Chapter 8 Supplement

Mechanical Advantage

The **mechanical advantage** (MA) of a machine or tool is defined as the ratio of the output force (load) F_2 to the applied or input force F_1 exerted on the machine.

$$\text{MA} = \frac{\text{load}}{\text{applied force}} = \frac{F_2}{F_1} \qquad \text{(8-S1)}$$

In other words, the mechanical advantage is the factor by which the input force is amplified.

Ideally, if the machine itself does not store or dissipate energy, then the work output is equal in magnitude to the work input. Let $d_{\text{out}} = v_{\text{out}}\,\Delta t$ be the displacement of the load during a time interval Δt caused by a displacement d_{in} during the same interval. Then

$$F_{\text{out}}d_{\text{out}} = F_{\text{in}}d_{\text{in}} \qquad \text{(8-S2)}$$

Combining Equations (8-S1) and (8-S2), we find

$$\text{MA} = \frac{F_{\text{out}}}{F_{\text{in}}} = \frac{d_{\text{in}}}{d_{\text{out}}} = \frac{v_{\text{in}}}{v_{\text{out}}} \qquad \text{(8-S3)}$$

Equation (8-S3) shows that there is a trade-off involved: a machine can amplify a force only by reducing the displacement of the force by the same factor. However, force amplification is not the only reason to use a machine. Sometimes a reduction in speed is desired, in which case the mechanical advantage is less than 1.

For machines that rotate, we define the MA in terms of torques and angular speeds:

$$\text{MA} = \frac{\tau_2}{\tau_1} = \frac{\Delta\theta_1}{\Delta\theta_2} = \frac{\omega_1}{\omega_2} \qquad \text{(8-S4)}$$

Using Equations (8-S1) and (8-S4), we can find the mechanical advantage of various machines. Here are a few examples.

Lever For an ideal lever, by using the fulcrum as the axis of rotation and setting the net torque on the lever equal to zero, we find $F_1\ell_1 = F_2\ell_2$. Therefore, the mechanical advantage is equal to the ratio of the lever arms of the two forces:

$$\text{MA} = \frac{\ell_1}{\ell_2} \qquad \text{(8-S5)}$$

Inclined Plane An inclined plane (Fig. 8.S1) can be used to lift an object by sliding it along an incline of length L instead of lifting it straight up to a height h. For an ideal, frictionless incline, the work done is the same either way. Then $F_1L = F_2h$, where F_1 is the applied force and F_2 is the weight of the object. If the incline is at an angle θ to the horizontal, then

$$MA = \frac{L}{h} = \frac{1}{\sin\theta} \qquad \text{(8-S6)}$$

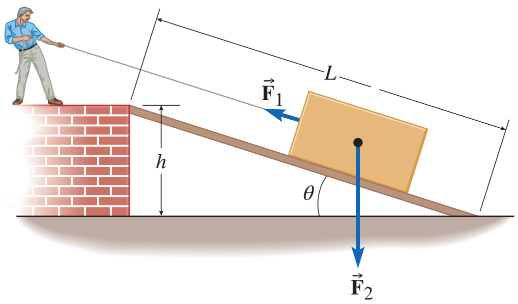

Figure 8.S1 An inclined plane being used to lift a heavy crate. Ignoring friction, the magnitude of the applied force $\vec{\mathbf{F}}_1$ is equal to the component of the crate's weight along the incline: $F_1 = F_2 \sin\theta$. The mechanical advantage is $\text{MA} = F_2/F_1 = 1/(\sin\theta)$.

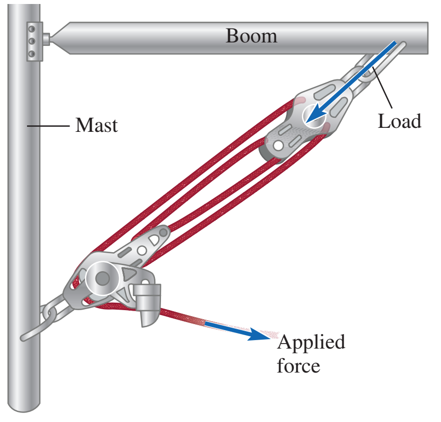

Figure 8.S2 A block and tackle used on a sailboat. In this case, the load is the force exerted on the boom. Four rope segments pull on the block connected to the boom, so the mechanical advantage is 4 (the force exerted on the boom is 4 times the applied force).

Block and Tackle In a block and tackle (Fig. 8.S2), one or more pulleys are assembled on a single axle to form each block. One block is fixed in place and the other exerts the load force. A single rope is threaded through all the pulleys. Ignoring friction and the weight of the rope, the tension T of the rope is the same everywhere. If N rope segments support the load, the input force is T and the load is NT, so the mechanical advantage is N.

Gears and Belt Drives The contact point of two meshing gears (Fig. 8.S3) has to move at the same speed. If r represents the radius of a gear to the contact point, then $v = r_1\omega_1 = r_2\omega_2$. From Eq. (8-S4), we have MA $= r_2/r_1$. To mesh together properly, the tooth size has to be the same. If N is the number of teeth on each gear, then $2\pi r_1/N_1 = 2\pi r_2/N_2$. The mechanical advantage can then be written in terms of the gear ratio N_2/N_1:

$$\text{MA} = \frac{r_2}{r_1} = \frac{N_2}{N_1} \tag{8-S7}$$

The output torque can be greater or less than the input torque, depending on whether the gear ratio is greater than or less than 1.

A belt drive is similar: two pulleys of different radii (r_1 and r_2) are connected by a belt. The tangential speeds of the pulleys are equal if the belt is not slipping. Then $v = r_1\omega_1 = r_2\omega_2$ and

$$\text{MA} = \frac{r_2}{r_1} \tag{8-S8}$$

Calculating Rotational Inertia

Four principles can be used along with Table 8.1 to find the rotational inertias for rigid objects about various axes: the sum and stretch rules and parallel and perpendicular axis theorems

Sum Rule Rotational inertia is defined as a sum: $I = \Sigma_{n=1}^{N} m_n r_n^2$. Therefore, an object can be broken into parts and the rotational inertia of the object is the sum of the rotational inertias of the parts.

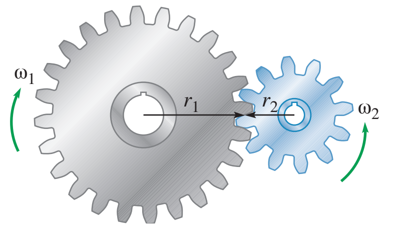

Figure 8.S3 Two meshing gears. The input torque τ_1 is applied to gear 1, and gear 2 supplies the output torque τ_2. The gear ratio is $N_2/N_1 = 24/12 = 2$. The mechanical advantage is MA $= 2$, so $\tau_2 = 2\tau_1$. Gear 2 turns half as fast as gear 1: $\omega_2/\omega_1 = \frac{1}{2}$.

Example 8S.1

Rotational Inertia of a Rod About Its Midpoint

The rotational inertia of a thin rod with the axis of rotation perpendicular to its length and passing through one end is $I = \frac{1}{3}ML^2$. From this expression, derive the rotational inertia of a rod with mass M and length L that rotates about a perpendicular axis through its *midpoint* (Fig. 8.S4).

Strategy In general, the same object rotating about a different axis has a different rotational inertia. With the axis at the midpoint, the rotational inertia is smaller than for the axis at the end, since the mass is closer to the axis, on average. Imagine performing the sum $\Sigma_{n=1}^{N} m_n r_n^2$; for the axis at the end, the values of r_n range from 0 to L, whereas with the axis at the midpoint, r_n is never larger than $\frac{1}{2}L$.

Imagine cutting the rod in half; then there are two rods, each with its axis of rotation at one of its *ends*. Then, since rotational inertia is additive, the rotational inertia for two such rods is just twice the value for one rod.

Solution Each of the halves has mass $\frac{1}{2}M$ and length $\frac{1}{2}L$ and rotates about an axis at its endpoint. We know that $I = \frac{1}{3}ML^2$ for a rod with mass M and length L rotating about its end, so each of the halves has

$$I_{\text{half}} = \tfrac{1}{3} \times \text{mass of half} \times (\text{length of half})^2$$
$$= \tfrac{1}{3} \times (\tfrac{1}{2}M) \times (\tfrac{1}{2}L)^2 = \tfrac{1}{3} \times \tfrac{1}{2} \times \tfrac{1}{4} \times ML^2 = \tfrac{1}{24}ML^2$$

Since there are two such halves, the total rotational inertia is twice that:

$$I = I_{\text{half}} + I_{\text{half}} = \tfrac{1}{12}ML^2$$

Discussion The rotational inertia is less than $\frac{1}{3}ML^2$, as expected. That it is $\frac{1}{4}$ as much is a result of the distances r_n being *squared* in the definition of rotational inertia. The various particles that compose the rod are at distances that range from 0 to $\frac{1}{2}L$ to from the rotation axis, instead of from 0 to L. Think of it as if the rod were compressed to half its length, still pivoting about the endpoint. All the distances r_n are half as much as before; since each r_n is squared in the sum, the rotational inertia is $(\frac{1}{2})^2 = \frac{1}{4}$ times its former value.

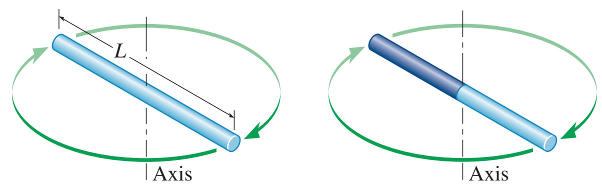

Figure 8.S4 (a) A rod rotating about a vertical axis through its center. (b) The same rod, viewed as two rods, each half as long, rotating about an axis through an end.

Stretch Rule The rotational inertia of a rigid object is not changed if the object is stretched or compressed in a direction parallel to the rotation axis. This rule follows directly from the definition of rotational inertia because moving a bit of mass parallel to the axis does not change its distance r_n from the rotation axis.

For example, you might want to calculate the rotational inertia of a door about the axis through its hinges. Mentally "compress" the door vertically (parallel to the axis) into a horizontal rod with the same mass, as in Fig. 8.S5. The rotational inertia is unchanged by the compression, since every particle maintains the same distance from the axis of rotation. Thus, the formula for the rotational inertia of a rod (listed in Table 8.1) can be used for the door.

Parallel Axis Theorem Suppose the rotational inertia of an object about an axis that passes through the center of mass is I_{CM}. Then the rotational inertia I about any axis *parallel* to that axis is

$$I = I_{\text{CM}} + Md^2 \qquad (8\text{-S9})$$

where M the object's mass and d is the perpendicular distance between the two axes (Fig. 8.S6).

Perpendicular Axis Theorem Suppose a flat rigid object lies entirely within the xy-plane. Define three mutually perpendicular rotation axes that pass through a single

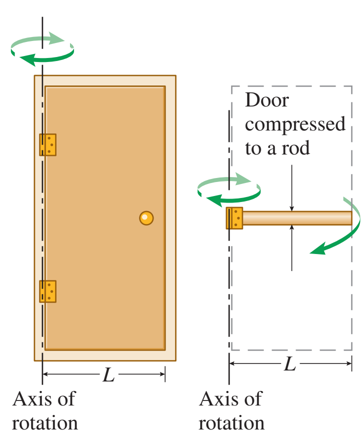

Figure 8.S5 The rotational inertias of a door and a rod are both given by $\frac{1}{3}ML^2$.

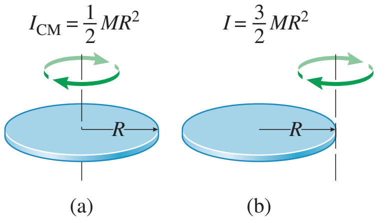

Figure 8.S6 From Table 8.1, the rotational inertia of a disk about an axis through the center of mass and perpendicular to the disk is $I_{CM} = \frac{1}{2}MR^2$. Then, using the parallel axis theorem, the rotational inertia of the disk about an axis through the edge and perpendicular to the disk is $I = I_{CM} + Md^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$.

point in the xy-plane, one parallel to each of the x-, y-, and z-axes (Fig. 8.S7). Call the rotational inertias of the object about these axes I_x, I_y, and I_z. Then

$$I_z = I_x + I_y \tag{8-S10}$$

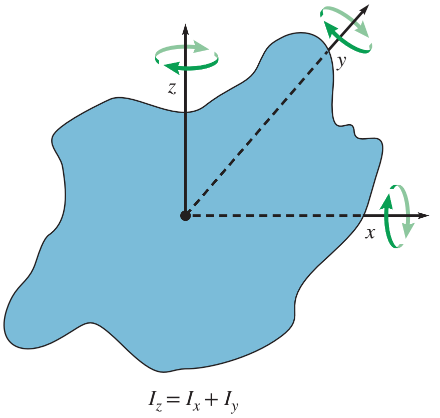

Figure 8.S7 The perpendicular axis theorem relates the rotational inertias of a flat object about three mutually perpendicular axes that all pass through the same point in the plane of the object and one of which is perpendicular to the plane of the object.

Problems

1. Alex uses a lever to lift a 230 kg stone. She applies a 280 N force to the end of the lever while moving it 15 cm. How far does the stone move?
2. In its "highest gear", the chain on Trey's bike goes around a gear connected to the pedals that has 44 teeth and a radius of 10 cm. The chain also goes around a gear connected to the rear wheel that has 11 teeth. Trey's feet apply a net torque of 65 N·m to the pedals. (a) What is the tension in the chain? Note that only the upper part of the chain is under tension; the lower part is slack. (b) What is the torque applied by the chain to the rear wheel?
3. Find the rotational inertia of a uniform thin rod (mass M, length L) about a perpendicular axis a distance $L/3$ from one end.
4. What is the rotational inertia of a uniform circular disk (mass M, radius R) about a diameter?
5. ✦ The pendulum in a grandfather clock is a rod (mass M, length L) suspended at one end with a uniform solid sphere (mass $2M$, radius $L/4$) attached to the other end. (The total length of the pendulum is the length of the rod plus the diameter of the sphere, which is $3L/2$.) What is the rotational inertia of the pendulum?

Answers to Problems

1. 1.9 cm

3. $\frac{7}{36}ML^2$

5. $\frac{49}{30}ML^2$