Section 11.1E

5. Let $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ be a quantificationally consistent set of sentences, none of which contains the constant **a**. Then there is at least one interpretation **I** on which every member of $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is true. Because $(\exists \mathbf{x})\mathbf{P}$ is true on **I**, we know that for any variable assignment $\mathbf{d_I}$, there is a member **u** of the UD such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **P** on **I**. Let **I′** be the interpretation that is just like **I** except that $\mathbf{I'}(\mathbf{a}) = \mathbf{u}$. Because **a** does not occur in $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$, it follows from 11.1.7 that every member of $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is true on **I′**.

On our assumption that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **P** on **I**, it follows from 11.1.6 that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **P** on **I′**. By the way that we have constructed **I′**, **u** is $\mathbf{I'}(\mathbf{a})$, and so $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ is $\mathbf{d_I}[\mathbf{I'}(\mathbf{a})/\mathbf{x}]$. From result 11.1.1, we therefore know that $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on **I′**. By 11.1.3, then, every variable assignment on **I′** satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$, and so it is true on **I′**.

Because evey member of $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is true on **I′**, we conclude that the extended set is quantificationally consistent.

6. Assume that **I** is an interpretation on which each member of the UD is assigned to at least one individual constant and that every substitution instance of (∀**x** Let $\mathbf{d_I}$ be an arbitrary variable assignment. $\mathbf{d_I}$ satisfies **(∀x)Px** if and only) **P** is true on **I**. $(\forall \mathbf{x})\mathbf{P}$ is true on **I** if and only if every variable assignment satisfies $(\forall \mathbf{x})\mathbf{P}$. Let $\mathbf{d_I}$ be an ar if for every member **u** of the UD, $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **P**. By our assumption, every member **u** of the UD is such that $\mathbf{u} = \mathbf{I}(\mathbf{a})$ for some individual constant **a**. Also by assumption, every substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is true on **I**—so $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$. By 11.1.1, then, for every member **u** of the UD and constant **a** that designates that member, $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$, which is $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$, satisfies **P**. We conclude that for every member **u** of the UD, $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **P**, that $\mathbf{d_I}$ therefore satisfies $(\forall \mathbf{x})\mathbf{P}$, and that $(\forall \mathbf{x})\mathbf{P}$ is true on **I**.

Section 11.2E

4. To prove 11.2.5, we will make use of the following:

11.2.6. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be closed terms such that $\mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1) = \mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2)$, and let **t** be a term that contains $\mathbf{t}_1$. Then for any variable assignment $\mathbf{d_I}$, and any term $\mathbf{t}(\mathbf{t}_2//\mathbf{t}_1)$ that results from replacing one or more occurrences of $\mathbf{t}_1$ in **t** with $\mathbf{t}_2$, $\mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}(\mathbf{t}_2//\mathbf{t}_1)) = \mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t})$.

Proof. If $\mathbf{t}_1$ is **t**, then $\mathbf{t}(\mathbf{t}_2//\mathbf{t}_1)$ must be $\mathbf{t}_2$, and by assumption $\mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1) = \mathrm{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2)$.

For the case where **t** contains but is not identical to $\mathbf{t}_1$, we shall prove 11.2.6 by mathematical induction on the number of functors that occur in **t**—since **t** must be a complex term in this case.

Basis clause: If $\mathbf{t}$ contains one functor, then for any variable assignment $\mathbf{d_I}$, and any term $\mathbf{t}(\mathbf{t}_2//\mathbf{t}_1)$ that results from replacing one or more occurrences of $\mathbf{t}_1$ in $\mathbf{t}$ with $\mathbf{t}_2$, $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}(\mathbf{t}_2//\mathbf{t}_1)) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t})$.

Proof of basis clause: $\mathbf{t}$ has the form $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')$, where each $\mathbf{t}_\mathbf{i}'$ is a variable or constant. In this case, one or more of the $\mathbf{t}_\mathbf{i}$'s must be $\mathbf{t}_1$ and has been replaced by $\mathbf{t}_2$ to form $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1)$ and the remaining $\mathbf{t}_\mathbf{i}$'s are unchanged. By assumption, $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2)$. So the denotations of the arguments at the corresponding positions in $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')$ and $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1)$ are identical, and therefore $\text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')) = \text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1))$.

Inductive step: If 11.2.6 holds for every term $\mathbf{t}$ that contains $\mathbf{k}$ or fewer functors, then it also holds for every term $\mathbf{t}$ that contains $\mathbf{k} + 1$ functors.

Proof of inductive step: Assume the inductive hypothesis for an arbitrary integer $\mathbf{k}$. We must show that 11.2.6 holds for every term $\mathbf{t}$ that contains $\mathbf{k} + 1$ functors. In this case, $\mathbf{t}$ has the form $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')$, where each $\mathbf{t}_\mathbf{i}'$ contains $\mathbf{k}$ or fewer functors and one or more of the $\mathbf{t}_\mathbf{i}$'s that is identical to or contains $\mathbf{t}_1$ has had one or more occurrences of $\mathbf{t}_1$ replaced by $\mathbf{t}_2$ to form $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1)$ and the remaining $\mathbf{t}_\mathbf{i}$'s are unchanged. In the cases where a replacement has occurred, it follows from the inductive hypothesis that the denotations of the arguments at the corresponding positions in $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')$ and $f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1)$ are identical, and therefore $\text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')) = \text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_\mathbf{n}')(\mathbf{t}_2//\mathbf{t}_1))$.

We can now use 11.2.6 in the

Proof of 11.2.5: We shall prove only the first half of 11.2.5, since the second half is proved in the same way with minor modifications. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be closed terms and let $\mathbf{P}$ be a sentence that contains $\mathbf{t}_1$. If $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\}$ is quantificationally inconsistent, then trivially, $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vDash \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$.

If $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\}$ is quantificationally consistent, then let $\mathbf{I}$ be an interpretation on which both $\mathbf{t}_1 = \mathbf{t}_2$ and $\mathbf{P}$ are true and therefore satisfied by every satisfaction assignment $\mathbf{d_I}$. We will show by mathematical induction on the number of occurrences of logical operators in a formula $\mathbf{P}$ that if $\mathbf{t}_1 = \mathbf{t}_2$ is satisfied by a satisfaction assignment $\mathbf{d_I}$ on an interpretation $\mathbf{I}$, then $\mathbf{P}$ is satisfied by $\mathbf{d_I}$ if and only if $\mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$ is satisfied by $\mathbf{d}$.

Basis clause: If $\mathbf{P}$ contains zero occurrences of logical operators and $\mathbf{t}_1 = \mathbf{t}_2$ is satisfied by a satisfaction assignment $\mathbf{d_I}$ on an interpretation $\mathbf{I}$ then $\mathbf{P}$ is satisfied by $\mathbf{d_I}$ if and only if $\mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$ is satisfied by $\mathbf{d}$ on $\mathbf{I}$.

Proof of basis clause: Since $\mathbf{P}$ contains $\mathbf{t}_1$, $\mathbf{P}$ must be either a formula of the form $\mathbf{A}\mathbf{t}_1' \ldots \mathbf{t}_\mathbf{n}'$ or a formula of the form $\mathbf{t}_1 = \mathbf{t}_2$.

If $\mathbf{P}$ has the form $\mathbf{At_1' \ldots t_n'}$ then $\mathbf{P(t_2//t_1)}$ is $\mathbf{At_1'' \ldots t_n''}$, where each $\mathbf{t_i''}$ is either $\mathbf{t_i'}$ or the result of replacing $\mathbf{t_1}$ in $\mathbf{t_i'}$ with $\mathbf{t_2}$. In the former case, $den_{\mathbf{I,d_I}}(\mathbf{t_i'}) = den_{\mathbf{I,d_I}}(\mathbf{t_i''})$ since $\mathbf{t_i''}$ is $\mathbf{t_i'}$. In the latter case, $den_{\mathbf{I,d_I}}(\mathbf{t_i'}) = den_{\mathbf{I,d_I}}(\mathbf{t_i''})$ by 11.2.6. So $<den_{\mathbf{I,d_I}}(\mathbf{t_1'}), den_{\mathbf{I,d_I}}(\mathbf{t_2'}), \ldots, den_{\mathbf{I,d_I}}(\mathbf{t_n'})> = <den_{\mathbf{I,d_I}}(\mathbf{t_1''}), den_{\mathbf{I,d_I}}(\mathbf{t_2''}), \ldots, den_{\mathbf{I,d_I}}(\mathbf{t_n''})>$ and therefore $<den_{\mathbf{I,d_I}}(\mathbf{t_1'}), den_{\mathbf{I,d_I}}(\mathbf{t_2'}), \ldots, den_{\mathbf{I,d_I}}(\mathbf{t_n'})> \in \mathbf{I(A)}$ if and only if $<den_{\mathbf{I,d_I}}(\mathbf{t_1''}), den_{\mathbf{I,d_I}}(\mathbf{t_2''}), \ldots, den_{\mathbf{I,d_I}}(\mathbf{t_n''})> \in \mathbf{I(A)}$. Consequently, $\mathbf{d_I}$ satisfies $\mathbf{At_1' \ldots t_n'}$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{At_1'' \ldots t_n''}$.

If $\mathbf{P}$ has the form $\mathbf{t_1' = t_2'}$ then $\mathbf{P(t_2//t_1)}$ is $\mathbf{t_1'' = t_2''}$, where each $\mathbf{t_i''}$ is either $\mathbf{t_i'}$ or the result of replacing $\mathbf{t_1}$ in $\mathbf{t_i'}$ with $\mathbf{t_2}$. In the former case, $den_{\mathbf{I,d_I}}(\mathbf{t_i'}) = den_{\mathbf{I,d_I}}(\mathbf{t_i''})$ since $\mathbf{t_i''}$ is $\mathbf{t_i'}$. In the latter case, $den_{\mathbf{I,d_I}}(\mathbf{t_i'}) = den_{\mathbf{I,d_I}}(\mathbf{t_i''})$ by 11.2.6. It follows that $den_{\mathbf{I,d_I}}(\mathbf{t_1'}) = den_{\mathbf{I,d_I}}(\mathbf{t_2'})$ if and only if $den_{\mathbf{I,d}}(\mathbf{t_1''}) = den_{\mathbf{I,d}}(\mathbf{t_2''})$. Since $\mathbf{d_I}$ satisfies $\mathbf{t_1' = t_2'}$ if and only if $den_{\mathbf{I,d_I}}(\mathbf{t_1'}) = den_{\mathbf{I,d_I}}(\mathbf{t_2'})$ and $\mathbf{d_I}$ satisfies $\mathbf{t_1'' = t_2''}$ if and only if $den_{\mathbf{I,d_I}}(\mathbf{t_1''}) = den_{\mathbf{I,d_I}}(\mathbf{t_2''})$, it follows that $\mathbf{d_I}$ satisfies $\mathbf{t_1' = t_2'}$ if and only if it satisfies $\mathbf{t_1'' = t_2''}$.

Inductive step: If 11.2.5 is true of every formula $\mathbf{P}$ that contains $\mathbf{k}$ or fewer occurrences of logical operators, then 11.2.5 is also true of every formula $\mathbf{P}$ that contains $\mathbf{k} + 1$ occurrences of logical operators.

Proof of inductive step: Assume that the inductive hypothesis holds for an arbitrary integer $\mathbf{k}$. Let $\mathbf{P}$ be a formula that contains $\mathbf{k} + 1$ logical operators. We shall show that if $\mathbf{t_1 = t_2}$ is satisfied by a satisfaction assignment $\mathbf{d_I}$ on an interpretation $\mathbf{I}$, then $\mathbf{P}$ is satisfied by $\mathbf{d_I}$ if and only if $\mathbf{P(t_2//t_1)}$ is also satisfied by $\mathbf{d_I}$, by considering each form that $\mathbf{P}$ might have.

Case 1. $\mathbf{P}$ is a formula of the form $\sim \mathbf{Q}$. Then $\mathbf{P}$ is satisfied by $\mathbf{d_I}$ if and only if $\mathbf{Q}$ is not satisfied by $\mathbf{d_I}$. Since $\mathbf{Q}$ contains $\mathbf{k}$ logical operators, it follows by the inductive hypothesis that $\mathbf{Q}$ is not satisfied by $\mathbf{d_I}$ if and only if $\mathbf{Q(t_2//t_1)}$ is not satisfied by $\mathbf{d_I}$, and this is the case if and only if $\sim \mathbf{Q(t_2//t_1)}$, which is $\mathbf{P(t_2//t_1)}$, is satisfied by $\mathbf{d_I}$.

Cases 2–5. $\mathbf{P}$ has one of the forms $(\mathbf{Q} \;\&\; \mathbf{R})$, $(\mathbf{Q} \vee \mathbf{R})$, $(\mathbf{Q} \supset \mathbf{R})$, or $(\mathbf{Q} \equiv \mathbf{R})$. Similar to case 1.

Case 6. $\mathbf{P}$ has the form $(\forall \mathbf{x})\mathbf{Q}$. Then $\mathbf{P}$ is satisfied by $\mathbf{d}$ if and only if every variable assignment $\mathbf{d_I}'$ that is like $\mathbf{d_I}$ except possibly in the value assigned to $\mathbf{x}$ satisfies $\mathbf{Q}$. Since $\mathbf{t_1}$ and $\mathbf{t_2}$ are closed terms, every such variable assignment $\mathbf{d_I}'$ will satisfy $\mathbf{t_1 = t_2}$ since $den_{\mathbf{I,d_I}}(\mathbf{t_1}) = den_{\mathbf{I,d_I'}}(\mathbf{t_1})$ and $den_{\mathbf{I,d_I}}(\mathbf{t_2}) = den_{\mathbf{I,d_I'}}(\mathbf{t_2})$ by 11.2.2. Because $\mathbf{Q}$ contains $\mathbf{k}$ occurrences of logical operators, it follows by the inductive hypothesis that every such variable assignment $\mathbf{d_I}'$ will satisfy $\mathbf{Q}$ if and only if it also satisfies $\mathbf{Q(t_2//t_1)}$, and every such variable assignment $\mathbf{d_I}'$ will satisfy $\mathbf{Q(t_2//t_1)}$ if and only if $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q(t_2//t_1)}$, which is $\mathbf{P(t_2//t_1)}$ ($\mathbf{t_1}$, being a closed term, is not the variable $\mathbf{x}$).

Case 7. $\mathbf{P}$ has the form $(\exists \mathbf{x})\mathbf{Q}$. Similar to case 6.

Section 11.3E

1.a. Assume that an argument of *PL* is valid in *PD*. Then the conclusion is derivable in *PD* from the set consisting of the premises. By Metatheorem 11.3.1, it follows that the conclusion is quantificationally entailed by the set consisting of the premises. Therefore the argument is quantificationally valid.

b. Assume that a sentence **P** is a theorem in *PD*. Then $\varnothing \vdash \mathbf{P}$. So $\varnothing \vDash \mathbf{P}$, by Metatheorem 11.3.1, and **P** is quantificationally true.

2. Our induction will be on the number of occurrences of logical operators in **P**.

Basis clause: Thesis 11.3.4 holds for every atomic formula of *PL*.

> **Proof:** Assume that **P** is an atomic formula and that **Q** is a subformula of **P**. Then **P** and **Q** are identical. For any formula $\mathbf{Q}_1$, then, $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is simply $\mathbf{Q}_1$. It is trivial that the thesis holds in this case.

Inductive step: Let **P** be a formula with $\mathbf{k} + 1$ occurrences of logical operators, let **Q** be a subformula of **P**, and let $\mathbf{Q}_1$ be a formula related to **Q** as stipulated. Assume (the inductive hypothesis) that 11.3.4 holds for every formula with **k** or fewer occurrences of logical operators. We now establish that 11.3.4 holds for **P** as well. Suppose first that **Q** and **P** are identical. In this case, that 11.3.4 holds for **P** and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is established as in the proof of the basis clause. So assume that **Q** is a subformula of **P** that is not identical with **P** (in which case we say that **Q** is a *proper subformula* of **P**). We consider each form that **P** may have.

(i) **P** is of the form $\sim \mathbf{R}$. Since **Q** is a proper subformula of **P**, **Q** is a subformula of **R**. Therefore $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is $\sim [\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$. Since **R** has fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that, on any interpretation, a variable assignment satisfies **R** if and only if it satisfies $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$, and therefore that on any interpretation a variable assignment satisfies $\sim \mathbf{R}$ if and only if it satisfies $\sim [\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$.

(ii)–(v) **P** is of the form $\mathbf{R} \,\&\, \mathbf{S}$, $\mathbf{R} \vee \mathbf{S}$, $\mathbf{R} \supset \mathbf{S}$, or $\mathbf{R} \equiv \mathbf{S}$. These cases are handled similarly to case (ii) in the inductive proof of Lemma 6.1 (in Chapter 6), with obvious adjustments as in case (i).

(vi) **P** is of the form $(\forall \mathbf{x})\mathbf{R}$. Since **Q** is a proper subformula of **P**, **Q** is a subformula of **R**. Therefore $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is $(\forall \mathbf{x})[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$. Since **R** has fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows, by the inductive hypothesis, that on any interpretation a variable assignment satisfies **R** if and only if that assignment satisfies $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$. Now $(\forall \mathbf{x})\mathbf{R}$ is satisfied by a variable assignment $\mathbf{d_I}$ if and only if for each member **u** of the UD, $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies **R**. The latter is the case just in case $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ is satisfied by every variant $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$. And this is the case if and only if $(\forall \mathbf{x})[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ is satisfied by $\mathbf{d_I}$. Therefore on any interpretation $(\forall \mathbf{x})[\mathbf{R}$ is satisfied by a variable assignment if and only if $(\forall \mathbf{x})[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ is satisfied by that assignment.

(vii) **P** is of the form $(\exists \mathbf{x})\mathbf{R}$. This case is similar to case (vi).

3. $\mathbf{Q_{k+1}}$ is justified at position $\mathbf{k} + 1$ by Quantifier Negation. Then $\mathbf{Q_{k+1}}$ is derived as follows:

h	**S**	
k + 1	$\mathbf{Q_{k+1}}$	**h** QN

where some component **R** of **S** has been replaced by a component $\mathbf{R_1}$ to obtain $\mathbf{Q_{k+1}}$ and the four forms that **R** and $\mathbf{R_1}$ may have are

R:	$\mathbf{R_1}$:
$\sim (\forall \mathbf{x})\mathbf{P}$	$(\exists \mathbf{x}) \sim \mathbf{P}$
$(\exists \mathbf{x}) \sim \mathbf{P}$	$\sim (\forall \mathbf{x})\mathbf{P}$
$\sim (\exists \mathbf{x})\mathbf{P}$	$(\forall \mathbf{x}) \sim \mathbf{P}$
$(\forall \mathbf{x}) \sim \mathbf{P}$	$\sim (\exists \mathbf{x})\mathbf{P}$

Whichever pair **R** and $\mathbf{R_1}$ constitute, the two sentences contain exactly the same nonlogical constants. We first establish that on any interpretation variable assignment $\mathbf{d_I}$ satisfies **R** if and only if $\mathbf{d_I}$ satisfies $\mathbf{R_1}$.

(case i) Either **R** is $\sim (\forall \mathbf{x})\mathbf{P}$ and $\mathbf{R_1}$ is $(\exists \mathbf{x}) \sim \mathbf{P}$, or **R** is $(\exists \mathbf{x}) \sim \mathbf{P}$ and $\mathbf{R_1}$ is $\sim (\forall \mathbf{x})\ \mathbf{P}$. Assume that a variable assignment $\mathbf{d_I}$ satisfies $\sim (\forall \mathbf{x})\mathbf{P}$. Then $\mathbf{d_I}$ does not satisfy $(\forall \mathbf{x})\mathbf{P}$. There is then at least one variant $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ that does not satisfy **P**. Hence $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\sim \mathbf{P}$. It follows that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $(\exists \mathbf{x}) \sim \mathbf{P}$. Now assume that a variable assignment $\mathbf{d_I}$ satisfies $(\exists \mathbf{x}) \sim \mathbf{P}$. Then some variant $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\sim \mathbf{P}$. This variant does not satisfy **P**. Therefore $\mathbf{d_I}$ does not satisfy $(\forall \mathbf{x})\mathbf{P}$ and does satisfy $\sim (\forall \mathbf{x})\mathbf{P}$.

(case ii) Either **R** is $\sim (\exists \mathbf{x})\mathbf{P}$ and $\mathbf{R_1}$ is $(\forall \mathbf{x}) \sim \mathbf{P}$, or **R** is $(\forall \mathbf{x}) \sim \mathbf{P}$ and $\mathbf{R_1}$ is $\sim (\exists \mathbf{x})\mathbf{P}$. This case is similar to case (i).

R and $\mathbf{R_1}$ contain the same nonlogical symbols and variables, so it follows, by 11.3.4 (Exercise 2), that **S** is satisfied by a variable assignment if and only if $\mathbf{Q_{k+1}}$ is satisfied by that assignment. So on any interpretation **S** and $\mathbf{Q_{k+1}}$ have the same truth-value.

By the inductive hypothesis, $\Gamma_\mathbf{k} \vDash \mathbf{S}$. But $\Gamma_\mathbf{k}$ is a subset of $\Gamma_{\mathbf{k}+1}$, and so $\Gamma_{\mathbf{k}+1} \vDash \mathbf{S}$, by 11.3.2. Since **S** and $\mathbf{Q_{k+1}}$ have the same truth-value on any interpretation, it follows that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q_{k+1}}$.

Section 11.4E

2. Assume that $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *PD*. Then there is a derivation of the following sort, where $\mathbf{Q_1}, \ldots, \mathbf{Q_n}$ are members of Γ:

1	$\mathbf{Q_1}$	Assumption
.	.	
n	$\mathbf{Q_n}$	Assumption
n + 1	$\sim \mathbf{P}$	Assumption
.	.	
m	**S**	
.	.	
p	$\sim \mathbf{S}$	

We construct a new derivation as follows:

1	$\mathbf{Q}_1$		Assumption
.	.		
n	$\mathbf{Q}_n$		Assumption
n + 1		~ **P**	Assumption
.		.	A / ~ **E**
m		**S**	
.		.	
p		~ **S**	
p + 1	**P**		**n** + 1 − **p** ~ **E**

where lines 1 to **p** are as in the original derivation, except that ~ **P** is now an auxiliary assumption. This shows that Γ ⊨ **P**.

3.a. Assume that an argument of *PL* is quantificationally valid. Then the set consisting of the premises quantificationally entails the conclusion. By Metatheorem 11.4.1, the conclusion is derivable from that set in *PD*. Therefore the argument is valid in *PD*.

b. Assume that a sentence **P** is quantificationally true. Then ∅ ⊨ **P**. By Metatheorem 11.4.1, ∅ ⊨ **P**. So **P** is a theorem in *PD*.

4. We shall associate a numeral with each symbol of *PL* as follows. With each symbol of *PL* that is a symbol of *SL*, associate the two-digit numeral that is associated with that symbol in the enumeration of Section 6.4. With the symbol ′ (the prime), associate the numeral ‘66’. With the nonsubscripted lowercase letters ‘a’, ‘b’, . . . , ‘z’, associate the numerals ‘67’, ‘68’, . . . , ‘92’, respectively. With the symbols ‘∀’ and ‘∃’ associate the numerals ‘93’ and ‘94’, respectively. (Note that the numerals ‘66’ to ‘94’ are not associated with any symbol of *SL*.) We then associate with each sentence of *PL* the numeral that consists of the associated numerals of each of the symbols that occur in the sentence, in the order in which the symbols occur. We now enumerate the sentences of *PL* in the order of the integers designated by their associated numerals.

5. Assume that Γ ⊢ **P**. Then there is a derivation

1	$\mathbf{Q}_1$
.	.
n	$\mathbf{Q}_n$
.	.
m	**P**

where $\mathbf{Q}_1$, . . . , $\mathbf{Q}_n$ are all members of Γ. The primary assumptions are all members of any superset Γ′ ⊢ of Γ, and so Γ′ ⊢ **P** as well.

6.a. Assume that **a** does not occur in any member of the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ and that the set is consistent in *PD*. Assume, contrary to what we want to prove, that $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is *in*consistent in *PD*. Then there is a derivation of the sort

1	$\mathbf{Q}_1$
.	.
$\mathbf{n}$	$\mathbf{Q}_\mathbf{n}$
$\mathbf{n} + 1$	$(\exists \mathbf{x})\mathbf{P}$
$\mathbf{n} + 2$	$\mathbf{P}(\mathbf{a}/\mathbf{x})$
$\mathbf{m}$	$\mathbf{R}$
.	.
$\mathbf{p}$	$\sim \mathbf{R}$

where $\mathbf{Q}_1, \ldots, \mathbf{Q}_\mathbf{n}$ are all members of Γ. We may convert this into a derivation showing that $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is inconsistent in *PD*, contradicting our initial assumption:

1	$\mathbf{Q}_1$	
.	.	
$\mathbf{n}$	$\mathbf{Q}_\mathbf{n}$	
$\mathbf{n} + 1$	$(\exists \mathbf{x})\mathbf{P}$	
$\mathbf{n} + 2$		$\mathbf{P}(\mathbf{a}/\mathbf{x})$	
$\mathbf{n} + 3$			$(\exists \mathbf{x})\mathbf{P}$	
.			.	
$\mathbf{m} + 1$			$\mathbf{R}$	
.			.	
$\mathbf{p} + 1$			$\sim \mathbf{R}$	
$\mathbf{p} + 2$		$\sim (\exists \mathbf{x})\mathbf{P}$	$\mathbf{n} + 3 - \mathbf{p} + 1$ $\sim$I
$\mathbf{p} + 3$	$\sim (\exists \mathbf{x})\mathbf{P}$	$\mathbf{n} + 2 - \mathbf{p} + 2$ $\exists$E
$\mathbf{p} + 4$	$(\exists \mathbf{x})\mathbf{P}$	$\mathbf{n} + 1$ R

(Note that use of $\exists$E is legitimate at line $\mathbf{p} + 3$ because **a**, by our initial hypothesis, does not occur in $(\exists \mathbf{x})\mathbf{P}$ or in any member of Γ.)

We conclude that if the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is consistent in *PD* and **a** does not occur in any member of that set, then $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is also consistent in *PD*.

b. Let Γ^* be constructed as in our proof of Lemma 11.4.4. Assume that $(\exists \mathbf{x})\mathbf{P}$ is a member of Γ^* and that $(\exists \mathbf{x})\mathbf{P}$ is the **i**th sentence in our enumeration of the sentences of *PL*. Then, by the way each member of the infinite sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is constructed, $\Gamma_{\mathbf{i}+1}$ contains $(\exists \mathbf{x})\mathbf{P}$ and a substitution instance of $(\exists \mathbf{x})\mathbf{P}$ if $\Gamma_\mathbf{i} \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is consistent in *PD*. Since each member of the infinite sequence is consistent in *PD*, $\Gamma_\mathbf{i}$ is consistent to *PD*. So assume that $\Gamma_\mathbf{i} \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is inconsistent in *PD*. Then, since we assumed that $\mathbf{P}_\mathbf{i}$, that is, $(\exists \mathbf{x})\mathbf{P}$, is a member of Γ^* and since every member of $\Gamma_\mathbf{i}$ is a member of Γ^*, it follows that Γ^*

is inconsistent in *PD*. But this contradicts our original assumption, and so $\Gamma_{\mathbf{i}} \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is consistent in *PD*. Hence $\Gamma_{\mathbf{i}+1}$ is $\Gamma_{\mathbf{i}} \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ for some constant **a**, and so some substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is a member of $\Gamma_{\mathbf{i}+1}$ and thus of Γ^*.

7. We shall prove that the sentence at each position **i** in the new derivation can be justified by the same rule that was used at position **i** in the original derivation.

Basis clause: Let **i** = 1. The sentence at position 1 of the original derivation is an assumption, and so the sentence at position 1 of the new sequence can be justified similarly.

Inductive step: Assume (the inductive hypothesis) that at every position **i** prior to position **k** + 1, the new sequence contains a sentence that may be justified by the rule justifying the sentence at position **i** of the original derivation. We now prove that the sentence at position **k** + 1 of the new sequence can be justified by the rule justifying the sentence at position **k** + 1 of the original derivation. We shall consider the rules by which the sentence at position **k** + 1 of the original derivation could have been justified:

1. **P** is justified at position **k** + 1 by Assumption. Obviously, **P*** can be justified by Assumption at position **k** + 1 of the new sequence.

2. **P** is justified at position **k** + 1 by Reiteration. Then **P** occurs at an accessible earlier position in the original derivation. Therefore **P*** occurs at an accessible earlier position in the new sequence, so **P*** can be justified at position **k** + 1 by Reiteration.

3. **P** is a conjunction **Q** & **R** justified at position **k** + 1 by Conjunction Introduction. Then the conjuncts **Q** and **R** of **P** occur at accessible earlier positions in the original derivation. Therefore **Q*** and **R*** occur at accessible earlier positions in the new sequence. So **P***, which is just **Q*** & **R***, can be justified at position **k** + 1 by Conjunction Introduction.

4–12. **P** is justified by one of the other truth-functional connective introduction or elimination rules. These cases are as straightforward as case 3, so we move on to the quantifier rules.

13. **P** is a sentence $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ justified at position **k** + 1 by ∀E, appealing to an accessible earlier position with $(\forall \mathbf{x})\mathbf{Q}$. Then $(\forall \mathbf{x})\mathbf{Q}^*$ occurs at the accessible earlier position of the new sequence, and $\mathbf{Q}(\mathbf{a}/\mathbf{x})^*$ occurs at position **k** + 1. But $\mathbf{Q}(\mathbf{a}/\mathbf{x})^*$ is just a substitution instance of $(\forall \mathbf{x})\mathbf{Q}^*$. So $\mathbf{Q}(\mathbf{a}/\mathbf{x})^*$ can be justified at position **k** + 1 by ∀E.

14. **P** is a sentence $(\exists \mathbf{x})\mathbf{Q}$ and is justified at position **k** + 1 by ∃I. This case is similar to case 13.

15. **P** is a sentence $(\forall \mathbf{x})\mathbf{Q}$ and is justified at position **k** + 1 by ∀I. Then some substitution instance occurs at an accessible earlier position **j**,

where **a** is a constant that does not occur in any open assumption prior to position **k** + 1 or in (∀**x**)**Q**. **Q**(**a**/**x**)* and (∀**x**)**Q*** occur at positions **j** and **k** + 1 of the new sequence. **Q**(**a**/**x**)* is a substitution instance of (∀**x**)**Q***. The instantiating constant **a** in **Q**(**a**/**x**) is some $\mathbf{a_i}$, and so the instantiating constant in **Q**(**a**/**x**)* is $\mathbf{b_i}$. Since $\mathbf{a_i}$ did not occur in any open assumption before position **k** + 1 or in (∀**x**)**Q** in the original derivation and $\mathbf{b_i}$ does not occur in the original derivation, $\mathbf{b_i}$ does not occur in any open assumption prior to position **k** + 1 of the new sequence or in (∀**x**)**Q***. So (∀**x**)**Q*** can be justified by ∀I at position **k** + 1 in the new sequence.

16. **P** is justified at position **k** + 1 by ∃E. This case is similar to case 15.

Since every sentence in the new sequence can be justified by a rule of *PD*, it follows that the new sequence is indeed a derivation of *PD*.

10. We required that Γ* be ∃-complete so that we could construct an interpretation I* for which we could *prove* that every member of Γ* is true on I*. In requiring that Γ be ∃-complete in addition to being maximally consistent in *PD*, we were guaranteed that Γ* had property g of sets that are both maximally consistent in *PD* and ∃-complete; and we used this fact in case 7 of the proof that every member of Γ* is true on I*.

11. To prove that *PD** is complete for predicate logic, it will suffice to show that with ∀E* instead of ∀E, every set Γ* of *PD** that is both maximally consistent in *PD** and ∃-complete has property f (i.e., (∀**x**)**P** ∈ Γ* if and only if for every constant **a**, **P**(**a**/**x**) ∈ Γ*)—the properties a to e and g can be shown to characterize such sets by appealing to the rules of *PD** that are rules of *PD*. Here is our proof:

> **Proof**: Assume that (∀**x**)**P** ∈ Γ*. Then, since {(∀**x**)**P**} ⊢ ~ (∃**x**) ~ **P** by ∀E*, it follows from 11.3.3 that ~ (∃**x**) ~ **P** ∈ Γ*. Then (∃**x**) ~ **P** ∉ Γ*, by a. Assume that for some substitution instance **P**(**a**/**x**) of (∀**x**)**P**, **P**(**a**/**x**) ∉ Γ*. Then, by a, ~ **P**(**a**/**x**) ∈ Γ*. Since {~ **P**(**a**/**x**)} ⊢ (∃**x**) ~ **P** (without use of ∀E), it follows that (∃**x**) ~ **P** ∈ Γ*. But we have just shown that (∃**x**) ~ **P** ∉ Γ*. Hence, if (∀**x**)**P** ∈ Γ*, then every substitution instance **P**(**a**/**x**) of (∀**x**)**P** is a member of Γ*.
>
> Now assume that (∀**x**)**P** ∉ Γ*. Then, by a, ~ (∀**x**)**P** ∈ Γ*. But then, since {~ (∀**x**)**P**} ⊢ (∃**x**) ~ **P** (without use of ∀E), it follows that (∃**x**) ~ **P** ∈ Γ*. Since Γ* is ∃-complete, some substitution instance ~ **P**(**a**/**x**) of (∃**x**) ~ **P** is a member of Γ*. By a, **P**(**a**/**x**) ∉ Γ*.

13. Assume that the sentence **P** is not quantificationally false. Then **P** is true on at least one interpretation, so {**P**} is quantificationally consistent. Now suppose that {**P**} is inconsistent in *PD*. Then a pair of sentences **Q** and ~ **Q** are derivable from {**P**} in *PD*. By Metatheorem 11.3.1, it follows that {**P**} ⊨ **Q** and {**P**} ⊨ ~ **Q**. But then **P** cannot be true on any interpretation, contrary to our assumption. So {**P**} is consistent in *PD*. By 11.4.3 and 11.4.4, {$\mathbf{P_e}$}—the set resulting from doubling the subscript of every individual constant in **P**—is a subset of a set Γ* that is both maximally consistent in *PD* and ∃-complete. It follows from Lemma 11.4.8 that Γ* is quantificationally consistent. But, in proving 11.4.8, we actually showed more—for the characteristic interpretation **I*** that we constructed for Γ* has the set of positive integers as UD. Hence every member of Γ* is true on some interpretation with the set of positive integers as UD, and thus $\mathbf{P_e}$ is true on some interpretation with the set of positive integers as UD. **P** can also be shown true on some interpretation with that UD, using 11.1.13.

16. We shall prove 11.4.1 by mathematical induction on the number of functors occurring in **t**.

> *Basis clause:* 11.4.1 holds of every complex closed term that contains one occurrence of a functor.
>
> **Proof of basis clause:** If **t** contains one functor, then **t** is $f(\mathbf{t}_1, \ldots, \mathbf{t_n})$, where each $\mathbf{t_i}$ is a constant. Let **a** be the alphabetically earliest constant such that $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{a} \in \Gamma^*$. It follows from clause 4 of the definition of **I*** that $<\mathbf{I}^*(\mathbf{t}_1), \ldots, \mathbf{I}^*(\mathbf{t_n}), \mathbf{I}^*(\mathbf{a})> \in \mathbf{I}^*(f)$ and so $\text{den}_{\mathbf{I}^*,\mathbf{d_I}}(f(\mathbf{t}_1, \ldots, \mathbf{t_n})) = \mathbf{I}^*(\mathbf{a})$.
>
> *Inductive step:* If 11.4.1 holds of every complex closed term that contains **k** or fewer occurrences of functors, then 11.4.1 holds of every complex closed term that contains **k** occurrences of functors.
>
> **Proof of inductive step:** Assume the inductive hypothesis: that 11.4.1 holds of every complex closed term that contains **k** or fewer occurrences of functors. Let **t** be a term that contains **k** + 1 occurrences of functors; we will show that 11.4.1 holds of **t** as well.
>
> **t** has the form $f(\mathbf{t}_1, \ldots, \mathbf{t_n})$, where each $\mathbf{t_i}$ is a closed term containing **k** or fewer occurrences of functors. Let **a** be the alphabetically earliest constant such that $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{a} \in \Gamma^*$. It follows from the inductive hypothesis that for each $\mathbf{t_i}$, $\text{den}_{\mathbf{I}^*,\mathbf{d_{I^*}}}(\mathbf{t_i}) = \mathbf{I}^*(\mathbf{a_i})$, where $\mathbf{a_i}$ is the alphabetically earliest constant such that $\mathbf{t_i} = \mathbf{a_i}$ is a member of Γ*. It follows from property (**i**) of maximally consistent, ∃-complete sets that $f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{a} \in \Gamma^*$, and it follows from clause 4 of the definition of **I*** that $<\mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n}), \mathbf{I}^*(\mathbf{a})> \in \mathbf{I}^*(f)$. So $\text{den}_{\mathbf{I}^*,\mathbf{d_{I^*}}}(f(\mathbf{t}_1, \ldots, \mathbf{t_n})) = \text{den}_{\mathbf{I}^*,\mathbf{d_{I^*}}}(f(\mathbf{a}_1, \ldots, \mathbf{a_n})) = \mathbf{I}^*(\mathbf{a})$.

17. Consider the sentence '(∀x)(∀y)x = y'. This sentence is not quantificationally false; it is true on every interpretation with a one-member UD. In addition, however, it is true on *only* those interpretations that have one-member UDs. (This is because for any variable assignment and any members $\mathbf{u}_1$ and $\mathbf{u}_2$ of a UD, $\mathbf{d_I}[\mathbf{u}_1/x, \mathbf{u}_2/y]$ satisfies 'x = y' as required for the truth of '(∀x) (∀y)x = y' if and only if $\mathbf{u}_1$ and $\mathbf{u}_2$ are the same object.) So there can be no interpretation with the set of positive integers as UD on which the sentence is true.

Section 11.5E

2.a. Assume that for some sentence **P**, {**P**} has a closed truth-tree. Then, by 11.5.1, {**P**} is quantificationally inconsistent. Hence there is no interpretation on which **P**, the sole member of {**P**}, is true. Therefore **P** is quantificationally false.

b. Assume that for some sentence **P**, {~ **P**} has a closed truth-tree. Then, by 11.5.1, {~ **P**} is quantificationally inconsistent. Hence there is no interpretation on which ~ **P** is true. So **P** is true on every interpretation; that is, **P** is quantificationally true.

d. Assume that $\Gamma \cup$ {~ **P**} has a closed truth-tree. Then, by 11.5.1, $\Gamma \cup$ {~ **P**} is quantificationally inconsistent. Hence there is no interpretation on which every member of Γ is true and ~ **P** is also true. That is, there is no interpretation on which every member of Γ is true and **P** is false. But then $\Gamma \vDash$ **P**.

3.a. **P** is obtained from ~ ~ **P** by ~ ~ D. It is straightforward that {~ ~ **P**} $\vDash$ **P**.

d. **P** or ~ **Q** is obtained from ~ (**P** ⊃ **Q**) by ~ ⊃D. On any interpretation on which ~ (**P** ⊃ **Q**) is true, **P** ⊃ **Q** is false—hence **P** is true and **Q** is false and ~ **Q** is true. Thus {~ (**P** ⊃ **Q**)} $\vDash$ **P** and {~ (**P** ⊃ **Q**)} $\vDash$ ~ **Q**.

e. **P**(**a**/**x**) is obtained from (∀**x**)**P** by ∀D. It follows, from 11.1.4, that {(∀**x**)**P**} $\vDash$ **P**(**a**/**x**).

4.a. ~ **P** and ~ **Q** are obtained from ~ (**P** & **Q**) by ~ &D. On any interpretation on which ~ (**P** & **Q**) is true, **P** & **Q** is false. But then either **P** is false and ~ **P** is true, or **Q** is false and ~ **Q** is true.

5. The path is extended to form two paths to level **k** + 1 as a result of applying one of the branching rules ≡D or ~ ≡D to a sentence **P** on $\Gamma_{\mathbf{k}}$. We consider four cases.

a. Sentences **P** and ~ **P** are entered at level **k** + 1 as the result of applying ≡D to a sentence **P** ≡ **Q** on $\Gamma_{\mathbf{k}}$. On any interpretation on which **P** ≡ **Q** is true, so is either **P** or ~ **P**. Therefore either **P** and all the sentences on $\Gamma_{\mathbf{k}}$ are true on $\mathbf{I}_{\Gamma_{\mathbf{k}}}$, which is a path variant of **I** for the new path containing **P**, or ~ **P** and all the sentences on $\Gamma_{\mathbf{k}}$ are true on $\mathbf{I}_{\Gamma_{\mathbf{k}}}$, which is a path variant of **I** for the new path containing ~ **P**.

b. Sentence **Q** (or ~ **Q**) is entered at level **k** + 1 as the result of applying ≡D to a sentence **P** ≡ **Q** on $\Gamma_{\mathbf{k}}$. Then **P** (or ~ **P**) occurs on $\Gamma_{\mathbf{k}}$ at level **k** (application of ≡D involves making entries at two levels, and **Q** and ~ **Q** are entries made on the second of these levels). Since {**P** ≡ **Q**, **P**} $\vDash$ **Q** (and {**P** ≡ **Q**, ~ **P**} $\vDash$ ~ **Q**), it follows that **Q** and all the sentences on $\Gamma_{\mathbf{k}}$ (~ **Q** and all the sentences on $\Gamma_{\mathbf{k}}$) are all true on $\mathbf{I}_{\Gamma_{\mathbf{k}}}$, which is a path variant of **I** for the new path containing **Q** (~ **Q**).

c. Sentences **P** and ~ **P** are entered at level **k** + 1 as the result of applying ~ ≡D to a sentence ~ (**P** ≡ **Q**) on $\Gamma_{\mathbf{k}}$. This case is similar to (a).

d. Sentence **Q** (or ~ **Q**) is entered at level **k** + 1 as the result of applying ~ ≡D to a sentence ~ (**P** ≡ **Q**) on $\Gamma_{\mathbf{k}}$. This case is similar to (b).

6. Yes. Dropping a rule would not make the method unsound, for, with the remaining rules, it would still follow that if a branch on a tree for a set Γ closes, then Γ is quantificationally inconsistent. That is, the remaining rules would still be consistency-preserving.

7. To prove that the tree method for *SL* is sound, there are obvious adjustments that must be made in the proof of Metatheorem 11.5.1. First, not all the tree rules for *PL* are tree rules for *SL*. In proving Lemma 11.5.2, then, we take only the tree rules for *SL* into consideration. And in the case of *SL* we would be proving that certain sets are truth-functionally consistent or inconsistent, rather than quantificationally consistent or inconsistent. With these stipulations, the proof of Metatheorem 11.5.1 can be converted straight-forwardly into a proof of the parallel metatheorem for *SL*.

Section 11.6E

1.a. Assume that the sentence **P** is quantificationally false. Then {**P**} is quantificationally inconsistent. It follows from Metatheorem 11.6.1 that every systematic tree for {**P**} closes.

b. Assume that the sentence **P** is quantificationally true. Then ~ **P** is quantificationally false, and {~ **P**} is quantificationally inconsistent. It follows from Metatheorem 11.6.1 that every systematic tree for {~ **P**} closes.

d. Assume that Γ ⊨ **P**. Then on every interpretation on which every member of Γ is true, **P** is true and ~ **P** is therefore false. So Γ ∪ {~ **P**} is quantificationally inconsistent. It follows from Metatheorem 11.6.1 that every systematic tree for Γ ∪ {~ **P**} closes.

2.a. The lengths are 6, 2, and 6, respectively.

b. Assume that the length of a sentence ~ (**Q** & **R**) is **k**. Then since ~ (**Q** & **R**) contains an occurrence of the tilde and an occurrence of the ampersand that neither **Q** nor **R** contains, the length of **Q** is **k** − 2 or less and the length of **R** is **k** − 2 or less. Hence the length of ~ **Q** is **k** − 1 or less, and the length of ~ **R** is **k** − 1 or less.

d. Assume that the length of a sentence ~ (∀**x**)**Q** is **k**. Then the length of the formula **Q** is **k** − 2. Hence the length of **Q**(**a**/**x**) is **k** − 2, since **Q**(**a**/**x**) differs from **Q** only in containing **a** wherever **Q** contains **x** and neither constants nor variables are counted in computing the length of a formula. Hence the length of ~ **Q**(**a**/**x**) is **k** − 1.

3.a. **P** has the form **Q** ∨ **R**. Assume that **P** ∈ Γ. Then, by e, either **Q** ∈ Γ or **R** ∈ Γ. If **Q** ∈ Γ, then **I**(**Q**) = **T**, by the inductive hypothesis. If **R** ∈ Γ, then **I**(**R**) = **T**, by the inductive hypothesis. Either way, it follows that **I**(**Q** ∨ **R**) = **T**.

c. **P** has the form **Q** ⊃ **R**. Assume that **P** ∈ Γ. Then, by g, either ~ **Q** ∈ Γ or **R** ∈ Γ. By the inductive hypothesis, then, either **I**(~ **Q**) = **T** or **I**(**R**) = **T**. So either **I**(**Q**) = **F** or **I**(**R**) = **T**. Consequently, **I**(**Q** ⊃ **R**) = **T**.

f. **P** has the form ~ (**Q** ≡ **R**). Assume that **P** ∈ Γ. Then, by j, either both **Q** ∈ Γ and ~ **R** ∈ Γ, or both ~ **Q** ∈ Γ and **R** ∈ Γ. In the former case, **I**(**Q**) = **T** and **I**(~ **R**) = **T**, by the inductive hypothesis; so **I**(**Q**) = **T** and **I**(**R**) = **F**. In the latter case, **I**(~ **Q**) = **T** and **I**(**R**) = **T**, by the inductive hypothesis; hence **I**(**Q**) = **F** and **I**(**R**) = **T**. Either way, it follows that **I**(**Q** ≡ **R**) = **F**, and so **I**(~ (**Q** ≡ **R**)) = **T**.

g. **P** has the form (∃**x**)**Q**. Assume that **P** ∈ Γ. Then, by m, there is some constant **a** such that **Q**(**a**/**x**) ∈ Γ. By the inductive hypothesis, **I**(**Q**(**a**/**x**)) = **T**. By 11.1.5, {**Q**(**a**/**x**)} ⊢ (∃**x**)**Q**. So **I**((∃**x**)**Q**) = **T** as well.

5. Clauses 7 and 9. First consider clause 7. Suppose that **Q** ⊃ **R** has **k** occurrences of logical operators. Then **Q** certainly has fewer than **k** occurrences of logical operators, and so does **R**. But, in the proof for case 7, once we assume that **Q** ⊃ **R** ∈ Γ, we know that either ~ **Q** or **R** is a member of Γ by property g of Hintikka sets. The problem is that we cannot apply the inductive hypothesis to ~ **Q** since ~ **Q** might contain **k** occurrences of logical operators. In the sentence '(Am & Bm) ⊃ Bm', for instance, this happens. The entire sentence has two occurrences of logical operators, but so does the negation of the antecedent '~ (Am & Bm)'. However, it can easily be shown that the *length* of ~ **Q** is less than the *length* of **Q** ⊃ **R**.

Similarly, in the case of clause 9 we know that if **Q** ≡ **R** ∈ Γ, then either both **Q** ∈ Γ and **R** ∈ Γ or both ~ **Q** ∈ Γ and ~ **R** ∈ Γ. But then we are not guaranteed that either ~ **Q** or ~ **R** has fewer occurrences of logical operators than does **Q** ≡ **R**. For instance, '~ Am' and '~ Bm' each contain one occurrence of a logical operator, and so does 'Am ≡ Bm'.

6. If ∃D were not included, then we could not be assured that the set of sentences on each open branch of a systematic tree has property m of Hintikka sets. And in the inductive proof that every Hintikka set is quantificationally consistent we made use of this property in steps (12) and (13).

7. Yes, it would. For let us trace those places in our proof of Metatheorem 11.6.1 where we appealed to the rule ~ ∀D. We used it to establish that the set of sentences on an open branch of a systematic tree has property 1 of Hintikka sets, and we appealed to property 1 in step (12) of our inductive proof of 11.6.4. So let us first replace property 1 by the following:

> 1*. If ~ (∀**x**)**P** ∈ Γ, then, for some constant **a** that occurs in some sentence in Γ, ~ **P**(**a**/**x**) ∈ Γ.

It is then easily established that every open branch of a systematic tree has properties a to k, 1*, and m to n. In our inductive proof of Lemma 11.6.4, change step (12) to the following:

12*. **P** has the form ~ (∀**x**)**Q**. Assume that **P** ∈ Γ. Then, by 1*, there is a constant **a** such that ~ **Q**(**a**/**x**) ∈ Γ. By the inductive hypothesis,

$\mathbf{I}(\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})) = \mathbf{T}$ and so $\mathbf{I}(\mathbf{Q}(\mathbf{a}/\mathbf{x})) = \mathbf{F}$. Since $\{(\forall \mathbf{x})\mathbf{Q}\} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$, by 11.1.4, it follows that $\mathbf{I}((\forall \mathbf{x})\mathbf{Q}) = \mathbf{F}$ and $\mathbf{I}(\sim (\forall \mathbf{x})\mathbf{Q}) = \mathbf{T}$.

8. Certain adjustments are obvious if we are to convert the proof of Metatheorem 11.6.1 into a proof that the tree method for *SL* is complete for sentential logic. The tree method for *SL* contains only some of the rules of the tree method for *PL*; hence we have fewer rules to work with. We replace talk of quantificational concepts (consistency and the like) with talk of truth-functional concepts, hence talk of interpretations with talk of truth-value assignments.

A Hintikka set of *SL* will have only properties a to j of Hintikka sets for *PL*. And trees for *SL* are *all* finite, so we have only finite open branches to consider in this case. (Thus Lemma 11.6 would not be used in the proof for *SL*.) Finally, the construction of the characteristic truth-value assignment for a Hintikka set of *SL* requires only clause 2 of the construction of the characteristic interpretation for a Hintikka set of *PL*.

9. We must first show that a set Γ^* that is both maximally consistent in *PD* and ∃-complete has the 14 properties of Hintikka sets. We list those properties here. (And we refer to the 7 properties a to g of sets that are both maximally consistent in *PD* and ∃-complete as 'M(a)', 'M(b)', . . . , 'M(g)'.)

a. For any atomic sentence **P**, not both **P** and $\sim$ **P** are members of Γ^*.

Proof: This follows immediately from property M(a) of Γ^*.

b. If $\sim \sim$ **P** is a member of Γ^*, then **P** is a member of Γ^*.

Proof: If $\sim \sim \mathbf{P} \in \Gamma^*$, then $\sim \mathbf{P} \notin \Gamma^*$, by M(a), and $\mathbf{P} \in \Gamma^*$, by M(a).

c. If $\mathbf{P}$ & $\mathbf{Q} \in \Gamma^*$, then $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.

Proof: This follows from property M(b) of Γ^*.

d. If $\sim (\mathbf{P}$ & $\mathbf{Q}) \in \Gamma^*$, then either $\sim \mathbf{P} \in \Gamma^*$ or $\sim \mathbf{Q} \in \Gamma^*$.

Proof: If $\sim (\mathbf{P}$ & $\mathbf{Q}) \in \Gamma^*$, then $\mathbf{P}$ & $\mathbf{Q} \notin \Gamma^*$, by M(a). By M(b), either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$. By M(a), either $\sim \mathbf{P} \in \Gamma^*$ or $\sim \mathbf{Q} \in \Gamma^*$.

e. to j. are established similarly.

k. If $(\forall \mathbf{x})\mathbf{P} \in \Gamma$, then at least one substitution instance of $(\forall \mathbf{x})\mathbf{P}$ is a member of Γ and for every constant **a** that occurs in some sentence of Γ, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma$.

Proof: This follows from property M(f) of Γ^*.

l. If ~ $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$, then $(\exists \mathbf{x})$ ~ $\mathbf{P} \in \Gamma^*$.

Proof: If ~ $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$, then $(\forall \mathbf{x})\mathbf{P} \notin \Gamma^*$, by M(a). Then, for some constant **a**, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$, by M(f). Then ~ $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$, by M(a). So $(\exists \mathbf{x})$ ~ $\mathbf{P} \in \Gamma^*$, by M(g).

m. If $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$, then, for at least one constant **a**, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$.

Proof: This follows from property M(g) of Γ^*.

n. If ~ $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$, then $(\forall \mathbf{x})$ ~ $\mathbf{P} \in \Gamma^*$.

Proof: If ~ $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$, then $(\exists \mathbf{x})\mathbf{P} \notin \Gamma^*$, by M(a). Then, for every constant **a**, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$, by M(g). So, for every constant **a**, ~ $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$, by M(a). And $(\forall \mathbf{x})$ ~ $\mathbf{P} \in \Gamma^*$, by M(f).

Second, that every Hintikka set is ∃-complete follows from property m of Hintikka sets.

Third, we show that some Hintikka sets are *not* maximally consistent in *PD*. Here is an example of such a set:

{(∀x)Fx, (∃y)Fy, Fa}

It is easily verified that this set is a Hintikka set. And the set is of course consistent in *PD*. But this set is *not* such that the addition to the set of any sentence that is not already a member will create an inconsistent set. For instance, the sentence ‘Fb’ may be added, and the resulting set is also consistent in *PD*:

{(∀x)Fx, (∃y)Fy, Fa, Fb}

Hence the set is not maximally consistent in *PD*.