## CHAPTER EIGHT

### Section 8.1E

**1.**a. **F**
c. **T**
e. **F**
g. **T**

**2.**a. **T**
c. **T**
e. **F**
g. **F**

**3.**a. One interpretation is

UD: The set of all people
N: $\{\langle \mathbf{u}_1, \mathbf{u}_2\rangle$: $\mathbf{u}_1$ is the mother of $\mathbf{u}_2\}$
a: Jane Doe
d: Jay Doe

c. One interpretation is

UD: The set of all U.S. cities
L: $\{\langle \mathbf{u}\rangle$: $\mathbf{u}$ is in California$\}$
C: $\{\langle \mathbf{u}_1, \mathbf{u}_2\rangle$: $\mathbf{u}_1$ is to the north of $\mathbf{u}_2\}$
h: San Francisco
m: Los Angeles

e. One interpretation is

UD: The set of positive integers
M M: $\{\langle \mathbf{u}\rangle$: $\mathbf{u}$ is odd$\}$
N N: $\{\langle \mathbf{u}\rangle$: $\mathbf{u}$ is even$\}$
a: 1
b: 2

**4.**a. One interpretation is

UD: The set of positive integers
C: $\{\langle \mathbf{u}_1, \mathbf{u}_2\rangle$: $\mathbf{u}_1$ equals $\mathbf{u}_2$ squared$\}$
r: 2
s: 3

c. One interpretation is

UD: The set of all people
L: $\varnothing$
i: Serena Williams

j: Edgar Allen Poe
m: Margaret Mead

e. One interpretation is

UD: The set of positive integers
J: $\{\langle \mathbf{u} \rangle$: $\mathbf{u}$ is even$\}$
a: 1
b: 2
c: 3
d: 4

**5.**a. One interpretation is

UD: The set of all people
F: $\{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$: $\mathbf{u}_1$ is the mother of $\mathbf{u}_2\}$
a: Liza Minelli
b: Judy Garland (Liza Minelli's mother)

On this interpretation, ‘Fab $\supset$ Fba’ is true and ‘Fba $\supset$ Fab’ is false.

c. One interpretation is

UD: The set of planets
C: $\{(\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$: The orbit of $\mathbf{u}_1$ is between the orbit of $\mathbf{u}_2$ and the orbit of $\mathbf{u}_3$
M: $\{\langle \mathbf{u} \rangle$: $\mathbf{u}$ is inhabited by human life$\}$
a: Earth
p: Venus
q: Pluto
r: Mars

On this interpretation, ‘$\sim$ Ma $\vee$ Cpqr’ is false and ‘Capq $\vee$ $\sim$ Mr’ is true.

e. One interpretation is

UD: The set of positive integers
L: $\{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$: $\mathbf{u}_1$ is less than $\mathbf{u}_2\}$
M: $\{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$: $\mathbf{u}_1 = \mathbf{u}_2\}$
j: 1
k: 1

On this interpretation the first sentence is true and the second, false.

**6.**a. False. For consider any person w who is over 40 years old. It is true that that person is over 40 years old but false that some person is her own sister. So that person w is *not* such that <u>if</u> w is over 40 years old <u>then</u> some person is her own sister.

c. False. The sentence says that there is at least one person x such that every person y is either a child or a brother of x, which is obviously false.

e. True. The antecedent, '$(\exists x)Cx$', is true. At least one person is over 40 years old. And the consequent, '$((\exists x)(\exists y)Fxy \supset (\exists y)By)$', is also true: '$(\exists x)(\exists y)Fxy$' is true, and '$(\exists y)By$' is true.

g. True. The antecedent, '$(\forall x)Bx$', is false, so the conditional sentence is true.

i. True. The sentence says that there is at least one person x such that either x is over 40 years old or x and some person y are sisters and y is over 40 years old. Both conditions are true.

**7.**a. True. Every U.S. president held office after George Washington's first term. Note that for the sentence to be true, George Washington too must have held office after George Washington's first term of office. He did—he was in office for two terms.

c. True. George Washington was the first U.S. president, and at least one U.S. president y held office after Washington.

e. True. Each U.S. president y is such that if y is a U.S. citizen (which every U.S. president y is) then at least one U.S. president held office before or after y's first term.

g. False. Every U.S. president x held office after George Washington's first term, but, for any such president x, no non-U.S. citizen has held office before x (because every U.S. president *is* a U.S. citizen).

i. True (in 2003!). The sentence says that a disjunction is not the case and therefore that each disjunct is false. The first disjunct, 'Bg', is false—George Washington was not a female. The second disjunct, which says that there is a U.S. president who held office after every U.S. president's first term of office, is false (there is no one yet who has held office after George W. Bush's first term).

**8.**a. True. The first conjunct, 'Bb', is true. The second conjunct is also true since no positive integer that is greater than 2 is equal to 2.

c. True. No positive integer x is equal to any number than which it is greater.

e. True. The antecedent is true since it is not the case that every positive integer is greater than every positive integer. But 'Mcba' is also true: $3 - 2 = 1$.

g. True. No positive integer z that is even is such that the result of subtracting 1 from z is also even.

i. False. Not every positive integer (in fact, *no* positive integer) is such that it equals itself if and only if there are not two positive integers of which it is the difference. Every positive integer equals itself, but every positive integer is also the difference between two positive integers.

**Section 8.2E**

**1.**a. The sentence is false on the following interpretation:

UD: The set of positive integers
Fx: x is divisible by 4
Gx: x is even

Every positive integer that is divisible by 4 is even, but not every positive integer is even.

c. The sentence is false on the following interpretation:

UD: The set of positive integers
Bxy: x is less than y

Every positive integer is less than at least one positive integer, but there is no single positive integer that every positive integer is less than.

e. The sentence is false on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is prime

The antecedent, '$(\forall x)Fx \supset (\forall w)Gw$', is true since *its* antecedent, '$(\forall x)Fx$', is false. But the consequent, '$(\forall z)(Fz \supset Gz)$', is false since at least one odd positive integer is not prime (the integer 9, for example).

g. The sentence is false on the following interpretation:

UD: The set of positive integers
Gx: x is negative
Fxy: x equals y

No positive integer is negative, but not every positive integer is such that if it equals itself (which every one does) then it is negative.

**2.**a. The sentence is true on the following interpretation:

UD: The set of positive integers
Bxy: x equals y

The sentence to the left of '$\equiv$' is true since it is not the case that all positive integers equal one another; and the sentence to the right of '$\equiv$' is true since each positive integer is equal to itself.

c. The sentence is true on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is even

At least one positive integer is odd, and at least one positive integer is even, but no positive integer is both odd and even.

e. The sentence is true on the following interpretation:

UD: The set of positive integers
Fx: x is negative
Gx: x is odd

Trivially, every negative positive integer is odd since no positive integer is negative; and every positive integer that is odd is not negative.

g. The sentence is true on the following interpretation:

UD: The set of positive integers
Bx: x is prime
Hx: x is odd

The antecedent is false—not every positive integer is such that it is prime if and only if it is odd, and the consequent is true—at least one positive integer is both prime and odd.

i. The sentence is true on the following interpretation:

UD: The set of positive integers
Bxy: x is less than y

The less-than relation is transitive, making the first conjunct true; for every positive integer there is a greater one, making the second conjunct true; and the less-than relation is irreflexive, making the third conjunct true.

**3.**a. The sentence is true on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is prime

At least one positive integer is both odd and prime, but also at least one positive integer is neither odd nor prime.

The sentence is false on the following interpretation:

UD: The set of positive integers
Fx: x is positive
Gx: x is prime

At least one positive integer is both positive and prime, but no positive integer is neither positive nor prime.

c. The sentence is true on the following interpretation:

UD: The set of positive integers
Bxy: x is evenly divisible by y
n: 9

The antecedent, '$(\forall x)Bnx$', is false on this interpretation; 9 is not evenly divisible by every positive integer.

The sentence is false on the following interpretation:

UD: The set of positive integers
Bxy: x is less than or equal to y
n: 1

The number 1 is less than or equal to every positive integer, so the antecedent is true and the consequent false.

e. The sentence is true on the following interpretation:

UD: The set of positive integers
Nxy: x equals y

Each positive integer x is such that each positive integer w that is equal to x is equal to itself.

The sentence is false on the following interpretation:

UD: The set of positive integers
Nxy: x is greater than y

No positive integer x is such that every positive integer w that is greater or smaller than x is greater than itself.

g. The sentence is true on the following interpretation:

UD: The set of positive integers
Cx: x is greater than 0
Dx: x is prime

Every positive integer is either greater than 0 or prime (because every positive integer is greater than 0), and at least one positive integer is both greater than 0 and prime. The biconditional is therefore true on this interpretation.

The sentence is false on the following interpretation:

UD: The set of positive integers
Cx: x is even
Dx: x is odd

Every positive integer is either even or odd, but no positive integer is both. The biconditional is therefore false on this interpretation.

**4.**a. If the antecedent is true on an interpretation, then at least one member x of the UD, let's assume a, stands in the relation B to every member y of the UD. But then it follows that for every member y of the UD, there is at least one member x that stands in the relation B to y—namely, a. So the consequent is also true. If the antecedent is false on an interpretation, then the conditional is trivially true. So the sentence is true on every interpretation.

c. If 'Fa' is true on an interpretation, then 'Fa $\vee$ [$(\forall x)Fx \supset Ga$]' is true. If 'Fa' is false on an interpretation, then '$(\forall x)Fx$' is false, making '$(\forall x)Fx \supset Ga$' true. Either way, the disjunction is true.

e. If '$(\exists x)Hx$' is true on an interpretation, then the disjunction is true on that interpretation. If '$(\exists x)Hx$' is false on an interpretation, then no member of the UD is H. In this case, every member of the UD is such that <u>if</u> it is H (which it is not) <u>then</u> it is J, and so the second disjunct is true, making the disjunction true as well. Either way, then, the disjunction is true.

**5.**a. No member of any UD is such that it is in the extension of 'B' if and only if it isn't in the extension of 'B'. So the existentially quantified sentence is false on every interpretation.

c. The second conjunct is true on an interpretation if and only if no member of the UD is G and no member of the UD is not F—that is, every member of the UD *is* F. But then the first conjunct must be false, because its antecedent is true but its consequent is false. Thus there is no interpretation on which the entire conjunction is true; it is quantificationally false.

e. The third conjunct is true on an interpretation if and only if at least one member **u** of the UD is A but is not C. For the first conjunct to be true, **u** must also be B since it is A; and for the second conjunct to be true, **u** must also be C since it is B. But that means that the conjunction is true if and only if at least one member **u** of the UD is both C and not C. This latter is impossible; so there is no interpretation on which the sentence is true, i.e., it is quantificationally false.

**6.**a. The sentence is quantificationally indeterminate. It is true on the interpretation

UD: The set of positive integers
Gx: x is odd
Hx: x is even

since at least one positive integer is odd and at least one is even, and at least one positive integer (in fact, every positive integer) is not both odd and even.

The sentence is false on the interpretation

UD: The set of positive integers
Gx: x is less than zero
Hx: x is even

since the first conjunct is false: no positive integer is less than zero.

c. The sentence is quantificationally true. If every member of the UD that is F is also G, then every member of the UD that fails to be G must also fail to be F.

e. The sentence is quantificationally indeterminate. It is true on the interpretation

UD: The set of positive integers
Dx: x is odd
Hxy: x is greater than or equal to y

because the consequent, which says that there is a positive integer z such that every odd positive integer is greater than or equal to z, is true. The positive integer 1 satisfies this condition.

The sentence is false on the interpretation

UD: The set of positive integers
Dx: x is odd
Hxy: x equals y

because the antecedent, which says that for every odd positive integer there is at least one positive integer to which it is equal, is true; but the consequent, which says that there is some one positive integer to which every odd positive integer is equal, is false.

**Section 8.3E**

**1.**a. The first sentence is false and the second true on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is prime
a: 4

Some positive integer is odd and 4 is not prime, so '$(\exists x)Fx \supset Ga$' is false. But any even positive integer is such that if that integer is odd (which it is not) then 4 is prime; so '$(\exists x)(Fx \supset Ga)$' is true.

c. The first sentence is false and the second true on the following interpretation:

UD: The set of positive integers
Fx: x is a multiple of 2
Gx: x is an odd number

It is false that either every positive integer is a multiple of 2 or every positive integer is odd, but it is true that every positive integer is either a multiple of 2 or odd.

e. The first sentence is false and the second true on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is prime

An odd prime (e.g., 3) is not such that it is even if and only if it is prime. But '$(\exists x)Fx \equiv (\exists x)Gx$' is true since '$(\exists x)Fx$' and '$(\exists x)Gx$' are both true.

g. The first sentence is true and the second false on the following interpretation:

UD: The set of positive integers
Bx: x is less than 5
Dxy: x is divisible by y without remainder

The integer 1 is less than 5 and divides every positive integer without remainder. But '$(\forall x)(Bx \supset (\forall y)Dyx)$' is false, for 2 is less than 5 but does not divide any odd integer without remainder.

i. The first sentence is false and the second true on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Kxy: x is smaller than y

The integer 1 does not satisfy the condition that <u>if</u> it is odd (which it is) <u>then</u> there is a positive integer that is smaller than it. But at least one positive integer does satisfy the condition—in fact, all other positive integers do.

**2.**a. Suppose that '$(\forall x)Fx \supset Ga$' is true on an interpretation. Then either '$(\forall x)Fx$' is false or 'Ga' is true. If '$(\forall x)Fx$' is false, then some member of the UD is not in the extension of 'F'. But then that object is trivially such that <u>if</u> it is F (which it is not) then a is G. So '$(\exists x)(Fx \supset Ga)$' is true. If 'Ga' is true, then trivially every member x of the UD is such that if x is F then a is G; so '$(\exists x)(Fx \supset Ga)$' is true in this case as well.

Now suppose that '$(\forall x)Fx \supset Ga$' is false on some interpretation. Then '$(\forall x)Fx$' is true, and 'Ga' is false. Every object in the UD is then in the extension of 'F'; hence no member x is such that if it is F (which it is) then a is G (which is false). So '$(\exists x)(Fx \supset Ga)$' is false as well.

c. Suppose that '$(\exists x)(Fx \vee Gx)$' is true on an interpretation. Then at least one member of the UD is either in the extension of 'F' or in the extension of 'G'. This individual therefore does not satisfy '$\sim Fy \,\&\, \sim Gy$', and so '$(\forall y)(\sim Fy \,\&\, \sim Gy)$' is false and its negation true.

Now suppose that '$(\exists x)(Fx \vee Gx)$' is false on an interpretation. Then no member of the UD satisfies '$Fx \vee Gx$'—no member of the UD is in the extension of 'F' or in the extension of 'G'. In this case, every member of the UD satisfies '$\sim Fy \,\&\, \sim Gy$'; so '$(\forall y)(\sim Fy \,\&\, \sim Gy)$' is true and its negation false.

e. Suppose that '$(\forall x)(\forall y)Gxy$' is true on an interpretation. Then each pair of objects in the UD is in the extension of 'G'. But then '$(\forall y)(\forall x)Gxy$' must also be true. The same reasoning establishes the reverse.

**3.**a. The sentences are not quantificationally equivalent. The first sentence is true and the second false on the following interpretation:

UD: The set of positive integers
Fx: x is greater than 4
Gx: x is less than 10

At least one positive integer is either greater than 4 or less than 10, but it is false that every positive integer fails to be both greater than 4 and less than 10.

c. The sentences are not quantificationally equivalent. The first sentence is false and the second true on the following interpretation:

UD: The set of positive integers
Gxy: x equals y

It is false that each pair of positive integers is such that either the first equals the second or vice versa, but it is true that each pair of positive integers is such that either the first member equals itself (which is always true) or it is equal to the second.

**4.**a. All the set members are true on the following interpretation:

UD: The set of positive integers
Bx: x is odd
Cx: x is prime

At least one positive integer is odd, and at least one positive integer is prime, and some positive integers are neither odd nor prime.

c. All the set members are true on the following interpretation:

UD: The set of positive integers
Fx: x is greater than 10
Gx: x is greater than 5
Nx: x is smaller than 3
Mx: x is smaller than 5

Every positive integer that is greater than 10 is greater than 5, every positive integer that is smaller than 3 is smaller than 5, and no positive integer that is greater than 5 is also smaller than 5.

e. All the set members are true on the following interpretation:

UD: The set of positive integers
Nx: x is negative
Mx: x equals 0
Cxy: x is greater than 7

The two sentences are trivially true, the first because no positive integer is negative and the second because no positive integer equals 0.

g. All the set members are true on the following interpretation:

UD: The set of positive integers
Nx: x is prime
Mx: x is even

The first sentence is true because 3 is prime but not even. Hence not all primes are even numbers. The second is true because any nonprime integer is such that <u>if</u> it is prime (which it is not) <u>then</u> it is even. Hence it is false that all positive integers fail to satisfy this condition.

i. All the set members are true on the following interpretation:

UD: The set of positive integers
Fxy: x evenly divides y
Gxy: x is greater than y
a: 1

At least one positive integer is evenly divisible by 1, at least one positive integer is such that 1 is not greater than that integer, and every positive integer is either evenly divisible by 1 or such that 1 is greater than it.

**5.**a. If the set is quantificationally consistent, then there is an interpretation on which both set members are true. But if '$(\exists x)(Bx \& Cx)$' is true on an interpretation, then at least one member x of the UD is in the extensions of both 'B' and 'C'. That member is *not* neither B nor C, so, if '$(\exists x)(Bx \& Cx)$' is true, then '$(\forall x) \sim (Bx \vee Cx)$' is false. There is no interpretation on which both set members are true.

c. If the first set member is true on an interpretation, then every pair x and y of members of the UD is such that either x stands in the relation B to y or y stands in the relation B to x. In particular, each pair consisting of a member of the UD and itself must satisfy the condition and so must stand in the relation B to itself. This being so, the second set member is false on such an interpretation. Thus there can be no interpretation on which both set members are true.

e. If the first sentence is true on an interpretation, then there is at least one member of the UD that stands in the relation G to every member of the UD. In that case it is false that every pair of members of the UD fail to satisfy 'Gxy', so the second sentence must be false. Thus there can be no interpretation on which both set members are true.

**6.**a. The set is quantificationally inconsistent. If the third member is true, then something in the UD is F. If the first member is also true, then, because the antedent will be true, the consequent will also be true: everything in the UD will be F. But then the second sentence must be false: there is nothing that is not F. Thus there can be no interpretation on which all three set members are true.

c. The set is quantificationally consistent, as the following interpretation shows:

UD: The set of positive integers
Gxy: x equals y

The first sentence is true because each positive integer fails to be equal to all positive integers; and the second sentence is true because every positive integer is equal to itself. Thus both members of the set are true on at least one interpretation.

**7.** Suppose that **P** and **Q** are quantificationally equivalent. Then on every interpretation **P** and **Q** have the same truth-value. Thus the biconditional $\mathbf{P} \equiv \mathbf{Q}$ is true on every interpretation (since a biconditional is true when its immediate components have the same truth-value); hence it is quantificationally true.

Suppose that $\mathbf{P} \equiv \mathbf{Q}$ is quantificationally true. Therefore it is true on every interpretation. Then **P** and **Q** have the same truth-value on every interpretation (since a biconditional is true only if its immediate components have the same truth-value) and are quantificationally equivalent.

**Section 8.4E**

**1.**a. The set members are true and '$(\exists x)(Hx \,\&\, Fx)$' false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Fx: | x is evenly divisible by 2 |
| Hx: | x is odd |
| Gx: | x is greater than or equal to 1 |

Every positive integer that is evenly divisible by 2 is greater than or equal to 1, every odd positive integer is greater than or equal to 1, but no positive integer is both evenly divisible by 2 and odd.

c. The set member is true and 'Fa' is false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Fx: | x is even |
| a: | 1 |

At least one positive integer is even, but 1 is not even.

e. The set members are true and '$(\exists x)Bx$' is false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Bx: | x is negative |
| Cx: | x is prime |

Every positive integer is trivially such that if it is negative then it is prime, for no positive integer is negative; and at least one positive integer is prime. But no positive integer is negative.

g. The set member is true and '$(\forall x) \sim Lxx$' is false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Lxy: | x is greater than or equal to y |

Every positive integer x is such that for some positive integer y, x is not greater than or equal to y. But it is false that every positive integer is not greater than or equal to itself.

**2.**a. The premises are true and the conclusion false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Fx: | x is positive |

Gx: x is negative
Nx: x equals 0

The first premise is true since its antecedent is false. The second premise is trivially true because no positive integer equals 0. The conclusion is false for no positive integer satisfies the condition of being either not positive or negative.

c. The premises are true and the conclusion false on the following interpretation:

UD: The set of positive integers
Fx: x is prime
Gx: x is even
Hx: x is odd

There is an even prime positive integer (2), and at least one positive integer is odd and prime, but no positive integer is both even and odd.

e. The premises are true and the conclusion false on the following interpretation:

UD: The set of positive integers
Fx: x is negative
Gx: x is odd

The first premise is trivially true, for no positive integer is negative. For the same reason, the second premise is true. But at least one positive integer is odd, and so the conclusion is false.

g. The premises are true and the conclusion false on the following interpretation:

UD: The set of positive integers
Gx: x is prime
Dxy: x equals y

Some positive integer is prime, and every prime number equals itself, but there is no prime number that is equal to every positive integer.

i. The premises are true and the conclusion false on the following interpretation:

UD: The set of positive integers
Fx: x is odd
Gx: x is positive
Hx: x is prime

Every odd positive integer is positive, and every prime positive integer is positive, but not every positive integer is odd or prime.

**3.**a. A symbolization of the first argument is

$$\frac{(\forall x)Bx}{(\exists x)Bx}$$

To see that this argument is quantificationally valid, assume that '$(\forall x)Bx$' is true on some interpretation. Then every member of the UD is B. Since every UD is nonempty, it follows that there is at least one member that is B. So '$(\exists x)Bx$' is true as well.

A symbolization of the second argument is

$$\frac{(\forall x)(Px \supset Bx)}{(\exists x)(Px \ \& \ Bx)}$$

The premise is true and the conclusion false on the following interpretation:

UD: The set of positive integers
Px: x is negative
Bx: x is prime

c. One symbolization of the first argument is

$$\frac{(\exists x)(\forall y)Lxy}{(\forall y)(\exists x)Lxy}$$

To see that the argument is quantificationally valid, assume that the premise is true on some interpretation. Then some member x of the UD—let's call it a—stands in the relation L to every member of the UD. Thus for each member y of the UD, there is some member—namely, a—that stands in the relation L to y. So the conclusion is true as well.

A symbolization of the second argument is

$$\frac{(\forall x)(\exists y)Lyx}{(\exists y)(\forall x)Lyx}$$

The following interpretation makes the premise true and the conclusion false:

UD: The set of positive integers
Lxy: x is larger than y

For each positive integer, there is a larger one, but no positive integer is the largest.

e. A symbolization of the first argument is

$$\frac{(\exists x)(Tx \mathbin{\&} Sx) \mathbin{\&} (\exists x)(Tx \mathbin{\&} {\sim} Hx)}{(\exists x)(Tx \mathbin{\&} (Sx \vee {\sim} Hx))}$$

To see that this argument is quantificationally valid, assume that the premise is true on some interpretation. Then at least one member of the UD—let's call it a—is both T and S and at least one member of the UD is both T and not H. a satisfies the condition of being both T and either S or H, and so the conclusion is true as well.

A symbolization of the second argument is

$$\frac{(\forall x)(Tx \supset Sx) \mathbin{\&} {\sim} (\exists x)(Tx \mathbin{\&} Hx)}{(\exists x)(Tx \mathbin{\&} (Sx \vee {\sim} Hx))}$$

The following interpretation makes the premise true and the conclusion false:

| | |
|---|---|
| UD: | The set of positive integers |
| Tx: | x is negative |
| Sx: | x is odd |
| Hx: | x is prime |

Every negative positive integer (there are none) is odd, and there is no positive integer that is negative and prime. But it is false that some positive integer is both negative and either odd or not prime.

g. A symbolization of the first argument is

$$\frac{(\forall x)(Ax \supset Cx) \mathbin{\&} (\forall x)(Cx \supset Sx)}{(\forall x)(Ax \supset Sx)}$$

To see that the argument is quantificationally valid, assume that the premise is true on some interpretation. Then every member of the UD that is A is also C, and every member of the UD that is C is also S. So if a member of the UD is A, it is C and therefore S as well, which is what the conclusion says.

A symbolization of the second argument is

$$\frac{(\forall x)(Sx \supset Cx) \mathbin{\&} (\forall x)(Cx \supset Ax)}{(\forall x)(Ax \supset Sx)}$$

The premise is true and the conclusion false on the following interpretation:

UD: The set of positive integers
Ax: x is positive
Cx: x is greater than 1
Sx: x is even

Every even positive integer is greater than 1, and every positive integer that is greater than 1 is positive. But not every positive integer that is positive is even—some positive integers are odd.

**4.**a. The argument is quantificationally invalid. The premises are true and the conclusion false on the following interpretation:

UD: The set of positive integers
Dx: x is odd
Fx: x is greater than 10
Lx: x is greater than 9

Every odd positive integer that is greater than 9 is greater than 10; at least one odd positive integer is not greater than 10; but it is false that no positive integer is greater than 9.

c. The argument is quantificationally invalid. The premise is true and the conclusion false on the following interpretation:

UD: The set of positive integers
Hx: x is less than 0
Rx: x is less than $-1$
Sx: x is less than $-2$

There is at least one positive integer such that it is less than 0 if and only if it is less than both $-1$ and $-2$; every positive integer has this property. But there is no positive integer that is either less than 0 and less than $-1$ or less than 0 and less than $-2$.

**Section 8.5E**

**1.**a. Ca ⊃ Daa
c. Ba ∨ Faa
e. Ca ⊃ (Faa ⊃ Ba)
g. Ba ⊃ Ca
i. Ca ∨ (Daa ∨ Ca)

**2.** Remember that, in expanding a sentence containing the individual constant ‘g’, we must use that constant.

a. Dag & Dgg

c. [Aa & (Daa ∨ Dba)] ∨ [Ab & (Dab ∨ Dbb)]

e. [Ua ⊃ ((Daa ∨ Daa) ∨ (Dab ∨ Dba))]
& [Ub ⊃ ((Dba ∨ Dab) ∨ (Dbb ∨ Dbb))]

g. [Dag ⊃ ((~ Ua & Daa) ∨ (~ Ug & Dag))]
& [Dgg ⊃ ((~ Ua & Dga) ∨ (~ Ug & Dgg))]

i. ~ (Bg ∨ ((Dgg & Dga) ∨ (Dag & Daa)))

**3.**a. [(Ga ⊃ Naa) & (Gb ⊃ Nbb)] & (Gc ⊃ Ncc)

c. ((Na ≡ Ba) ∨ (Na ≡ Bb)) ∨ (Na ≡ Bc)

**4.** The truth-table for an expansion for the set {‘a’} is

| | | | | ↓ | |
|---|---|---|---|---|---|
| Fa | (Fa | & | ~ Fa) | ⊃ | ~ Fa |
| **T** | **T** | **F** | **F T** | **T** | **F T** |
| **F** | **F** | **F** | **T F** | **T** | **T F** |

This truth-table shows that the the sentence

((∃x)Fx & (∃y) ~ Fy) ⊃ (∀x) ~ Fx

is true on every interpretation with a one-member UD. The truth-table for an expansion for the set {‘a’, ‘b’} is

| | | | | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | [(Fa | ∨ | Fb) | & | (~ Fa | ∨ | ~ Fb)] | ⊃ | (~ Fa | & | ~ Fb) |
| **T** | **T** | **T** | **T** | **T** | **F** | **F T** | **F** | **F T** | **T** | **F T** | **F** | **F T** |
| **T** | **F** | **T** | **T** | **F** | **T** | **F T** | **T** | **T F** | **F** | **F T** | **F** | **T F** |
| **F** | **T** | **F** | **T** | **T** | **T** | **T F** | **T** | **F T** | **F** | **T F** | **F** | **F T** |
| **F** | **F** | **F** | **F** | **F** | **F** | **T F** | **T** | **T F** | **T** | **T F** | **T** | **T F** |

This truth-table shows that the sentence

((∃x)Fx & (∃y) ~ Fy) ⊃ (∀x) ~ Fx

is true on at least one interpretation with a two-member UD and false on at least one interpretation with a two-member UD.

**5.**a. One assignment to its atomic components for which the expansion

[Naa ∨ (Naa ∨ Nan)] & [Nnn ∨ (Nna ∨ Nnn)]

is true is

| | | | | | | | | | ↓ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Naa | Nan | Nna | Nnn | [Naa | ∨ | (Naa | ∨ | Nan)] | & | [Nnn | ∨ | (Nna | ∨ | Nnn)] |
| **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** |

Using this information, we shall construct an interpretation with a two-member UD such that the relation N holds between any two members of the UD:

UD: The set {1, 2}
N: {<1, 1>, <1, 2>, <2, 1>, <2, 2>}

c. There is only one assignment to its atomic components for which the expansion 'Saan & Snnn' is true.

| | | | ↓ | |
|---|---|---|---|---|
| Saan | Snnn | Saan | & | Snnn |
| **T** | **T** | **T** | **T** | **T** |

Using this information, we construct an interpretation with a two-member UD:

UD: The set {1, 2}
S: {<2, 2, 1>, <1, 1, 1>}
a: 2
n: 1

**6.**a.

| | | | | | ↓ | |
|---|---|---|---|---|---|---|
| Fa | Ga | (Fa | ⊃ | Ga) | ⊃ | Ga |
| **F** | **F** | **F** | **T** | **F** | **F** | **F** |

c.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Baa | Bab | Bba | Bbb | [(Baa | ∨ | Bab) | & | (Bba | ∨ | Bbb)] |
| **T** | **F** | **F** | **T** | **T** | **T** | **F** | **T** | **F** | **T** | **T** |

| ↓ | | | | | | | |
|---|---|---|---|---|---|---|---|
| ⊃ | [(Baa | & | Bba) | ∨ | (Bab | & | Bbb)] |
| **F** | **T** | **F** | **F** | **F** | **F** | **F** | **T** |

e.

| Fa Ga Fb Gb | [(Fa & Fb) ⊃ (Ga & Gb)] ⊃ [(Fa ⊃ Ga) & (Fb ⊃ Gb)] |
|---|---|
| T F F T | T F F T F F T F T F F F F T T |

g.

| Faa Ga | ~ Ga ⊃ (Faa ⊃ Ga) |
|---|---|
| T F | T F F T F F |

**7.**a.

| Baa Bab Bba Bbb | ~ [(Baa & Bab) & (Bba & Bbb)] ≡ (Baa & Bbb) |
|---|---|
| T F F T | T T F F F F F T T T T T |

c.

| Fa Fb Ga Gb | [(Fa ∨ Fb) & (Ga ∨ Gb)] |
|---|---|
| T F F T | T T F T F T T |

| & ~ [(Fa & Ga) ∨ (Fb & Gb)] |
|---|
| T T T F F F F F T |

e.

| Fa Ga | (Fa ⊃ Ga) & (Ga ⊃ ~ Fa) |
|---|---|
| F T | F T T T T T T F |

g.

| Ba Ha | (Ba ≡ Ha) ⊃ (Ba & Ha) |
|---|---|
| T T | T T T T T T T |

i. Sneaky. This one can't be done because, as pointed out in Section 8.2, the sentence is false on all interpretations with finite UDs.

**8.**a.

| Fa Fb Ga Gb | ((Fa & Ga) ∨ (Fb & Gb)) |
|---|---|
| T T F F | T F F F T F F |

| ⊃ (~ (Fa ∨ Ga) ∨ ~ (Fb ∨ Gb)) |
|---|
| T F T T F F F T T F |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Ga | Gb | ((Fa | & | Ga) | ∨ | (Fb | & | Gb)) |
| **T** | **F** | **T** | **T** | **T** | **T** | **T** | **T** | **F** | **F** | **T** |

| ↓ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ⊃ | (~ | (Fa | ∨ | Ga) | ∨ | ~ | (Fb | ∨ | Gb)) |
| **F** | **F** | **T** | **T** | **T** | **F** | **F** | **F** | **T** | **T** |

c.

| | | ↓ | | |
|---|---|---|---|---|
| Bnn | Bnn | ⊃ | ~ | Bnn |
| **F** | **F** | **T** | **T** | **F** |

| | | ↓ | | |
|---|---|---|---|---|
| Bnn | Bnn | ⊃ | ~ | Bnn |
| **T** | **T** | **F** | **F** | **T** |

e.

| | | | | ↓ | |
|---|---|---|---|---|---|
| Naa | (Naa | ∨ | Naa) | ⊃ | Naa |
| **T** | **T** | **T** | **T** | **T** | **T** |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Naa | Nab | Nba | Nbb | [[(Naa | ∨ | Naa) | ⊃ | Naa] |
| **T** | **T** | **T** | **F** | **T** | **T** | **T** | **T** | **T** |

| | | | | | | ↓ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| & | [(Nba | ∨ | Nab) | ⊃ | Nbb]] | & | [[(Nab | ∨ | Nba) | ⊃ | Naa] |
| **F** | **T** | **T** | **T** | **F** | **F** | **F** | **T** | **T** | **T** | **T** | **T** |

| | | | | | |
|---|---|---|---|---|---|
| & | [(Nbb | ∨ | Nbb) | ⊃ | Nbb]] |
| **T** | **F** | **F** | **F** | **T** | **F** |

g.

| | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|
| Ca | Da | (Ca | ∨ | Da) | ≡ | (Ca | & | Da) |
| **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** |

| | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|
| Ca | Da | (Ca | ∨ | Da) | ≡ | (Ca | & | Da) |
| **T** | **F** | **T** | **T** | **F** | **F** | **T** | **F** | **F** |

**10.** The expanded sentence 'Ga & ~ Ga' is a truth-functional compound. It is false on every truth-value assignment, so it is quantificationally false. But the fact that this sentence is quantificationally false only shows that '(∃y)Gy & (∃y) ~ Gy' is not true on any interpretation that has a one-member UD—for it is an expansion using only one constant. The sentence is in fact not quantificationally false, for it is true on some interpretations with larger universes of discourse. We may expand the sentence for the set {'a', 'b'} to show this:

| | | | | | | ↓ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ga | Gb | | (Ga | ∨ | Gb) | & | (~ | Ga | ∨ | ~ | Gb) |
| **T** | **F** | | **T** | **T** | **F** | **T** | **F** | **T** | **T** | **T** | **F** |

**11.**a.

| | | | | | | | ↓ | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Ga | | (Fa | ∨ | Fb) | ⊃ | Ga | (Fa | ⊃ | Ga) | ∨ | (Fb | ⊃ | Ga) |
| **T** | **F** | **F** | | **T** | **T** | **F** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **T** | **F** |

c.

| | | | | | | | | ↓ | | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Ga | Gb | | (Fa | & | Fb) | ∨ | (Ga | & | Gb) | (Fa | ∨ | Ga) | & | (Fb | ∨ | Gb) |
| **T** | **F** | **F** | **T** | | **T** | **F** | **F** | **F** | **F** | **F** | **T** | **T** | **T** | **F** | **T** | **F** | **T** | **T** |

e.

| | | | | | | | | ↓ | | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Ga | Gb | | (Fa | ≡ | Ga) | & | (Fb | ≡ | Gb) | (Fa | ∨ | Fb) | ≡ | (Ga | ∨ | Gb) |
| **T** | **F** | **F** | **T** | | **T** | **F** | **F** | **F** | **F** | **F** | **T** | **T** | **T** | **F** | **T** | **F** | **T** | **T** |

g.

| | | | | | | | | | | | | ↓ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ba | Bb | Daa | Dab | Dba | Dbb | | (Ba | & | (Daa | & | Dba)) | ∨ | (Bb | & | (Dab | & | Dbb)) |
| **F** | **F** | **T** | **T** | **T** | **T** | | **F** | **F** | **T** | **T** | **T** | **F** | **F** | **F** | **T** | **T** | **T** |

| | | | | | ↓ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| (Ba | ⊃ | (Daa | & | Dba)) | & | (Bb | ⊃ | (Dab | & | Dbb)) |
| **F** | **T** | **T** | **T** | **T** | **T** | **F** | **T** | **T** | **T** | **T** |

i.

| | | | | | | | | | | | | | | ↓ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Kaa | Kab | Kba | Kbb | | ((Fa | ⊃ | Kaa) | ∨ | (Fa | ⊃ | Kba)) | & | ((Fb | ⊃ | Kab) | ∨ | (Fb | ⊃ | Kbb)) |
| **T** | **T** | **F** | **T** | **F** | **T** | | **T** | **F** | **F** | **F** | **T** | **F** | **F** | **F** | **T** | **T** | **T** | **T** | **T** | **T** | **T** |

| | | | | | | | ↓ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ((Fa | ⊃ | Kaa) | ∨ | (Fa | ⊃ | Kba)) | ∨ | ((Fb | ⊃ | Kab) | ∨ | (Fb | ⊃ | Kbb)) |
| **T** | **F** | **F** | **F** | **T** | **F** | **F** | **T** | **T** | **T** | **T** | **T** | **T** | **T** | **T** |

**12**.a.

| | | | | | ↓ | | | ↓ | | ↓ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ba | Bb | Ca | Cb | Ba | ∨ | Bb | Ca | ∨ | Cb | ~ | [(Ba | ∨ | Ca) | & | (Bb | ∨ | Cb)] |
| **T** | **F** | **T** | **F** | **T** | **T** | **F** | **T** | **T** | **F** | **T** | **T** | **T** | **T** | **F** | **F** | **F** | **F** |

c.

| | | | | | ↓ | | | ↓ | | | ↓ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Ga | Ma | Na | Fa | ⊃ | Ga | Na | ⊃ | Ma | Ga | ⊃ | ~ | Ma |
| **F** | **F** | **F** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **F** | **T** | **T** | **F** |

e.

| | | | | ↓ | | | | | ↓ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Caa | Ma | Na | Na | ⊃ | (Ma | & | Caa) | Ma | ⊃ | ~ | Caa |
| **T** | **F** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **T** | **F** | **T** |

g.

| | | | | ↓ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ma | Mb | Na | Nb | ~ | [(Na | ⊃ | Ma) | & | (Nb | ⊃ | Mb)] |
| **F** | **T** | **T** | **T** | **T** | **T** | **F** | **F** | **F** | **T** | **T** | **T** |

| ↓ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ~ | [~ | (Na | ⊃ | Ma) | & | ~ | (Nb | ⊃ | Mb)] |
| **T** | **T** | **T** | **F** | **F** | **F** | **F** | **T** | **T** | **T** |

i.

| | | ↓ | ↓ | | | ↓ | |
|---|---|---|---|---|---|---|---|
| Faa | Gaa | Faa | ~ | Gaa | Faa | ∨ | Gaa |
| **T** | **F** | **T** | **T** | **F** | **T** | **T** | **F** |

**14**.a.

| | | | | | | ↓ | | | ↓ | | | | ↓ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Ga | Na | (Fa | ⊃ | Ga) | ⊃ | Na | Na | ⊃ | Ga | ~ | Fa | ∨ | Ga |
| **T** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **F** |

c.

| | | | | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fa | Fb | Ga | Gb | Ha | Hb | (Fa | & | Ga) | ∨ | (Fb | & | Gb) |
| **T** | **T** | **T** | **F** | **F** | **T** | **T** | **T** | **T** | **T** | **T** | **F** | **F** |

| | | | ↓ | | | | | | | ↓ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (Fa | & | Ha) | ∨ | (Fb | & | Hb) | (Ga | & | Ha) | ∨ | (Gb | & | Hb) |
| **T** | **F** | **F** | **T** | **T** | **T** | **T** | **T** | **F** | **F** | **F** | **F** | **F** | **T** |

e.

| | | | ↓ | | ↓ | | ↓ | |
|---|---|---|---|---|---|---|---|---|
| Fa | Ga | Fa | ⊃ | Ga | ~ | Fa | ~ | Ga |
| **F** | **T** | **F** | **T** | **T** | **T** | **F** | **F** | **T** |

g.

| Daa | Dab | Dba | Dbb | Ga | Gb | Ga | ∨ | Gb | (Ga | ⊃ | Daa) | & | (Gb | ⊃ | Dbb) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | ↓ | | | | | ↓ | | | |
| **T** | **F** | **F** | **T** | **F** | **T** | **F** | **T** | **T** | **F** | **T** | **T** | **T** | **T** | **T** | **T** |

| [(Ga | & | Daa) | & | (Ga | & | Dab)] | ∨ | [(Gb | & | Dba) | & | (Gb | & | Dbb)] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | ↓ | | | | | | | |
| **F** | **F** | **T** | **F** | **F** | **F** | **F** | **F** | **T** | **F** | **F** | **F** | **T** | **T** | **T** |

i.

| Fa | Ga | Ha | Fa | ⊃ | Ga | Ha | ⊃ | Ga | Fa | ∨ | Ha |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | ↓ | | | ↓ | | | ↓ | |
| **F** | **F** | **F** | **F** | **T** | **F** | **F** | **T** | **F** | **F** | **F** | **F** |

**Section 8.6E**

**1.**a. **F**
c. **T**
e. **T**
g. **F**
i. **F**

**2.**a. The sentence is false on the following interpretation:

UD: The set of positive integers

There is no positive integer that is identical to every positive integer.

c. The sentence is false on the following interpretation:

UD: The set {1, 2, 3}

It is not true that for any three members of the UD, at least two are identical.

e. The sentence is false on the following interpretation:

UD: The set {1}
Gxy: x is greater than y

It is not true that there is a pair of members of the UD such that either the members of the pair are not identical or one member is greater than the other. The only pair of members of the UD consists of 1 and 1.

**3.**a. Consider any interpretation and any members x, y, and z of its UD. If x and y are not the same member or if y and z are not the same member, then these members do not satisfy the condition specified by ‘(x = y & y = z)’, and so they do satisfy '$[(x = y \& y = z) \supset x = z]$'. On the other hand, if x and y are the same and y and z are the same, then x and z must be the same, satisfying the consequent '$x = z$'. In this case as well, then, x, y, and z satisfy '$[(x = y \& y = z) \supset x = z]$'. Therefore the universal claim is true on every interpretation.

c. Consider any interpretation and any members x and y of its UD. If x and y are not the same, they do not satisfy '$x = y$' and so do satisfy '$[x = y \supset (Gxy \equiv Gyx)]$'. If x and y are the same, and hence satisfy '$x = y$', they must satisfy '$(Gxy \equiv Gyx)$' as well—the pair consisting of the one object and itself is either in the extension or not. Therefore the universal claim must be true on every interpretation.

**4.**a. The first sentence is true and the second false on the following interpretation:

UD: The set of positive integers

Every positive integer is identical to at least one positive integer (itself), but not even one positive integer is identical to every positive integer.

c. The first sentence is false and the second is true on the following interpretation:

UD: The set of positive integers
a: 1
b: 1
c: 2
d: 3

**5.**a. The sentences are all true on the following interpretation:

UD: The set of positive integers
a: 1
b: 1
c: 1
d: 2

c. The sentences are all true on the following interpretation:

UD: The set of positive integers

The first sentence is true because there are at least two positive integers. The second sentence is true because for any positive integer x, we can find a pair of positive integers z and w such that either x is identical to z or x is identical to w—just let one of the pair be x itself.

**6.**a. The following interpretation shows that the entailment does not hold:

UD: The set {1, 2}

It is true that for any x, y, and z in the UD, at least two of x, y, and z must be identical. But it is not true that for any x and y in the UD, x and y must be identical.

c. The following interpretation shows that the entailment does not hold:

UD: The set {1, 2}
Gxy: x is greater than or equal to y

At least one member of the UD (the number 2) is greater than or equal to every member of the UD, and at least one member of the UD (the number 1) is not greater than or equal to any member of the UD other than itself. But no member of the UD is not greater than or equal to itself.

**7.**a. The argument can be symbolized as

$$\frac{(\forall x)[Mx \supset (\exists y)(\sim y = x \,\&\, Lxy)] \;\&\; (\forall x)[Mx \supset (\forall y)(Pxy \supset Lxy)]}{(\forall x)(Mx \supset \sim Pxx)}$$

The argument is quantificationally invalid, as the following interpretation shows:

UD: The set of positive integers
Mx: x is odd
Lxy: x is less than or equal to y
Pxy: x squared equals y

For every odd positive integer, there is at least one other positive integer that it is less than or equal to, and every odd positive integer is such that it is less than or equal to its square . However, the conclusion, which says that no odd positive integer is its own square, is false because the square of 1 is 1.

c. The argument can be symbolized as

$$\frac{(\forall x)[(Fx \,\&\, (\forall y)(Pxy \;\; Lxy)) \;\; Lxx]}{(\forall x)\; [Fx \supset (\exists y)(\exists z)((Lxy \,\&\, Lxz) \;\&\; \sim y = z)]}$$

The argument is quantificationally invalid, as the following interpretation shows:

UD: The set of positive integers
Fx: x is odd
Lxy: x is greater than y
Pxy: x is less than y

Trivially, every odd positive integer that is both less than and greater than some positive integer (there are none) is less than itself. But not all odd positive integers are greater than at least two positive integers—the integer 1 is not.

e. The argument may be symbolized as

$$(\forall x) \sim (\exists y)(\exists z)(\exists w)([[Pyz \;\&\; Pzx) \;\&\; Pwx]$$
$$\&\; [(\sim y = z \;\&\; \sim z = w) \;\&\; \sim w = y]]$$
$$\&\; (\forall x_1)[Px_1x \supset ((x_1 = y \vee x_1 = z) \vee x_1 = w)])$$
$$(\forall x)(\exists y)(\exists z)[(Pyx \;\&\; Pzx) \;\&\; \sim y = z)]$$
$$\overline{(\forall x)(\exists y)(\exists z)[((Pyx \;\&\; Pzx) \;\&\; \sim y = z) \;\&\; (\forall w)(Pwx \supset (w = y \vee w = z))]}$$

The argument is quantificationally invalid, as the following interpretation shows:

UD: The set of positive integers
Pxy: x is greater than y

No positive integer is less than exactly three positive integers (for any positive integer, there are infinitely many positive integers that are greater). Every positive integer is less than at least two positive integers. But no positive integer is less than exactly two positive integers.

**8.** a.

| a = a | ~ a = a |
|---|---|
| **T** | **F T** |

| a = a | a = b | b = a | b = b | (~ a = a ∨ ~ b = a) ∨ (~ a = b ∨ ~ b = b) |
|---|---|---|---|---|
| **T** | **F** | **F** | **T** | **F T  T T F  T  T F  T F T** |

c.

| a = a | Gaa | (Gaa ∨ Gaa) ∨ a = a | Gaa |
|---|---|---|---|
| **T** | **F** | **F  F F  T T** | **F** |

e.

| a = a | a = b | b = a | b = b | a = a & b = b | (~ a = a ∨ ~ a = b) ∨ (~ b = a ∨ ~ b = b) |
|---|---|---|---|---|---|
| **T** | **F** | **F** | **T** | **T  T T** | **F T  T T F  T  T F  T F T** |

| (~ a = a & ~ a = b) ∨ (~ b = a & ~ b = b) |
|---|
| **F T  F T F  F  T F  F F T** |

**9.**a. True. Every positive integer is less than its successor.

c. True. For any positive integer x, there is a positive integer that equals 2x.

e. False. The sum of any even integer and any odd integer is odd, not even.

g. True. For any positive integer x there is a positive integer z that satisfies the first disjunct, namely, x squared plus z is even.

**10.**a. The sentence is false on the following interpretation:

UD: The set of positive integers
Px: x is odd
$f$(x): the successor of x

It is false that a positive integer with an odd successor is itself odd.

c. The sentence is false on the following interpretation:

UD: The set of positive integers
$g$(x): the successor of x

There is no positive integer that is the successor of every positive integer.

e. The sentence is false on the following interpretation:

UD: The set of positive integers
$f$(x): x squared

Since $1 = 1^2$, not all positive integers fail to be equal to their squares.

**11.**a. The sentence is true on an interpretation if and only if every member x of the UD satisfies '(∃y) y = $f$($f$(x))', and that is the case if and only if for every member x of the UD, there is a member y such that y is identical to $f$($f$(x)). Since $f$ is a function that is defined for every member of the UD, there must be a member that is identical to $f$(x), and hence there must also be a member that is identical to $f$($f$(x)). Hence the sentence is true on every interpretation.

c. Assume that the antecedent is true on some interpretation. By the first conjunct, it must be the case that every member x of the UD stands in the relation H to $f$(x), and also that every member $f$(x) stands in the relation H to $f$($f$(x)). By the second conjunct it follows that every member x of the UD therefore stands in the relation H to $f$($f$(x)). The consequent must therefore be true as well. Since the consequent is true on every interpretation on which the antecedent is true, the sentence is quantificationally true.

**12.**a. The first sentence is true and the second false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| Lxyz: | x plus y equals z |
| $f$(x): | the successor of x |
| a: | 1 |
| b: | 2 |

The sum of 1 and 2 is 3, the successor of 2; but the sum of 1 and 3 is not 2.

c. The first sentence is true and the second false on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| $f$(x): | x squared |
| $g$(x): | the successor of x |

For any positive integer x, there is a positive integer that is equal to the square of the successor of x; but there is no positive integer that is equal to its own successor squared.

**13.**a. The members of the set are all true on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| $f$(x): | x squared |
| a: | 1 |
| b: | 1 |
| c: | 1 |

The integer 1 equals itself squared, which is what each of the three sentences in the set say on this interpretation.

c. The members of the set are all true on the following interpretation:

| | |
|---|---|
| UD: | The set of positive integers |
| $f$(x): | the smallest odd integer that is less than or equal to x |

There is a positive integer, namely 1, that is the smallest odd integer less than or equal to any positive integer, and there is at least one positive integer, for example 2, that fails to be the smallest odd integer less than or equal to even one positive integer.

**14.**a. The argument is quantificationally invalid, as the following interpretation shows:

| | |
|---|---|
| UD: | The set of positive integers |
| Fx: | x is odd |
| $g$(x): | the successor of x |

The premise, which says that every positive integer is such that either it or its successor is odd, is true on this interpretation. The conclusion, which says that every positive integer is such that either it or the successor of its successor is odd, is false—no even positive integer satisfies this condition.

c. The argument is quantificationally invalid, as the following interpretation shows:

| | |
|---|---|
| UD: | The set of positive integers |
| Lxyz: | x plus y equals z |
| $f$(x): | the successor of x |

The premise is true on this interpretation: every positive integer is such that its successor plus some positive integer equals a positive integer. The conclusion is false: there is no positive integer such that the sum of x and any integer's successor equals any integer's successor.

e. The argument is quantificationally valid. If the premise is true on an intepretation, then every member x of the UD that is a value of the function $g$ and that is B is such that nothing stands in the relation H to x. If the antecedent of the conclusion is true, then a is a value of the function $g$ (for the argument b), and is such that something stands in the relation H to a. It follows from the premise that the consequent of the conclusion must be true as well, i.e., a cannot be B. So the conclusion is true on any interpretation on which the premise is true.

**15.**a.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | ↓ | | ↓ |
| a = $g$(a) | Fa | F$g$(a) | Fa | ∨ | F$g$(a) | a = $g$(a) |
| **T** | **T** | **T** | **T** | **T** | **T** | **T** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | ↓ | | ↓ |
| a = $g$(a) | Fa | F$g$(a) | Fa | ∨ | F$g$(a) | a = $g$(a) |
| **T** | **F** | **F** | **F** | **F** | **F** | **T** |

c.

| $a = f(a)$ | $a = f(b)$ | $a = f(f(a))$ | $a = f(f(b))$ | $b = f(a)$ | $b = f(b)$ | $b = f(f(a))$ |
|---|---|---|---|---|---|---|
| **F** | **T** | **T** | **F** | **T** | **F** | **F** |

| | | | ↓ | | | | ↓ | |
|---|---|---|---|---|---|---|---|---|
| $b = f(f(b))$ | $\sim$ | $a = f(a)$ | $\&$ | $\sim$ | $b = f(b)$ | $a = f(a)$ | $\vee$ | $b = f(a)$ |
| **T** | **T** | **F** | **T** | **T** | **F** | **F** | **T** | **T** |

| | ↓ | | | ↓ | | | ↓ | |
|---|---|---|---|---|---|---|---|---|
| $a = f(b)$ | $\vee$ | $b = f(b)$ | $a = f(f(a))$ | $\vee$ | $b = f(f(a))$ | $a = f(f(b))$ | $\vee$ | $b = f(f(b))$ |
| **T** | **T** | **F** | **T** | **T** | **F** | **F** | **T** | **T** |