the change in x is negative: $\Delta x = 0 - 20 = -20$. Now divide the changes in y and x: $\Delta y / \Delta x = +80/-20 = -4$.

Moving along the line in Panel *C*, Δy is always zero for any change in x. Thus, the slope of the horizontal line is zero.

INTERPRETING THE SLOPE OF A LINE AS A RATE OF CHANGE

Students usually learn to calculate the slope of lines rather quickly. It is surprising, however, that many students who can correctly calculate the slope of a line cannot explain the meaning of slope or interpret the numerical value of slope. To understand the meaning and usefulness of slope in decision making, you must learn to think of slope as the rate of change in the dependent variable as the independent variable changes. Since the slope parameter is a ratio of two changes, Δy and Δx, slope can be interpreted as the *rate of change in y per unit change in x*.

In Panel A of Figure A.2, the slope of the line is 2.5 ($b = \Delta y / \Delta x = +2.5$). This means that y changes 2.5 units for every one-unit change in x, and y and x move in the same direction (since $b > 0$). Thus, if x increases by 2 units, y increases by 5 units. If x decreases by 6 units, then y decreases by 15 units. Y changes 2.5 times as much as x changes and in the same direction as x changes.

In Panel B of Figure A.2, the slope of the line is -4 ($b = \Delta y / \Delta x = -4$). This means that y changes 4 units for every one unit x changes, and y and x move in opposite directions (since $b < 0$). Thus, if x increases by 2 units, y decreases by 8 units. If x decreases by 6 units, then y increases by 24 units. Y changes 4 times as much as x changes but in the opposite direction of x.

When a linear function has more than one independent variable, each independent variable has a slope parameter that gives the rate of change in y per unit change in that particular independent variable *by itself*, or holding other independent variables constant. For example, consider a linear function with three independent variables x, y, and z:

$$w = f(x, y, z) = a + bx + cy + dz$$

The slope parameters b, c, and d give the impacts of one-unit changes in x, y, and z, respectively, on w. Each slope parameter measures the rate of change in w attributable to a change in a specific independent variable. For example, if

$$w = 30 + 2x - 3y + 4z$$

then each one unit increase in x causes w to increase by 2 units, each one-unit increase in y causes w to decrease by 3 units, and each one-unit increase in z causes w to increase by 4 units.

Exercises

4. Consider again the golf pro's demand for lessons, which is graphed in Figure A.1. Let the golf pro increase the price of her lessons from \$20 to \$25.
   a. The quantity of lessons given at a price of \$25 is _____ lessons per week.
   b. The change in p is Δp = ____.
   c. The change in q is Δq = ____.
   d. The slope of the demand line is ______.
   e. Since a \$1 increase in price causes the quantity of lessons each week to _______________ by ____ lessons per week, a \$5 increase in price causes the quantity of lessons per week to ________________ by ____ lessons per week.

5. In Panel A of Figure A.2, change the coordinates of point S to $x = 30$ and $y = 100$. Let point R continue to be at $x = 0$ and $y = 10$.
   a. Moving from point R to the new point S, Δy = ____ − ____ = _____.
   b. Moving from point R to the new point S, Δx = ____ − ____ = _____.
   c. The slope of the line = _______.
   d. The equation of the new line is ________________________.
   e. Along the new line, y changes ______ times as much as x changes and in the ____________ direction.
   f. If x increases by 4 units, y ____________ by _____ units.
   g. The parameter values for the new line are a = _____ and b = _____.

6. In Panel *B* of Figure *A*.2, change the coordinates of point *R* to $x = 40$ and $y = 40$. Let point *S* continue to be at $x = 0$ and $y = 180$.
   a. Moving from the new point *R* to point *S*, Δy = ____ − ____ = _____.
   b. Moving from the new point *R* to point *S*, Δx = ____ − ____ = _____.
   c. The slope of the line = _______.
   d. The equation of the new line is _______________________.
   e. Along the new line, *y* changes ______ times as much as *x* changes and in the ____________ direction.
   f. If *x* increases by 5 units, *y* ____________ by _____ units.
   g. The parameter values for the new line are a = _____ and b = _____.

7. Let $w = 30 + 2x - 3y + 4z$.
   a. Evaluate *w* at $x = 10$, $y = 20$, and $z = 30$.
   b. If *x* decreases by 1 unit, *w* ____________ by ______ units.
   c. If *y* increases by 1 unit, *w* ____________ by ______ units.
   d. If *z* increases by 3 units, *w* ____________ by ______ units.
   e. Evaluate *w* at $x = 9$, $y = 21$, and $z = 33$. Compare this value of *w* to the value of *w* in part *a*: the change in *w* equals _________.
   f. Add the individual impacts on *w* in parts *b*, *c*, and *d*. Does the sum of the individual impacts equal the total change in *w* found in part *e*?

Curvilinear Functions

Linear functions are easy to use because the rate of change in the dependent variable as the independent variable changes is constant. For many relations, however, the rate of change in *y* as *x* changes is not constant. Functions for which the rate of change, or slope, varies are called *curvilinear* functions. As the name suggests, the graph of a curvilinear function is a curve rather than a straight line. While some curvilinear functions can be difficult to use, the curvilinear functions used in managerial economics are generally quite easy to use and interpret.

Any relation between *y* and *x* that cannot be expressed algebraically in the form $y = a + bx$ is a curvilinear function. Examples of curvilinear functions include: $y = a + bx + cx^2$, $y = \sin x$, $y = \ln x$, and $y = e^x$. We begin this section with a discussion of how to measure and interpret the slope of a curvilinear function. Then we examine the properties of polynomial functions and logarithmic functions.

MEASURING SLOPE AT A POINT ON A CURVE

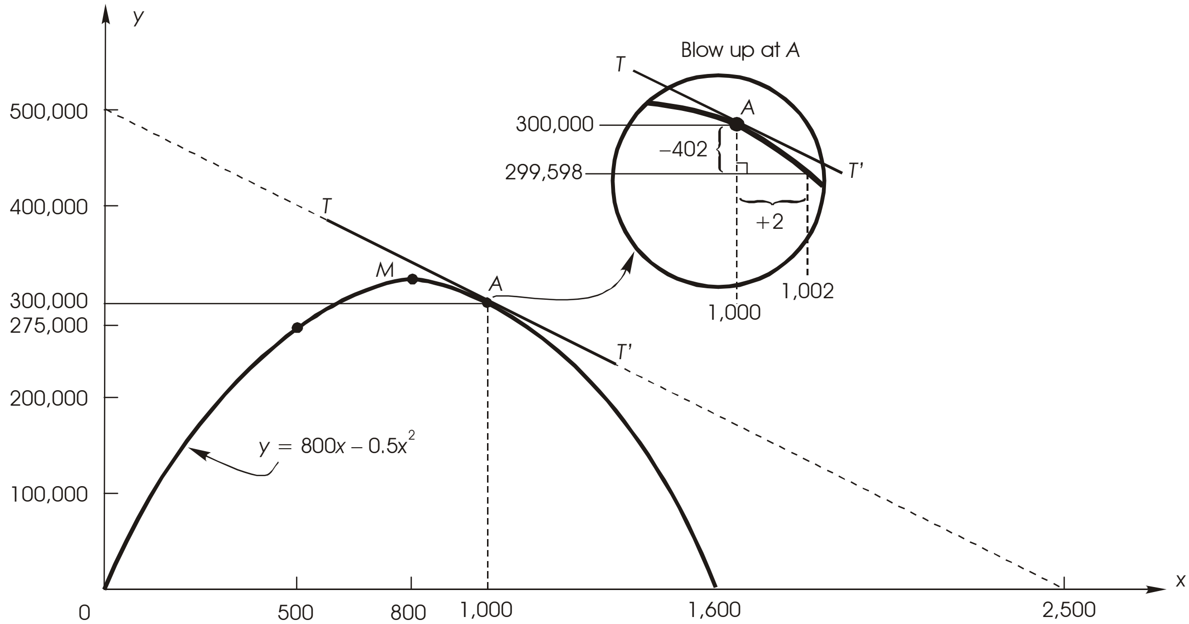
The slope of a curve varies continuously with movements along the curve. It is useful in studying decision making to be able to measure the slope of a curve at any one point of interest along the curve. You may recall from high school algebra or a pre-calculus class in college that the slope at a point on a curve can be measured by constructing the tangent line to the curve at the point.[5] The slope of a curve at the point of tangency is equal to the slope of the tangent line itself.

To illustrate how to measure the slope of a curve at a point, consider the curvilinear function $y = f(x) = 800x - 0.5x^2$, which is graphed in Figure *A*.3. As *x* increases, the value of *y* increases (the curve is positively sloped), reaches its peak value at $x = 800$

[5]When a line is tangent to a curve, it touches the curve at only one point. For smooth, continuous curves, there is one and only one line that is tangent to a curve at any point on the curve. Consequently, the slope of a curve at a point is unique and equal to the slope of the tangent line.

(point M), and then decreases until $y = 0$ at $x = 1{,}600$. Let's find the slope of this curve at point A, which is the point $x = 1{,}000$ and $y = 300{,}000$. First, a tangent line labeled TT' in the figure, is constructed at point A.[6] Then, the slope of TT' is calculated to be -200 $(= -500{,}000/2{,}500)$. The slope of the curve at point A indicates that y changes 200 times as much as x changes, and in the opposite direction because the slope is negative.

FIGURE A.3: Measuring slope at a point on a curve



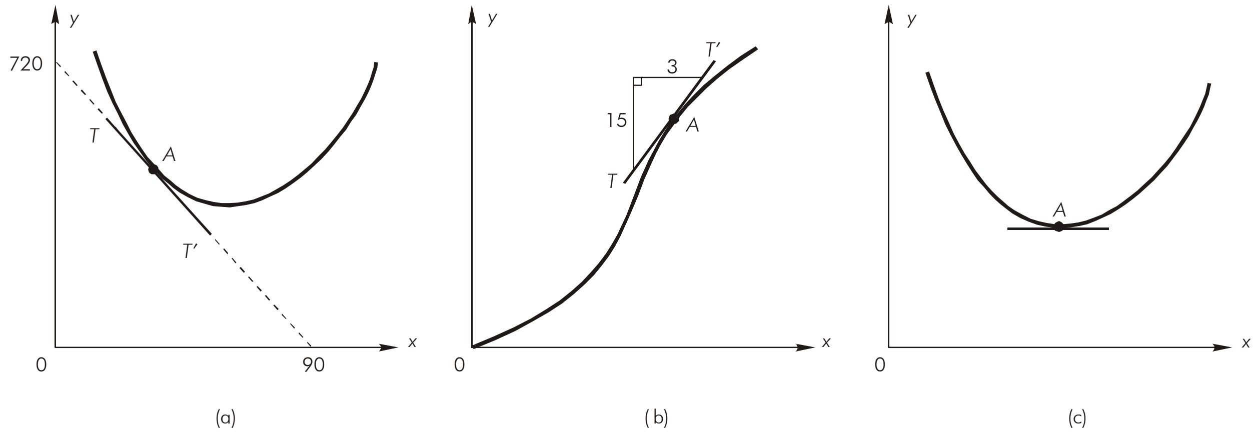
It is important to remember that the slope of a curve at a point measures the rate of change in y at precisely that point. If x changes by more than an infinitesimal amount, which it usually does in practical applications, the slope of the tangent measures only approximately the rate of change in y relative to the change in x. However, the smaller the change in x, the more precisely the change in y can be approximated by the slope of the tangent line.

To illustrate this rather subtle point, let x increase by 2 units from 1,000 (at point A) to 1,002. Since the slope of the curve at point A is -200, the increase in x of 2 units causes y to decrease by approximately 400 units. We say y decreases by "approximately" 400 units, because a 2-unit change in x, which is a rather small change when x is equal to 1,000, is still large enough to create a tiny amount of error. Since the change in x is quite small relative to the point of measure, the actual change in y will be quite close to -400 though not exactly -400. The blow up at point A in Figure A.3 confirms that the change in y is not exactly -400 but rather is -402. When the change in x is "small," we must emphasize that the tiny error can be ignored for practical purposes.

[6]A tangent line that you draw will only approximate an exact tangent line because your eyesight is probably not so sharp that you can draw precisely a line tangent to a curve. Fortunately, for the purposes of managerial economics, you only need to be able to sketch a line that is approximately tangent to a curve. If you have taken a course in calculus, you know that taking the derivative of a function and evaluating this derivative at a point is equivalent to constructing a tangent line and measuring its slope.

Exercises

8. Calculate the slope at point *A* on the following curves:



9. In Figure A.3, find the slope of the curve at $x = 500$. [Hint: Use a ruler and a sharp pencil to draw the appropriate tangent line.]

POLYNOMIAL FUNCTIONS

One of the simplest kinds of curvilinear functions is a *polynomial function.* A polynomial function takes the form

$$y = f(x) = a_0 + a_1 x + a_1 x^2 + ... + a_n x^n$$

where n is an integer and a_0, a_1, a_2, … , a_n are parameters. The highest power to which the independent variable x is raised is called the *degree* of the polynomial. Many of the curvilinear relations in managerial economics involve polynomial functions of degree 2 or degree 3. A polynomial of degree 1 is a linear function, which we have discussed.

When the relation between y and x is a polynomial of degree 2, the function is known as a *quadratic* function, which can be expressed as follows:

$$y = a + bx + cx^2$$

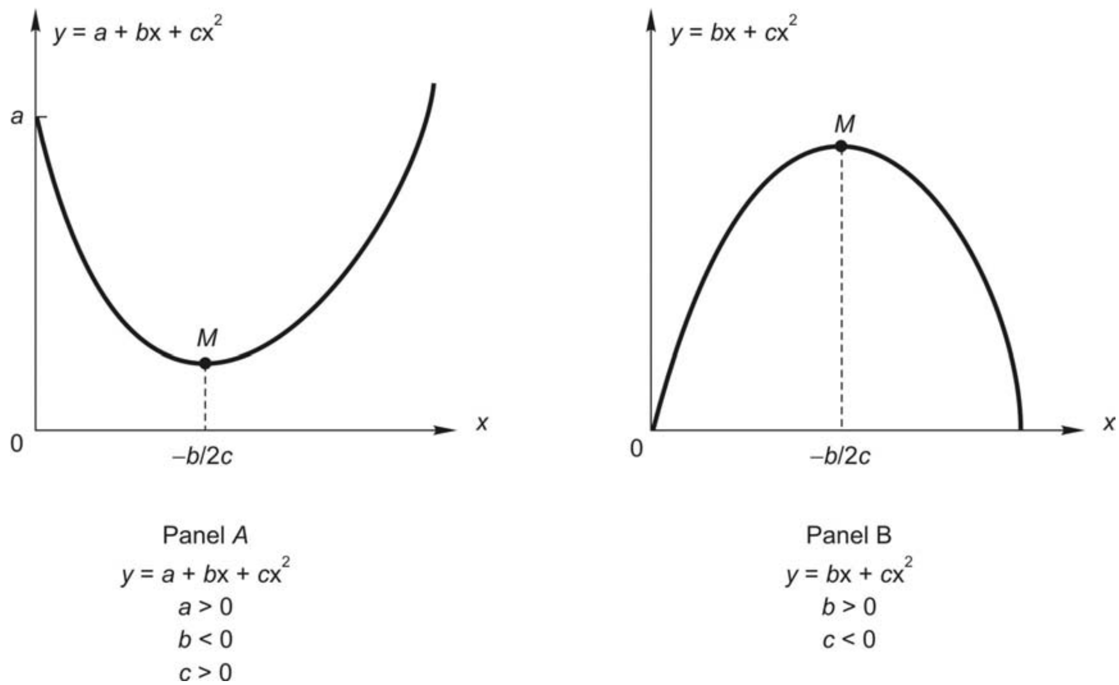
The graph of a quadratic function can be either $\cup$-shaped or $\cap$-shaped, depending on the algebraic signs of the parameters. Two of the most common quadratic functions in managerial economics have the properties shown in Panels *A* and *B* of Figure *A*.4.

We now discuss the restrictions on the algebraic signs of the parameters associated with the $\cup$-shaped or $\cap$-shaped curves shown in Figure *A*.4. As shown in Figure *A*.4, when b is negative (positive) and c is positive (negative), the quadratic function is $\cup$-shaped ($\cap$-shaped).[7] Point *M*, which denotes either the minimum point for a $\cup$-shaped curve in Panel *A* or the maximum point for a $\cap$-shaped curve in Panel *B*, occurs at $x = -b/2c$ in either case. As just mentioned, b and c are of opposite algebraic signs, thus x is positive when these curves reach either a minimum or a maximum.[8]

[7]In Panel *b*, the constant term is zero ($a = 0$) because the curve passes through the origin.

[8]Since economic variables generally take only positive values, an additional mathematical property, $b^2 < 4ac$, ensures that point *M* occurs at a positive, rather than negative, value of y. We do not emphasize this property here or in the textbook because, as it turns out, the condition is always met in practice when the parameters of the quadratic equation are estimated using economic data for which x and y take only positive values.

FIGURE A.4: Two common quadratic functions



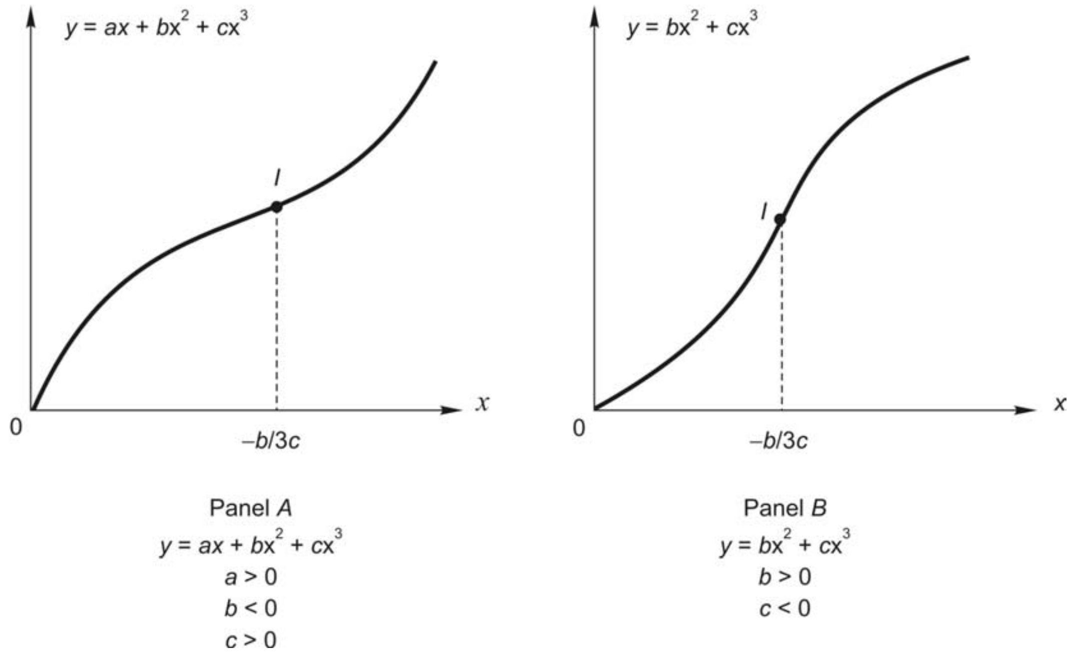
A polynomial function of degree 3 is often referred to as a *cubic* function. The specific form of a cubic function used in managerial economics takes the form:

$$y = ax + bx^2 + cx^3$$

Figure *A*.5 summarizes the shapes of two cubic functions that you will see later in this course. The graph of a cubic function is either *S*-shaped or reverse *S*-shaped, depending on the values of the parameters. In Panel *A*, the reverse *S*-shaped cubic function is characterized by parameters with algebraic signs $a > 0$, $b < 0$, and $c > 0$. In Panel *B*, the *S*-shaped cubic function has parameter restrictions $b > 0$ and $c < 0$. (Note that $a = 0$ in Panel *B*.)[9]

[9] For cubic equations, the condition $b^2 < 4ac$ ensures that the slope of the curve at the point of inflexion (I) is upward (positive). As explained in footnote 8, this condition is not usually of much concern in practice.

FIGURE A.5: Two common cubic functions



The *inflexion point* on an *S*-shaped or a reverse *S*-shaped curve is the point at which the *slope* of the curve reaches either its minimum value (Panel *A*) or its maximum value (Panel *B*). Consider Panel *A*. Beginning at zero and moving toward the point of inflexion (labeled "*I*" in both panels), the slope of the curve is decreasing. Then, beyond the inflexion point, the slope of the curve begins increasing.[10] Note that y is always *increasing* as x increases, but at first (over the range 0 to I) y increases at a decreasing rate, then beyond I, y increases at an increasing rate. It follows that the slope is smallest at point I in Panel *A*. In Panel *B*, y first increases at an increasing rate, and then beyond the inflexion point, y increases at a decreasing rate. Thus, the rate of change in y is greatest at inflexion point I in Panel *B*. The point of inflexion I occurs at $x = -b/3c$ for both *S*-shaped and reverse *S*-shaped curves.

Exercises

10. Consider the function $y = 10 - 0.03x + 0.00005x^2$.
    a. Is this function $\cup$-shaped or $\cap$-shaped? How do you know?
    b. The dependent variable y reaches a ____________(minimum, maximum) value when x = ________.
    c. The value of y at the minimum/maximum point found in part b is _________.

[10] To "see" that slope is decreasing over the segment of the curve from 0 to I, visualize a series of lines tangent to the curve at points along the curve between 0 and I. Since these "visualized" tangent lines are getting flatter between 0 and I, the slope of the curve is getting smaller. Moving beyond I, the "visualized" tangent lines are getting steeper, so the slope of the curve is rising.

11. Consider the function $y = -0.025x^2 + 1.45x$.

a. Is this function $\bigcup$-shaped or $\bigcap$-shaped? How do you know?

b. The dependent variable y reaches a ____________(minimum, maximum) value when x = ________.

c. The value of y at the minimum/maximum point found in part b is _________.

12. In Panel A of Figure A.5, choose 5 points along the curve and sketch the tangent lines at the points.

a. Visually verify that the tangent lines get flatter over the portion of the curve from 0 to I.

b. Visually verify that the tangent lines get steeper over the portion of the curve beyond I.

c. At point I, is the slope of the curve positive, negative, or zero? How can you tell?

13. Consider the function $y = 10x - 0.03x^2 + 0.00005x^3$.

a. Is this function S-shaped or reverse S-shaped? How do you know?

b. What is the value of the y-intercept?

c. The inflexion point occurs at x = __________. At the inflexion point, the slope of the curve reaches its _______________ (maximum, minimum) value.

d. What is the value of y when x = 1,000?

EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS

We now discuss two more curvilinear functions, exponential functions and natural logarithmic functions, which have mathematical properties that can be quite useful in many applications arising in managerial economics and finance.

Exponential Functions

An *exponential function* takes the form

$$y = f(x) = a^x$$

where a is any positive constant (called the "base" of the exponent), and the independent variable x is the power to which the base is raised. Exponential functions differ from polynomial functions, such as $y = ax + bx^2 + cx^3$, because x is an *exponent* in exponential functions but is a *base* in polynomial functions.

Exponential functions have numerous algebraic properties that are quite helpful in applications you will see later in this course.

1. $a^n = a \times a \times ... \times a \quad (n\ times)$

2. $a^0 = 1$

3. $a^{-n} = \dfrac{1}{a^n}$

4. $a^i a^j = a^{i+j}$

5. $(a^i)^j = a^{ij}$

6. $\dfrac{a^i}{a^j} = a^{i-j}$

A commonly used base in economics and finance applications is the constant e, which is the limiting value of the expression $(1 + \frac{1}{n})^n$, as n gets very large (i.e., approaches infinity):

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \cong 2.718$$

Nearly all hand calculators have a key labeled "*e*" or "e^x " that enters the value 2.718... into the calculator's display for exponential computations. Verify that you can use this key on your calculator by making the calculation $e^4 = 54.5981$.

Natural Logarithms

Natural logarithms are logarithms for which *e* is the base

$$y = e^x$$

where *x* is the power to which *e* must be raised to get *y*. For example, *e* must be raised to the power of 4 to obtain the number 54.5981. Thus, the natural logarithm of 54.5981 is 4. In general, the natural logarithm of any *positive* number *y* can be expressed as

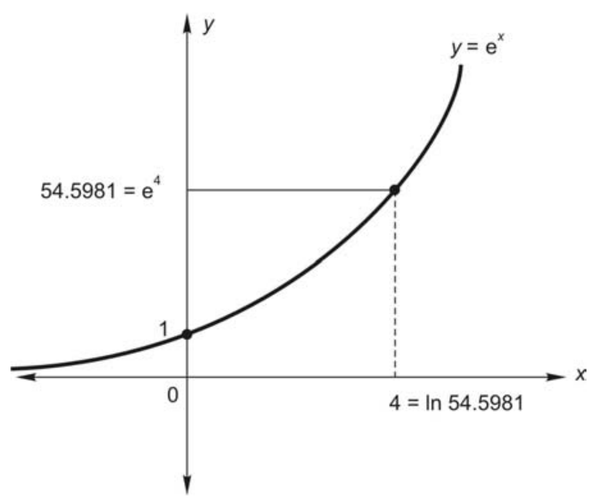
$$x = \ln y$$

The symbol "ln" is used to distinguish this base *e* logarithm from the base 10 logarithm that is used in some scientific applications.[11] Notice that the natural logarithmic function $x = \ln y$ is the inverse of the exponential function $y = e^x$. Figure *A*.6 illustrates the inverse nature of the two functions.

Natural logarithms have the following convenient algebraic properties that can be quite useful:

1. $\ln rs = \ln r + \ln s$
2. $\ln r^s = s \ln r$
3. $\ln\left(\frac{r}{s}\right) = \ln r - \ln s$

FIGURE A.6: The inverse relation between exponential and natural logarithmic functions



[11]The natural logarithm function key on your hand calculator is labeled "ln" or "ln *x*," whereas the key for base 10 logarithms is usually labeled "log" or "log10." You will always use the "ln" or "ln *x*" key in this Student Workbook (and Textbook).

Exercises

14. Evaluate the following:

a. 5^0

b. $2^3 \cdot 2^2$

c. $(K^a)^b$

d. L^{a+b}/L^b

e. $e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n$

f. e^{-4}

g. e^3

h. $\ln 8$

i. $\ln 1{,}000$

j. $\ln(ax^b)$

k. $\ln(K/L)$

l. e^0

Finding Points of Intersection

You may recall from your algebra courses that the point at which two lines intersect can be found mathematically by solving two equations containing the two variables. Similarly, the two points where a straight line crosses a quadratic function can be found using the quadratic formula. While you no doubt hoped to avoid seeing these techniques again, you will see that finding points of intersection plays a crucial role in solving many business decision-making problems. We now review this important algebraic skill.

FINDING THE INTERSECTION OF TWO LINES

Consider two lines represented by the two linear equations

$$y = a + bx$$
$$y = c + dx$$

The point at which these two lines intersect—assuming they are not parallel ($b = d$)—can be found algebraically by recalling that the values of y and x at the point of intersection solve both equations. Setting the two equations equal to each other and solving for x provides the value of x at the point of intersection ($\bar{x}$)

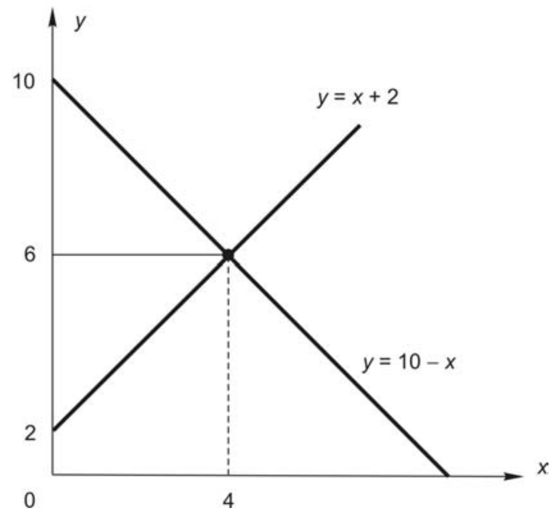
$$\bar{x} = \frac{a-c}{d-b}$$

To find the value of y at the point of intersection, $\bar{x}$ is substituted into either linear equation to find $\bar{y}$

$$\bar{y} = \frac{ad-bc}{d-b}$$

Figure A.8 illustrates how to find the point of intersection of the two linear equations: $y = 10 - x$ and $y = 2 + x$. In this example, a = 10, b = -1, c = 2, and d = 1. Substituting these parameter values into the above formulas, the solution is found to be $\bar{x}$ = 4 and $\bar{y}$ = 6. Verify that the point $\bar{x}$ = 4, $\bar{y}$ = 6 solves both equations.

FIGURE A.8: Finding the intersection of two lines



FINDING THE INTERSECTION OF A LINE AND A QUADRATIC CURVE

A line crosses a quadratic curve – either a ⋃-shaped or a ⋂-shaped curve– at two points. Figure *A*.9 shows the linear function $y = 56 - 0.02x$, as well as the quadratic function $y = 20 - 0.14x + 0.003x^2$. As in the case of intersecting lines, the values of *y* and *x* at the points of intersection between a line and a quadratic curve solve both equations. In general, to find the points of intersection, the equations for the line and the quadratic curve are set equal to each other and then solved for the two values of *x* at which the line and the curve cross.

To illustrate this technique, set the equation for the line, $y = 56 - 0.02x$, equal to the equation for the curve, $y = 20 - 0.14x + 0.003x^2$

$$56 - 0.02x = 20 - 0.14x + 0.0003x^2$$

Now there is one variable in one equation, and the equation is a quadratic equation. To solve a quadratic equation, the quadratic equation must be expressed in the form

$$0 = A + Bx + Cx^2$$

where *A* is the constant term, *B* is the coefficient on the linear term, and *C* is the coefficient on the quadratic term. After setting expressions equal, terms are rearranged to get zero on one side of the equality:

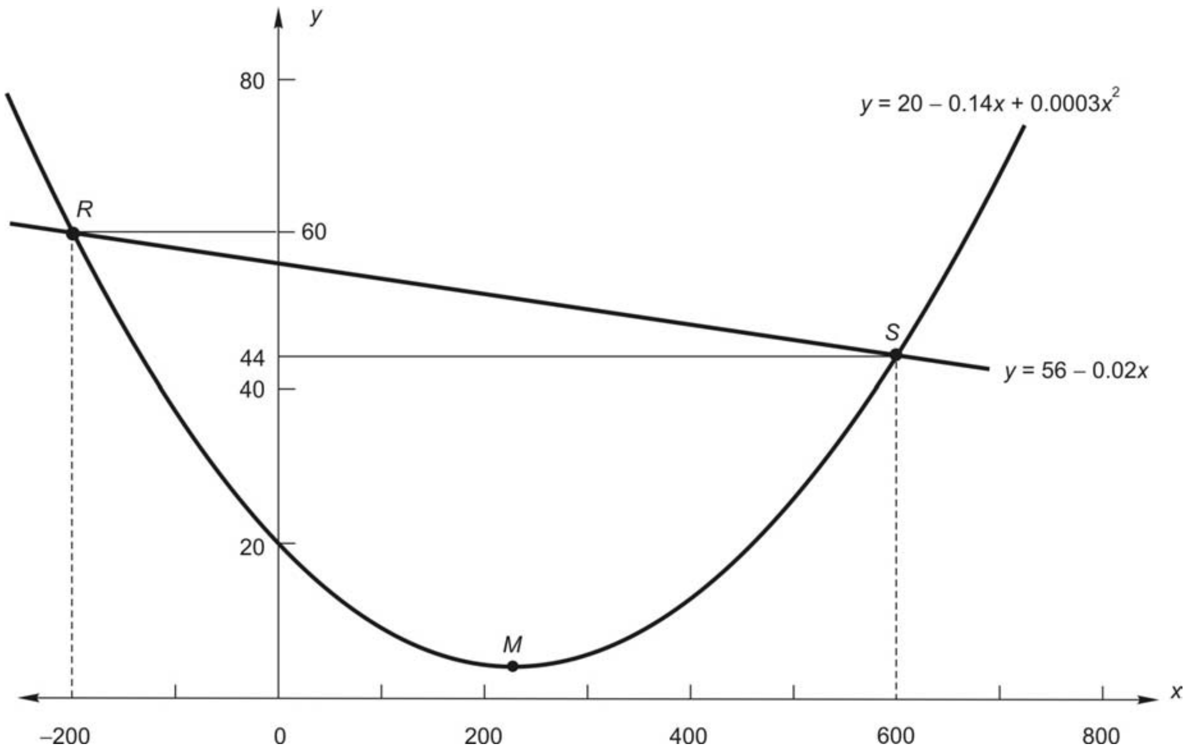
$$0 = -36 - 0.12x + 0.0003x^2$$

It is important to note that *A* and *B* are *not* equal to the parameter values *a* and *b* of the quadratic curve. In this problem, for example, $a = 20$ and $b = -0.14$, but $A = -36$ and $B = -0.12$. The solution to the quadratic equation $0 = A + Bx + Cx^2$ is the familiar *quadratic formula*:

$$x_1, x_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}$$

Substituting the values of *A*, *B*, and *C* into the quadratic formula yields the two solutions for *x*: $x_1 = -200$ and $x_2 = 600$.

FIGURE A.9: Finding the intersection of a line and a quadratic curve



Exercises

15. Find the point of intersection of the following two lines:

$$y = 1{,}000 - 20x \quad \text{and} \quad y = 25 + 5x$$

16. Consider the following linear function and quadratic function:

$$y = 32 - 0.02x$$

$$y = 50 - 0.2x + 0.00025x^2$$

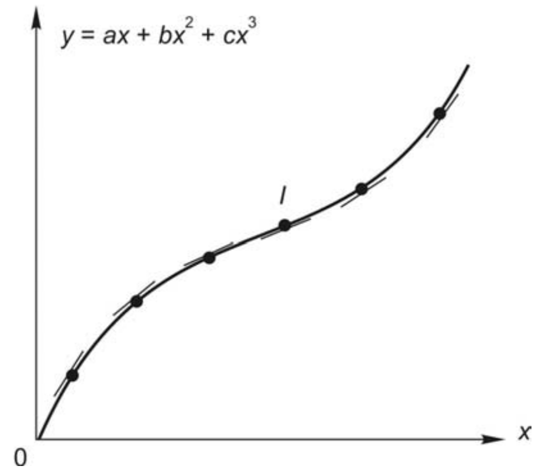
The line intersects the quadratic curve at two points: $x_1 =$ ______, $y_1 =$ ______ and $x_2 =$ ______, $y_2 =$ ______.

ANSWERS TO EXERCISES

1. a. $y = 100\ (= 300 - 200)$
   b. $y = 140\ (= 10 + 30 + 100)$
   c. $w = 10\ (= -20 + 30)$
   d. $s = 402\ (= 400 + 2)$
2. a. $x = f(y) = -30 + 0.067y$
   b. $p = f(q) = 40 - 0.04q$
3. a. $y = 650\ (= 625 + 25)$
   b. $z = 14{,}800\ (= 10{,}000 + 4{,}800)$
4. a. $50\ (= 100 - 2 \times 25)$
   b. $+5\ (= 25 - 20)$
   c. $-10\ (= 50 - 60)$
   d. $-2\ (= \Delta q / \Delta p = -10/+5)$
   e. decrease; 2; decrease; 10
5. a. $\Delta y = 100 - 10 = +90$
   b. $\Delta x = 30 - 0 = +30$
   c. $+3\ (= \Delta y / \Delta x = +90/+30)$
   d. $y = 10 + 3x$
   e. 3; same
   f. increases; 12
   g. $a = 10; b = 3$
6. a. $\Delta y = 180 - 40 = +140$
   b. $\Delta x = 0 - 40 = -40$
   c. $-3.5\ (= \Delta y / \Delta x = +140/-40)$
   d. $y = 180 - 3.5x$
   e. 3.5; opposite
   f. decreases; 17.5
   g. $a = 180; b = -3.5$
7. a. $110\ (= 30 + 2 \times 10 - 3 \times 20 + 4 \times 30)$
   b. decreases; 2
   c. decreases; 3
   d. increases; 12
   e. +7; because 30+(2 × 9)+(–3 × 21)+(4 × 33)=117, which is 7 units greater than the value of *w* in part *a*.
   f. (Δ *x* × 2)+(Δ *y* × –3)+(Δ *z* × 4) = –2+–3+12 = +7; yes, $\Delta w = +7$
8. a. slope at point *A* = slope of *TT'* = $-720/90 = -8$
   b. slope at point *A* = slope of *TT'* = $15/3 = +5$
   c. slope at point *A* = slope of tangent = 0 a the minimum point
9. When *x* = 500, *y* = 275,000 (confirm this with the equation). Construct a tangent line at the dot on the curve at *x* = 500 and *y* = 275,000, and extend the tangent line to cross the *y*-axis. If precisely drawn, the tangent line crosses the *y*-axis at 125,000. If your tangent line is precisely constructed, the slope of the tangent line is $+300\ [= (275{,}000 - 125{,}000)/500]$. Be satisfied if you came close to +300; calculus is required to get a precisely accurate measure of slope.
10. a. $\bigcup$-shaped; The polynomial is a quadratic and the parameters have the sign pattern associated with a $\bigcup$-shaped curve: $a > 0$, $b < 0$, $c > 0$.
    b. minimum; 300 $(= -b/2c = 0.03/0.0001)$
    c. 5.5 $[= 10 - 0.03(300) + 0.00005(300)^2]$

11. a. $\cap$-shaped; The polynomial is a quadratic and the parameters have the sign pattern associated with a $\cap$-shaped curve:, $b > 0$, $c < 0$.
b. maximum; 29 $(= -b/2c = -1.45/-0.05)$
c. 21.025 $[= -0.025(29)^2 + 1.45(29)]$

12. The figure shows the "sketched" tangent lines at five additional points along the curve in Panel *A* of Figure *A*.5.
a. You can verify visually that the three tangent lines between 0 and *I* get flatter as *x* increases.
b. You can verify visually that the two tangent lines to the right of point *I* get steeper as *x* increases.



c. At point *I*, the slope is positive because the sketched tangent line at point *I* slopes upward.

13. a. Reverse *S*-shaped, as in Panel *A* of Figure *A*.5. The parameters have the signs $a > 0$, $b < 0$, and $c > 0$.
b. 0
c. 200; minimum
d. 30,000 $[= 10(1{,}000) - 0.03(1{,}000)^2 + 0.00005(1{,}000)^3]$

14. a. 1
b. $32 = 2^5 = 2^{3+2}$
c. K^{ab}
d. $L^a = L^{a+b-b}$
e. $2.718... = e$
f. 0.01832
g. 20.0855
h. 2.0794
i. 6.9077
j. $\ln a + b \ln x$
k. $\ln K - \ln L$
l. 1, because $a^0 = 1$

15. $\overline{x}$ = (1,000 – 25) / (5 – (–20)) = 39 and $\overline{y}$ = 1,000 –(20×39) = 220 = 25 + (5×39)

16. Solve the quadratic equation, $0 = 18 - 0.18x + 0.00025x^2$, to get two solutions for x:
$x_1 = 600$, $y_1 = 20$
$x_2 = 120$, $y_2 = 29.6$