SUPPLEMENT 1 TO CHAPTER 18

Derivation of the Optimal Policy for the Stochastic Single-Period Model for Perishable Products

Consider the stochastic single-period model for perishable products presented in Sec. 18.7. We will derive the optimal inventory policy for this model (as stated in Sec. 18.7 without proof) for the case where the setup cost $K = 0$. Section 18.7 then shows how this optimal policy extends to the case where $K > 0$.

We will continue to use the following notation introduced in Sec. 18.7.

I = initial inventory

Q = order quantity

$S = I + Q$ = decision variable for the model

D = demand for the product (a random variable)

$f(x)$ = probability density function of D

$F(x)$ = cumulative distribution function

Initially consider the case where $I = 0$. For this case,

$C(D, S)$ = cost if the demand is D and S is stocked,

$$C(S) = E\{C(D, S)\},$$

where an expression for $C(S)$ is given in Sec. 18.7 in terms of $f(x)$ and the parameters of the model—c (unit purchase cost), h (unit holding cost), and p (unit shortage cost). Section 18.7 then points out that the optimal value of S that minimizes $C(S)$ is that value S^* which satisfies

$$F(S^*) = \frac{p - c}{p + h}.$$

We now will derive this result

Derivation When $I = 0$

For any positive constants c_1 and c_2, define $g(x, S)$ as

$$g(x, S) = \begin{cases} c_1(S - x) & \text{if } S > x \\ c_2(x - S) & \text{if } S \le x, \end{cases}$$

and let

$$G(S) = \int_0^{\infty} g(x, S)f(x)dx + cS,$$

where $c > 0$. Then $G(S)$ is minimized at $S = S^*$, where S^* is the solution to

$$F(S^*) = \frac{c_2 - c}{c_2 + c_1}.$$

To see why this value of S^* minimizes $G(S)$, note that, by definition,

$$G(S) = c_1 \int_0^S (S - x)f(x)dx + c_2 \int_S^{\infty} (x - S)f(x)dx + cS.$$

Taking the derivative (see the end of Appendix 3) and setting it equal to zero lead to

$$\frac{dG(S)}{dS} = c_1 \int_0^S f(x)dx - c_2 \int_S^{\infty} f(x)dx + c = 0.$$

Because

$$\int_0^{\infty} f(x)dx = 1,$$

this expression implies that

$$c_1F(S^*) - c_2[1 - F(S^*)] + c = 0.$$

Solving this equation results in

$$F(S^*) = \frac{c_2 - c}{c_2 + c_1}.$$

The solution of this equation minimizes $G(S)$ because

$$\frac{d^2G(S)}{dS^2} = (c_1 + c_2)f(S) \ge 0$$

for all S.

To apply this result, it is sufficient to show that

$$C(S) = cS + \int_S^{\infty} p(x - S)f(x)dx + \int_0^S h(S - x)f(x)dx$$

has the form of $G(S)$. Clearly, $c_1 = h$, $c_2 = p$, and $c = c$, so the optimal quantity to order S^* is that value which satisfies

$$F(S^*) = \frac{p - c}{p + h}.$$

Derivation When $I > 0$

To derive the results for the case where the initial inventory level is $I > 0$, recall that it is necessary to solve the relationship

$$\min_{S \ge I}\left\{-cI + \left[\int_S^{\infty} p(x - S)f(x)dx + \int_0^S h(S - x)f(x)dx + cS\right]\right\}.$$

Note that the expression in brackets has the form of $G(S)$, with $c_1 = h$. $c_2 = p$, and $c = c$. Hence, the cost function to be minimized can be written as

$$\min_{S \geq I}\{-cI + G(S)\}.$$

It is clear that $-cI$ is a constant, so that it is sufficient to find the S that satisfies the expression

$$\min_{S \geq I} G(S).$$

Therefore, the value of S^* that minimizes $G(S)$ satisfies

$$F(S^*) = \frac{p - c}{p + h}.$$

Furthermore, $G(S)$ must be a convex function because

$$\frac{dG^2(S)}{d^2S} \geq 0.$$

Also,

$$\lim_{S \to 0} \frac{dG(S)}{dS} = c - p,$$

which is negative,[1] and

$$\lim_{S \to \infty} \frac{dG(S)}{dS} = h + c,$$

which is positive. Therefore, $G(S)$ possesses a global minimum at some value $S = S^* > 0$. It follows that the optimal policy is the one shown below (as claimed in Sec. 18.7).

Optimal Inventory Policy

If $I < S^*$, order $S^* - I$ to bring the inventory level up to S^*.
If $I \geq S^*$, do not order,

where S^* again satisfies

$$F(S^*) = \frac{p - c}{p + h}.$$

The model presented in Sec. 18.7 assumes that the holding cost is proportional to the number of units being held and that the shortage cost also is proportional to the amount of unsatisfied demand. A similar argument can be constructed to obtain the optimal inventory policy for an extension of the model where the holding cost and shortage cost are *not* proportional to these respective quantities. This extension is considered below.

The Optimal Policy with Nonproportional Holding Costs and Shortage Costs

Denote the holding cost by

$$\begin{array}{ll} h[S - D] & \text{if } S \geq D, \\ 0 & \text{if } S < \mathrm{D}, \end{array}$$

where $h[\cdot]$ is a mathematical function, not necessarily linear.

[1]If $c - p$ is nonnegative, $G(S)$ will be a monotone increasing function. This implies that the item should not be stocked, that is, $S^* = 0$.

Similarly, the shortage cost can be denoted by

$$\begin{array}{ll} p[D - S] & \text{if } D \geq S, \\ 0 & \text{if } D < S, \end{array}$$

where p[·] is also a function, not necessarily linear.

Thus, the total expected cost is given by

$$c(S - I) + \int_S^\infty p[x - S]f(x)dx + \int_0^S h[S - x]f(x)dx,$$

where I is the amount on hand.

If $L(S)$ is defined as the *expected shortage plus holding cost,* i.e.,

$$L(S) = \int_S^\infty p[x - S]f(x)dx + \int_0^S h[S - x]f(x)dx,$$

then the total expected cost can be written as

$$c(S - I) + L(S).$$

The optimal policy is obtained by minimizing this expression, subject to the constraint that $S \geq I$, that is,

$$\min_{S \geq I}\{c(S - I) + L(S)\}.$$

If $L(S)$ is strictly convex[1] [a sufficient condition being that the shortage and holding costs each are convex and $f(x) > 0$ for all $x \geq 0$], then the optimal policy is the following:

Optimal Inventory Policy

If $I < S^*$, order $S^* - I$ to bring the inventory level up to S^*.
If $I \geq S^*$, do not order,

where S^* is the value of S that satisfies the equation

$$\frac{dL(S)}{dS} + c = 0.$$

PROBLEMS

18S1-1. The campus bookstore must decide how many textbooks to order for a course that will be offered only once. The number of students who will take the course is a random variable D, whose distribution can be approximated by a (continuous) uniform distribution on the interval [40, 60]. After the quarter starts, the value of D becomes known. If D exceeds the number of books available, the known shortfall is made up by placing a rush order at a cost of \$14 plus \$2 per book over the normal ordering cost. If D is less than the stock on hand, the extra books are returned for their original ordering cost less \$1 each. What is the order quantity that minimizes the expected cost?

18S1-2. Consider the following inventory model, which is a single-period model with known density of demand $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ elsewhere. There are two costs connected with the model. The first is the purchase cost, given by $c(S - I)$. The second is a cost p that is incurred once if there is *any* unsatisfied demand (independent of the amount of unsatisfied demand).

(a) If I units are available and goods are ordered to bring the inventory level up to S (if $I < S$), write the expression for the expected loss and describe completely the optimal policy.

(b) If a fixed cost K is also incurred whenever an order is placed, describe the optimal policy.

[1] See Appendix 2 for the definition of a strictly convex function.