CHAPTER 9 - SINUSOIDS AND PHASORS

List of topics for this chapter :

- Sinusoids
- Phasors
- Phasor Relationships for Circuit Elements
- Impedance and Admittance
- Impedance Combinations
- Applications

SINUSOIDS

Problem 9.1 Express the following as cosine functions.

(a) $5\sin(2\pi t)$

(b) $4.3\sin(\omega t - 47°)$

(c) $2\sin(\omega t - \pi/2)$

To convert the sine function to the cosine function, we will need a trigonometric identity.

$$\mp \sin(x) = \cos(x \pm 90°)$$

(a) $\underline{\mathbf{5\cos(2\pi t - 90°)}}$

(b) $4.3\cos(\omega t - 47° - 90°) = \underline{\mathbf{4.3\cos(\omega t - 137°)}}$

(c) $\pi/2 \text{ rad} = 90°$ and $\pi \text{ rad} = 180°$

$2\sin(\omega t - \pi/2) = 2\cos(\omega t - \pi/2 - \pi/2) = \underline{\mathbf{2\cos(\omega t - \pi)}}$

Because

$\sin(x \pm 90°) = \pm\cos(x)$ or $\cos(x \pm 180°) = -\cos(x)$

this can also be written as $\underline{\mathbf{-2\cos(\omega t)}}$

Problem 9.2 Find the magnitude, angular frequency, frequency, and phase angle of each of the following functions.

(a) $5\sin(10t)$

(b) $-2.5\cos(2\pi t)$

(c) $\sqrt{3}\cos(\omega t - 37°)$

(a) Consider $v(t) = V_m \sin(\omega t + \phi)$. Also, note that $\omega = 2\pi f$.

$V_m = \underline{\mathbf{5}}$ $\quad \omega = \underline{\mathbf{10}}$ $\quad f = \underline{\mathbf{10/2\pi}}$, $\quad \phi = \underline{\mathbf{0°}}$

(b) Consider $v(t) = V_m \cos(\omega t + \phi)$

$V_m = \underline{\mathbf{2.5}}$ $\quad \omega = \underline{\mathbf{2\pi}}$ $\quad f = \underline{\mathbf{1}}$, $\quad \phi = \underline{\mathbf{180°}}$

Note that $\phi = 180°$ due to the negative sign in front of the function.

(c) Consider $v(t) = V_m \cos(\omega t + \phi)$

$V_m = \underline{\mathbf{\sqrt{3}}}$ $\quad \omega = \underline{\omega}$ $\quad f = \underline{\omega/2\pi}$, $\quad \phi = \underline{\mathbf{-37°}}$

Problem 9.3 *[9.5]* Given $v_1 = 20\sin(\omega t + 60°)$ and $v_2 = 60\cos(\omega t - 10°)$, determine the phase angle between the two sinusoids and which one lags the other.

$$v_1 = 20\sin(\omega t + 60°) = 20\cos(\omega t + 60° - 90°) = 20\cos(\omega t - 30°)$$
$$v_2 = 60\cos(\omega t - 10°)$$

This indicates that the phase angle between the two signals is $\underline{\mathbf{20°}}$ and that $\underline{\mathbf{v_1 \text{ lags } v_2}}$.

PHASORS

Problem 9.4 Convert the following into phasors.

(a) $100\sin(\omega t)$
(b) $20\cos(\omega t)$
(c) $50\cos(\omega t - 80°)$
(d) $25\sin(\omega t + 45°)$

(a) $\underline{\mathbf{100\angle 0°}}$ assuming a reference of $A\sin(\omega t + \phi)$

(b) $\underline{\mathbf{20\angle 0°}}$ assuming a reference of $A\cos(\omega t + \phi)$

(c) $\underline{\mathbf{50\angle -80°}}$ assuming a reference of $A\cos(\omega t + \phi)$

(d) $\underline{\mathbf{25\angle 45°}}$ assuming a reference of $A\sin(\omega t + \phi)$

Problem 9.5 *[9.11]* Let $\mathbf{X} = 8\angle 40°$ and $\mathbf{Y} = 10\angle -30°$. Evaluate the following quantities and express your results in polar form.

(a) $(\mathbf{X}+\mathbf{Y})\mathbf{X}^*$

(b) $(\mathbf{X}-\mathbf{Y})^*$

(c) $(\mathbf{X}+\mathbf{Y})/\mathbf{X}$

(a)
$$\mathbf{X}+\mathbf{Y} = 8\angle 40° + 10\angle -30°$$
$$\mathbf{X}+\mathbf{Y} = (6.128 + j5.142) + (8.66 - j5)$$
$$\mathbf{X}+\mathbf{Y} = 14.79 + j0.142 = 14.79\angle 0.55°$$

$$(\mathbf{X}+\mathbf{Y})\mathbf{X}^* = (14.79\angle 0.55°)(8\angle -40°)$$
$$(\mathbf{X}+\mathbf{Y})\mathbf{X}^* = \underline{\mathbf{118.3\angle -39.45°}}$$

(b)
$$\mathbf{X}-\mathbf{Y} = 8\angle 40° - 10\angle -30°$$
$$\mathbf{X}-\mathbf{Y} = (6.128 + j5.142) - (8.66 - j5)$$
$$\mathbf{X}-\mathbf{Y} = -2.532 + j10.142 = 10.45\angle 104°$$

$$(\mathbf{X}-\mathbf{Y})^* = \underline{\mathbf{10.45\angle -104°}}$$

(c) From (a), $\mathbf{X}+\mathbf{Y} = 14.79\angle 0.55°$

$$\frac{\mathbf{X}+\mathbf{Y}}{\mathbf{X}} = \frac{14.79\angle 0.55°}{8\angle 40°} =$$
$$(\mathbf{X}+\mathbf{Y})/\mathbf{X} = \underline{\mathbf{1.849\angle -39.45°}}$$

Problem 9.6 If $A\sin(\omega t + \phi)$ is used as a common reference, what would be the phasors?

(a) $100\sin(\omega t)$

(b) $20\cos(\omega t)$

(c) $50\cos(\omega t - 80°)$

(d) $25\sin(\omega t + 45°)$

(a) $\underline{\mathbf{100\angle 0°}}$

(b) $\underline{\mathbf{20\angle 90°}}$

(c) $\underline{\mathbf{50\angle 10°}}$

(d) $\underline{\mathbf{25\angle 45°}}$

Problem 9.7 Given the circuit of Figure 9.1, find the steady-state value of $v_c(t)$ when $i_s(t) = 5\sin(1000t)$ A.

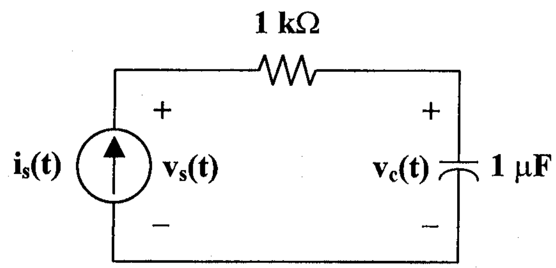


Figure 9.1

- **Carefully DEFINE the problem.**
  Each component is labeled completely. The problem is clear.

- **PRESENT everything you know about the problem.**
  In the time domain,

$$v_c = \frac{1}{C}\int i_c \, dt$$

If the circuit is transformed to its frequency domain equivalent, however, then

$$\mathbf{V}_c = \mathbf{I}\,\mathbf{Z}_c = \frac{1}{j\omega C}\mathbf{I}$$

The final answer can then be converted to the time domain.

- **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**
  The technique used to solve this problem is Ohm's law. The choice to be made is whether to use the time domain or the frequency domain. From what we know about the problem, converting the time-domain circuit into the frequency domain allows the use of algebra with complex numbers rather than calculus to analyze the circuit.

  Analysis of simple circuits can be done in the time-domain as a check of the answer.

- **ATTEMPT a problem solution.**
  Transforming the circuit to the frequency domain, the current source is $\mathbf{I}_s = 5\angle 0°$.
  The resistor converts to $\mathbf{Z}_R = 1\ \text{k}\Omega$, and the capacitor becomes

$$\mathbf{Z}_c = \frac{1}{j\omega C} = \frac{1}{j(1000)(10^{-6})} = -j1000 = 1000\angle -90°$$

Thus,

$$\mathbf{V}_c = (5\angle 0°)(1000\angle -90°) = 5000\angle -90°$$

or $$v_c(t) = 5000\sin(1000t - 90°)\text{ V}$$

- **EVALUATE the solution and check for accuracy.**
  Using KVL in the frequency domain,

$$\mathbf{V}_R = \mathbf{I}\,\mathbf{Z}_R = (5\angle 0°)(1000) = 5000\angle 0°\text{ V}$$

$$\mathbf{V}_c = \mathbf{I}\,\mathbf{Z}_c = 5000\angle -90°$$

$$\mathbf{V}_s = \mathbf{V}_R + \mathbf{V}_c = 5000\angle 0° + 5000\angle -90°$$

$$\mathbf{V}_s = 5000 - j5000 = 5000\sqrt{2}\angle -45°$$

or $$v_s(t) = 5000\sqrt{2}\sin(1000t - 45°)\text{ V}$$

Using KVL in the time domain,

$$v_c(t) = \frac{1}{C}\int i_c(t)\,dt = \frac{1}{10^{-6}}\int 5\sin(1000t)\,dt$$

$$v_c(t) = \frac{1}{10^{-6}\,10^{3}}(5)[-\cos(1000t)]$$

$$v_c(t) = 5000\sin(1000t - 90°)\text{ V}$$

$$v_R(t) = R\,i(t) = 5000\sin(1000t)\text{ V}$$

$$v_s(t) = v_R(t) + v_c(t)$$

$$v_s(t) = 5000\sin(1000t) + 5000\sin(1000t - 90°)$$

$$v_s(t) = 5000\sqrt{2}\sin(1000t - 45°)\text{ V}$$

Our check for accuracy was successful.

- **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to "ALTERNATIVE solutions" and continue through the process again.**
  This problem has been solved satisfactorily.

$$v_c(t) = \underline{\mathbf{5000\sin(1000t - 90°)\ V}}$$

Problem 9.8 Given $\omega = 100$, determine the frequency domain ($s = j\omega$) values for the following elements.

(a) $R = 1\Omega, 10\Omega, 1\text{k}\Omega, 1\text{M}\Omega, 100\text{M}\Omega$

(b) $L = 10\text{H}, 5\text{H}, 1\text{H}, 5\text{mH}, 40\mu\text{H}$

(c) $C = 2\text{mF}, 333\mu\text{F}, 5\mu\text{F}, 10\text{pF}$

(a) $R = \underline{\mathbf{1\Omega, 10\Omega, 1k\Omega, 1M\Omega, 100M\Omega}}$

(b) $\omega L = \underline{\mathbf{1000\Omega, 500\Omega, 100\Omega, 500m\Omega, 4m\Omega}}$

where the units are ohms of inductive reactance

(c) $\frac{1}{\omega C} = \underline{\mathbf{5\Omega, 30\Omega, 2k\Omega, 1G\Omega}}$

where the units are ohms of capacitive reactance

Problem 9.9 Given $R = 100\Omega$, $L = 1H$, and $C = 100\mu F$, calculate the values in the following table.

ω	R	X_L	X_C
1			
10			
100			
1000			
10000			

Clearly, $R = 100$

which is not dependent upon the frequency.

$L = 1$ and $C = 10^{-4}$ implies that

$$X_L = \omega L = \omega \quad \text{and} \quad X_C = \frac{1}{\omega C} = \frac{10^4}{\omega}$$

and the table becomes

ω	R	X_L	X_C
1	**100**	**1**	**10 k**
10	**100**	**10**	**1 k**
100	**100**	**100**	**100**
1000	**100**	**1k**	**10**
10000	**100**	**10k**	**1**

IMPEDANCE AND ADMITTANCE

Problem 9.10 Assume that $\mathbf{Z} = R + jX_L - jX_C$. For the values used in Problem 9.9, what would be the values of Z in rectangular coordinates?

ω	Z
1	
10	
100	
1000	
10000	

Insert the values of R , X_L, and X_C into

$$\mathbf{Z} = R + j(X_L - X_C)$$

and it is evident that

ω	Z
1	**100 – j 9999**
10	**100 – j 990**
100	**100 + j 0**
1000	**100 + j 990**
10000	**100 + j 9999**

Problem 9.11 *[9.43]* In the circuit of Figure 9.2, find $\mathbf{V}_s$ if $\mathbf{I}_o = 2\angle 0°\ A$.

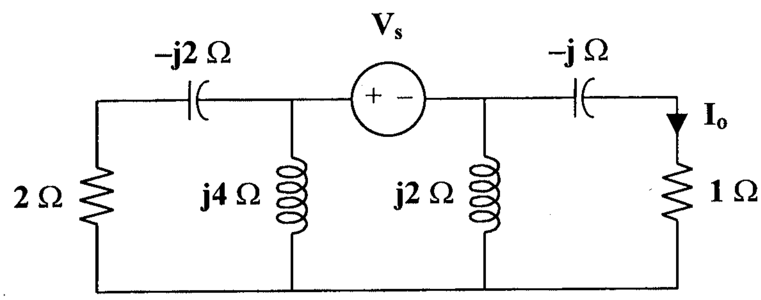
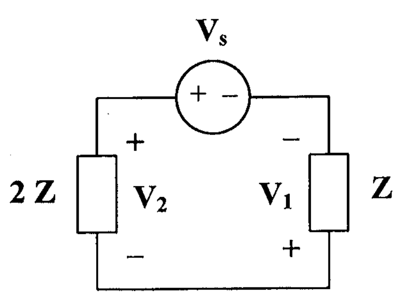


Figure 9.2

Since the left portion of the circuit is twice as large as the right portion, the equivalent circuit is shown below.



$$\mathbf{V}_1 = -\mathbf{I}_o(1 - j) = -2(1 - j)$$
$$\mathbf{V}_2 = 2\mathbf{V}_1 = -4(1 - j)$$

$$\mathbf{V}_s = \mathbf{V}_1 + \mathbf{V}_2 = 6(1 - j)$$
$$\mathbf{V}_s = \underline{\mathbf{8.485\angle 135° \ V}}$$

Problem 9.12 Using the values in Problem 9.10, what would be the values of Z in polar coordinates?

ω	Z
1	
10	
100	
1000	
10000	

ω	Z
1	**9999.5 ∠–89.4°**
10	**995 ∠–84.2°**
100	**100 ∠0°**
1000	**995 ∠84.2°**
10000	**9999.5 ∠89.4°**

IMPEDANCE COMBINATIONS

Problem 9.13 Given the circuit of Figure 9.3, find $\mathbf{Z}_{in}$ for $\omega = 1000$ rad/s.

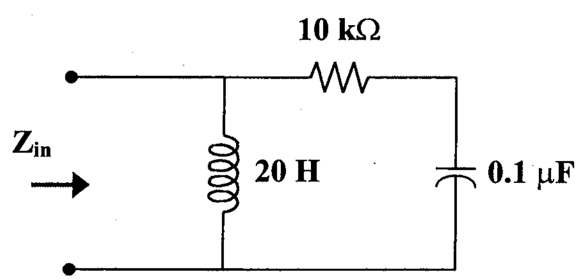


Figure 9.3

The phasor domain equivalent includes

$$\mathbf{Z}_R = 10 \text{ k}\Omega$$
$$\mathbf{Z}_L = j\omega L = j(1000)(20) = j20 \text{ k}\Omega$$
$$\mathbf{Z}_C = \frac{1}{j\omega C} = \frac{1}{j(1000)(0.1\times 10^{-6})} = \frac{-j}{10^{-4}} = -j10 \text{ k}\Omega$$

$$\mathbf{Z}_{in} = \mathbf{Z}_L \parallel (\mathbf{Z}_R + \mathbf{Z}_C) = j20k \parallel (10k - j10k)$$

$$\mathbf{Z}_{in} = \frac{(j20k)(10k - j10k)}{j20k + 10k - j10k} = \frac{j20k(10 - j10)}{10 + j10} = \frac{(200k)(1+j)}{(10)(1+j)}$$

$$\mathbf{Z}_{in} = \underline{\mathbf{20\ k\Omega}}$$

Problem 9.14 *[9.47]* Find $\mathbf{Z}_{eq}$ in the circuit of Figure 9.4.

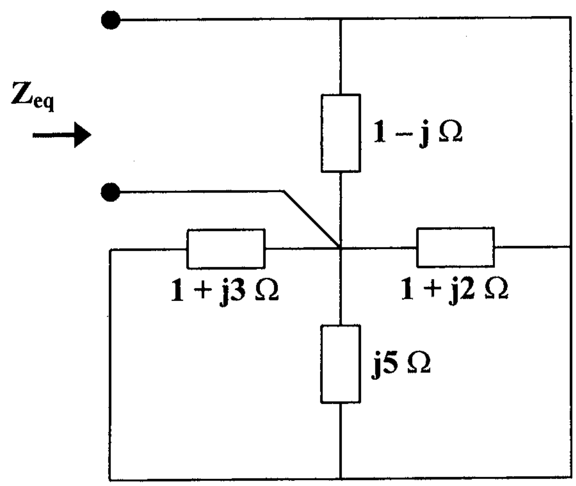


Figure 9.4

All of the impedances are in parallel.

Thus,

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{1}{1-j} + \frac{1}{1+j2} + \frac{1}{j5} + \frac{1}{1+j3}$$

$$\frac{1}{\mathbf{Z}_{eq}} = (0.5 + j0.5) + (0.2 - j0.4) + (\text{-}j0.2) + (0.1 - j0.3)$$

$$\frac{1}{\mathbf{Z}_{eq}} = 0.8 - j0.4$$

$$\mathbf{Z}_{eq} = \frac{1}{0.8 - j0.4}$$

$$\mathbf{Z}_{eq} = \underline{\mathbf{1 + j0.5\ \Omega}}$$

APPLICATIONS

Problem 9.15 The circuit shown in Figure 9.5 is used to make a simple low-pass filter. An important part of choosing the appropriate value of C is to determine the highest frequency to be passed and then choose a value of C such that the output voltage is $1/\sqrt{2}$ times the magnitude of the input at that frequency. What value of C makes this a low-pass filter for frequencies from $0\ \text{Hz}$ to $1000\ \text{Hz}$?

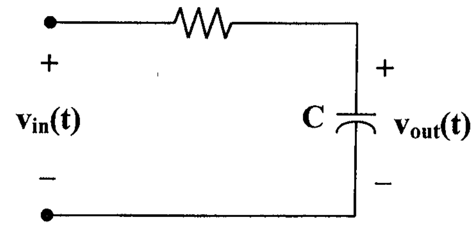
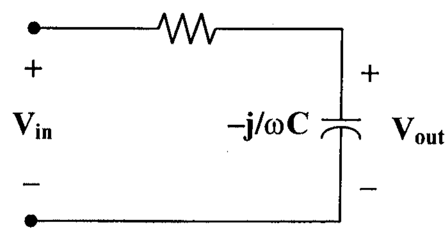


Figure 9.5

Transforming this circuit to the frequency domain yields



$$\mathbf{V}_{out} = \frac{-j/\omega C}{10k - j/\omega C}\mathbf{V}_{in}$$

$$|\mathbf{V}_{out}| = |\mathbf{V}_{in}|\left|\frac{-j/\omega C}{10k - j/\omega C}\right|$$

$f = 1000\ \text{Hz}$ is the upper frequency limit, called the corner frequency.

$$\left|\frac{-j/\omega C}{10k - j/\omega C}\right|_{\omega=2\pi(1000)} = \frac{1}{\sqrt{2}}$$

$$\left|\frac{-j/2\pi(1000)C}{10k - j/2\pi(1000)C}\right| = \frac{X_c}{\sqrt{10^8 + X_c^{\,2}}} = \frac{1}{\sqrt{2}}$$

$$\frac{X_c^{\,2}}{10^8 + X_c^{\,2}} = \frac{1}{2} \longrightarrow X_c^{\,2} = \frac{X_c^{\,2}}{2} + \frac{10^8}{2}$$

$$\frac{X_c^{\,2}}{2} = \frac{10^8}{2}$$

$$X_c = 10^4 = \frac{1}{j2\pi(1000)C}$$

$$C = \frac{1}{2\pi\,(10^3)(10^4)}$$

$$C = \frac{\mathbf{1}}{\mathbf{20\pi}}\ \mathbf{\mu F}$$

Problem 9.16 *[9.61]* Using the circuit of Figure 9.6,

(a) Calculate the phase shift.
(b) State whether the phase shift is leading or lagging (output with respect to input).
(c) Determine the magnitude of the output when the input is 120 V.

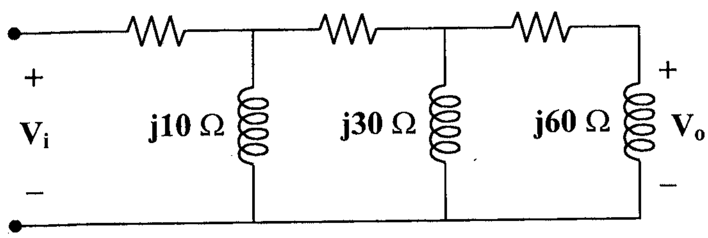
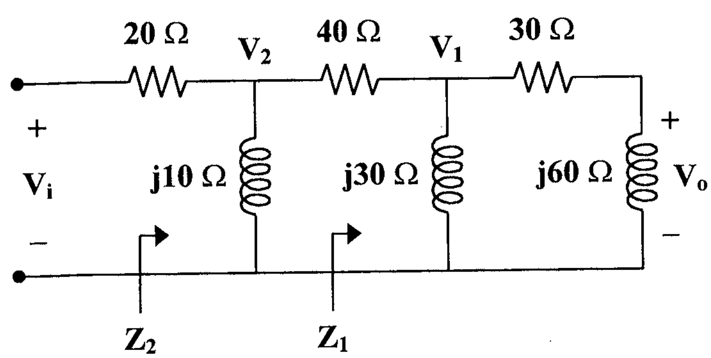


Figure 9.6

(a) Consider the circuit as shown.



$$\mathbf{Z}_1 = j30 \,||\, (30 + j60) = \frac{(j30)(30 + j60)}{30 + j90} = 3 + j21$$

$$\mathbf{Z}_2 = j10 \,||\, (40 + \mathbf{Z}_1) = \frac{(j10)(43 + j21)}{43 + j31} = 1.535 + j8.896 = 9.028\angle 80.21°$$

Let $\mathbf{V}_i = 1\angle 0°$.

$$\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_2 + 20}\mathbf{V}_i = \frac{(9.028\angle 80.21°)(1\angle 0°)}{21.535 + j8.896}$$
$$\mathbf{V}_2 = 0.3875\angle 57.77°$$

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + 40}\mathbf{V}_2 = \frac{3 + j21}{43 + j21}\mathbf{V}_2 = \frac{(21.213\angle 81.87°)(0.3875\angle 57.77°)}{47.85\angle 26.03°}$$
$$\mathbf{V}_1 = 0.1718\angle 113.61°$$

$$\mathbf{V}_o = \frac{j60}{30 + j60}\mathbf{V}_1 = \frac{j2}{1 + j2}\mathbf{V}_1 = \frac{2}{5}(2 + j)\mathbf{V}_1$$
$$\mathbf{V}_o = (0.8944\angle 26.56°)(0.1718\angle 113.6°)$$
$$\mathbf{V}_o = 0.1536\angle 140.2°$$

Therefore, the phase shift is **140.2°**

(b) The phase shift is **leading**.

(c) If $\mathbf{V}_i = 120\ \text{V}$, then

$$\mathbf{V}_o = (120)(0.1536\angle 140.2°) = 18.43\angle 140.2°\ \text{V}$$

and the magnitude is **18.43 V**.