BRIEF REVIEW OF DERIVATIVES AND OPTIMIZATION

In business decision making, the concept of a derivative can be used to find solutions to optimization problems—either maximization or minimization problems in either unconstrained or constrained situations. This review briefly covers the basic skills of taking derivatives and using derivatives to find solutions to optimization problems.

THE CONCEPT OF A DERIVATIVE

When y is a continuous, differentiable function of x—denoted as $y = f(x)$—the derivative of this function gives the rate of change in y as x changes. Derivatives are generally denoted using either one of following conventions:

$$\frac{dy}{dx} \equiv \frac{d}{dx} f(x) \equiv f'(x) \equiv f'$$

If, for example, at $x = 1,000$, $f'(x) = -200$, then y changes 200 times as much as x changes, but y changes in the opposite direction of the change in x.[1] Since the derivative is evaluated at $x = 1,000$, the rate of change, -200, applies to changes in x that occur "in the neighborhood" of $x = 1,000$; that is, changes in x that are "small" relative to $x = 1,000$. An increase in x of 2 units results in a decrease in y of (approximately) 400 units.[2] In the figure below, the function $y = f(x) = -0.5x^2 + 800x$ is graphed. For this function, the rate of change in y with respect to x ($= dy/dx$) is equal to -200 at $x = 1,000$ (see point A). The blow up at point A shows that a 2-unit increase in x from 1,000 to 1,002 does indeed cause y to decrease by (approximately) 400 units—in this case, the precise decrease is 402 units. (Later in this review we will show you how to find the derivative of a function and the numerical value of the derivative at any particular value of x.)

Since derivatives measure the rate of change in y as x changes, derivatives can also be interpreted as the slope of curve at a point on the curve. As you may recall from a course in algebra, the slope of a curve is measured by constructing a line tangent to the curve at the point of measure. The slope of the tangent line is then equal to the slope of the curve at the point of measure. In the figure below, the slope of the curve at point A is -200, which is the slope of the tangent line TT'. The procedure in calculus of taking a derivative of a function and evaluating that derivative at a point is equivalent to the procedure in geometry of constructing a line tangent to a curve and measuring its slope.

Without using calculus to find the derivative of a function, it is possible to determine visually, rather than mathematically, whether the derivative is positive, negative, or zero at points along a curve. In the figure above, you can visually verify—just "see" the tangent line at a point on the curve—that the derivative of the function $y = f(x) = -0.5x^2 + 800x$ is positive over the range of x from 0 to 800. At $x = 800$ (point M), y reaches its maximum value of 320,000, and the derivative is equal to zero. At either maximum or minimum points on a curve, the slope of the curve is zero at these points. The "blow up at M" shows how you can visualize the slope of $f(x)$

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[1] When a derivative is *positive* (*negative*), the variables y and x are *directly* (*inversely*) related. When the variables are directly related, y and x move in the same direction. When the variables are inversely related, y and x move in opposite directions.

[2] Actually, the 800 unit decrease in y is only approximately correct because, strictly speaking, derivatives measure rate of change in y for tiny or "infinitesimal" changes in x. Since a 2-unit change in x is a rather small change when x is equal to 1,000, the actual change in y will be quite close to -400 but not exactly -400. The smaller is the change in x, the more precisely the change in y is approximated.

at $x = 800$. For values of x greater than 800, the derivative is negative.
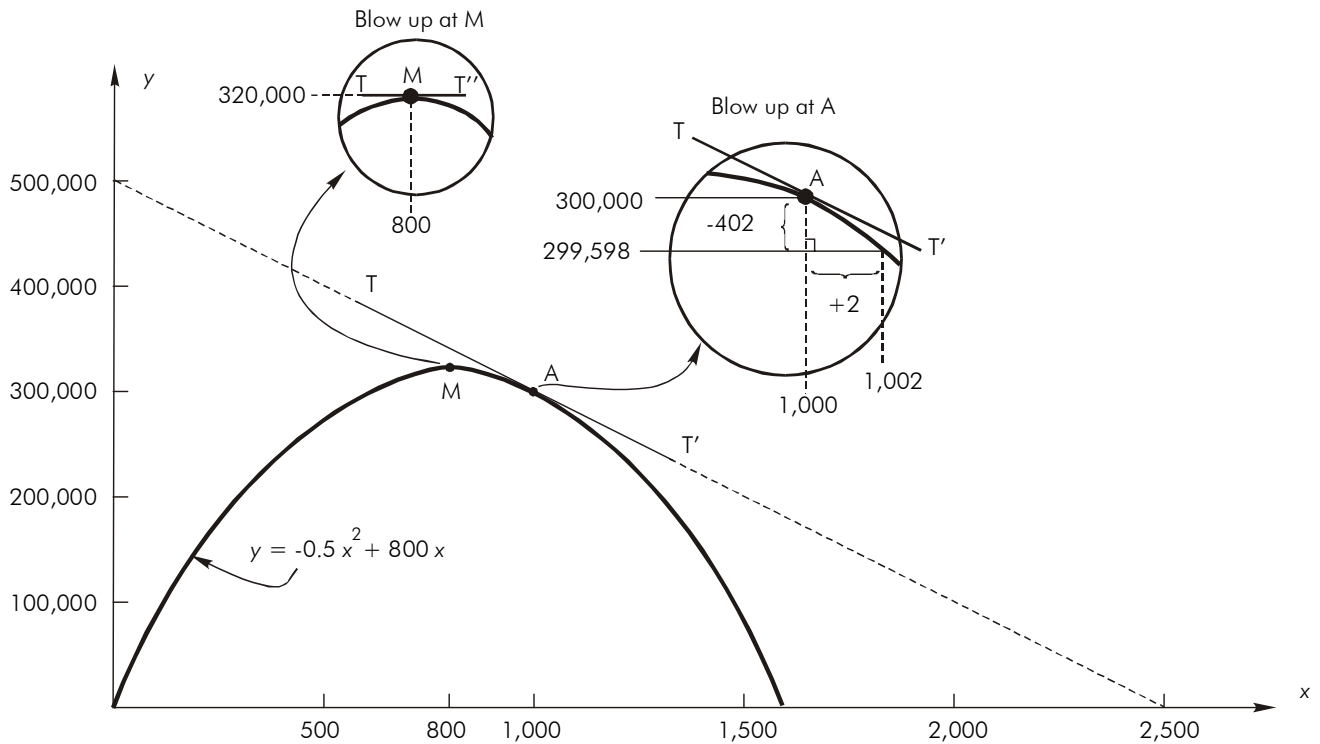
CALCULUS RULES FOR TAKING DERIVATIVES

This section presents six fundamental rules for taking derivatives. These six rules provide the tools for finding the derivatives of the polynomial functions typically encountered in business decision making. An application of the rule is given for each rule.

Power function rule:
> For any *n*,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Example: Let $y = f(x) = x^3$. Using the power function rule, the derivative is $dy/dx = 3x^2$. At $x = 0.5$, $dy/dx = 3(0.5)^2 = 0.75$, which means that the slope of the tangent at $x = 5$ is 0.75. In other words, for very small changes in x in the neighborhood of $x = 0.5$, y changes only 3/4 as much as x changes (and in the same direction).

Constant function rule:
> For any constant function $y = k$, where k is any constant value,

$$\frac{d}{dx}k = 0$$

Example: Let $y = 20$. Since y takes the value 20 for all values of x, dy/dx must equal zero.

Sum (difference) of functions rule:

For any two functions, $f(x)$ and $g(x)$, the derivative of the sum or difference of the two functions, $y = f(x) \pm g(x)$, is simply the sum or difference of the derivatives of the individual functions:

$$\frac{dy}{dx} = \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Example: Let $R(q) = -0.75q^2 + 200q$ and $C(q) = 500 - 10q + 2q^2$. If $P(q) = R(q) - C(q)$, then the derivative of $P(q)$, is $dP/dq = R'(q) - C'(q) = (-1.5q + 200) - (-10 + 4q) = -5.5q + 210$.

Product rule:

For any two functions, $f(x)$ and $g(x)$, the derivative of the product of the two functions, $y = f(x)g(x)$, is:

$$\frac{dy}{dx} = \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

Example: Let $f(x) = 10x^2$ and $g(x) = 2x$. The derivative of the product, $y = f(x) \cdot g(x)$, is $dy/dx = 20x(2x) + 2(10x^2) = 60x^2$.

Quotient rule:

For any two functions, $f(x)$ and $g(x)$, the derivative of the quotient of the two functions, $y = f(x)/g(x)$ is:

$$\frac{dy}{dx} = \frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Example: Let $f(x) = 10x^2$ and $g(x) = 2x$. The derivative of the quotient $y = f(x)/g(x)$, is $dy/dx = [20x(2x) - 2(10x^2)]/(2x)^2 = (40x^2 - 20x^2)/4x^2 = 5$. Note that it is much easier to find the derivative of the quotient when the quotient is first simplified algebraically: $f(x)/g(x) = 5x$, so $dy/dx = 5$.

Chain rule:

Let one function be composed of another function: $y = f(z)$ and $z = g(x)$. The derivative dy/dx is found by applying the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

Example: Let $R(Q) = 500Q - 20Q^2$ and $Q(L) = -2L^3 + 15L^2$. The rate of change in R as L changes, dR/dL, is easily found using the chain rule: $dR/dL = (500 - 40Q)(-6L^2 + 30L)$. In order to express the derivative as a function of L only, substitute $Q(L) = -2L^3 + 15L^2$ into the derivative and simplify algebraically to obtain $dR/dL = -480L^5 + 6,000L^4 - 18,000L^3 - 3,000L^2 + 15,000L$.

As a final example, find the derivative of the function in the figure above. Recall from the discussion of the figure that the curve is a plot of the function $y = -0.5x^2 + 800x$ (over the range of x from 0 to 1,600). The derivative is $dy/dx = -x + 800$. At point A, $x = 1,000$, and $dy/dx = -1,000 + 800 = -200$. At point M, $x = 800$, and $dy/dx = -800 + 800 = 0$.

UNCONSTRAINED OPTIMIZATION

Derivatives of functions can be used to find the value of x that maximizes or minimizes the value of y. The procedure involves two steps:

(1)   The derivative of $y = f(x)$, which is denoted by $f'(x)$, is set equal to zero and solved for the value(s) of x that satisfy the condition $f'(x) = 0$. The values of x for which $f'(x) = 0$ are called *extreme points* because these are the points on a curve at which the curve reaches either a maximum or minimum value. A function may have multiple extreme points. The condition $f'(x) = 0$, then, identifies values of x that may be either maximum points or minimum points, and this condition is generally referred to as the *first-order (necessary) condition* for either a maximum point or minimum point.

(2)   Both kinds of extreme points — maximum points and minimum points — occur where the derivative is zero. In order to determine whether an extreme point is a maximum or minimum point, the *second derivative* of the function, which is the derivative of $f'(x)$ and denoted by $f''(x)$, must be examined. The second derivative of a function indicates the rate at which the slope of the curve—the first-derivative $f'(x)$—is changing. At a maximum point, increasing x causes $f'(x)$ to change from a value of zero to a negative value; i.e. moving in a rightward direction from a maximum point necessarily causes $f'(x)$ to decrease. Consequently, the second derivative is negative when evaluated at a maximum point. Conversely, the second derivative is positive when evaluated at a minimum point. The condition required of the second derivative to establish either a maximum point or a minimum point is generally referred to as the *second-order (sufficient) condition*.

These two conditions for finding a maximum or minimum point can be summarized as follows:

	First-order condition	*Second-order condition*
Maximum point	$f'(x) = 0$	$f''(x_0) < 0$, where x_0 is an extreme point of $f(x)$
Minimum point	$f'(x) = 0$	$f''(x_0) > 0$, where x_0 is an extreme point of $f(x)$

To illustrate finding the value of x that maximizes a function $y = f(x)$, consider again the function graphed in the figure above: $y = f(x) = -0.5x^2 + 800x$. Setting the first-derivative equal to zero gives the first-order condition:

$$\frac{dy}{dx} = -x + 800 = 0$$

Solving the first-order condition for x reveals that $x = 800$ is an extreme point. To determine whether 800 is a maximum point or a minimum point—even though you can see that it is a maximum point in the figure—the second derivative is evaluated at $x = 800$:

$$\frac{d}{dx}(-x + 800) = f''(x) = -1$$

Since $f''(x)$ is a constant function equal to -1, the second-order condition for a maximum is met at $x = 800$.

CONSTRAINED OPTIMIZATION

In some situations, decision makers face constraints on the values of the *decision variables* (also called *choice variables*) that prevent reaching the absolute maximum or minimum point of an objective function (the function that is to be maximized or minimized). A typical constrained optimization problem might involve choosing values of x and y either to maximize or to minimize the function $z = f(x, y)$ subject to the constraint that $g(x, y) = k$, where k is any constant value. One technique of solving this type of constrained optimization problem is called the *Lagrange multiplier method*. In this review, we will focus exclusively on the first-order conditions for optimization—which are the same for maximization and minimization problems—and leave the more challenging second-order conditions for a course in mathematical economics.

The Lagrangian multiplier method of solving constrained optimization problems involves maximizing (or minimizing) the following "Lagrangian function":

$$L = L(x, y, \lambda) = f(x, y) + \lambda [k - g(x, y)]$$

where λ is called the Lagrange multiplier and is treated as a variable in the Lagrangian function. To obtain the first-order conditions for constrained optimization, *partial* derivatives of the Lagrangian function must be set equal to zero:[3]

$$L_x(x, y, \lambda) = f_x(x, y) - \lambda g_x(x, y) = 0$$
$$L_y(x, y, \lambda) = f_y(x, y) - \lambda g_y(x, y) = 0$$
$$L_\lambda(x, y, \lambda) = k - g(x, y) \qquad = 0$$

The optimal values of x, y, and λ are found by solving the system of first-order equations simultaneously. Notice that the first-order condition associated with the partial derivative of λ forces the constraint to be satisfied when x^*, y^*, and λ^* are found by solving the system of first-order equations simultaneously.

To illustrate the Lagrange multiplier technique, find the values of x and y that maximize the function $z = f(x, y) = xy + 5y + 2x + 10$ subject to the constraint that $g(x, y) = 15$ where $g(x, y) = x + 2y$. The Lagrangian function for this constrained optimization problem is:

$$L = L(x, y, \lambda) = f(x, y) + \lambda [k - g(x, y)]$$
$$= xy + 5y + 2x + 10 + \lambda(15 - x - 2y)$$

The first-order conditions for maximization (or minimization, for that matter) are:

$$L_x(x, y, \lambda) = y + 2 - \lambda = 0$$
$$L_y(x, y, \lambda) = x + 5 - 2\lambda = 0$$
$$L_\lambda(x, y, \lambda) = 15 - x - 2y = 0$$

Solving for x^*, y^*, and λ^* using either substitution or Cramer's rule produces the solution:

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[3]The partial derivative of $f(x, y)$ with respect to x measures the rate of change in z as x changes, *holding y constant*. Similarly, the partial derivative of $f(x, y)$ with respect to y is the rate of change in z with respect to y, *holding x constant*. Partial derivatives are denoted either by using the partial differentiation symbol "∂" in $\partial z/\partial x$ and $\partial z/\partial y$ or by using a subscript to indicate partial differentiation in $f_x(x, y)$ and $f_y(x, y)$.

To find a partial derivative with respect to a particular variable, employ the usual rules for differentiation, but treat all other variables in the function as constants. For example, the partial derivatives of $z = f(x, y) = 5x^2 - 2xy^2 + 3y^3$ are $f_x(x, y) = 10x - 2y^2$ and $f_y(x, y) = -4xy + 6y^2$.

$$x^* = 7$$
$$y^* = 4$$
$$\lambda^* = 6$$

The maximum value for z can be found by substitution: $z^* = f(x^*, y^*) = 7 \cdot 4 + 5 \cdot 4 + 2 \cdot 7 + 10 = 72$.