Marginal and average product

Consider a firm producing a product using just two inputs, labor (*L*) and capital (*K*). The firm's total product (*TP*, or "*Q*" for "quantity produced") is a function of its two inputs: $TP = Q = f(L, K)$. The *marginal product of labor* (MP_L) is defined as the change in total product from expanding labor input by one unit while holding capital constant. Mathematically then, $MP_L = \frac{\partial Q}{\partial L} = \frac{\partial f(L,K)}{\partial L}$. By assumption, this is greater than zero.

The *law of diminishing returns* states that adding additional amounts of labor to a fixed amount of capital will eventually reduce labor's marginal product. This law can be stated mathematically as $\frac{\partial MP_L}{\partial L} = \frac{\partial^2 f(L,K)}{\partial L^2} = \frac{\partial^2 Q}{\partial L^2} < 0$.

Average product (AP_L) is defined as output per unit of labor input: Q/L. What is the shape of AP_L and how is it related to MP_L? First, we will determine the slope of AP_L with respect to labor by taking the first derivative. Using the quotient rule, $\frac{\partial\left(f(L,K)/L\right)}{\partial L} = \frac{\partial(Q/L)}{\partial L} = \frac{L\frac{\partial Q}{\partial L} - Q}{L^2}$. In this form, this derivative is not too enlightening. Suppose, however, we divide both numerator and denominator by the number of workers, *L*. Then, $\frac{\partial(Q/L)}{\partial L} = \frac{\frac{\partial Q}{\partial L} - \frac{Q}{L}}{L}$. You will recognize the first term in the numerator as labor's marginal product, while the second term is labor's average product. Making the substitutions, we see that the slope of the average product function is $\frac{\partial(Q/L)}{\partial L} = \frac{MP_L - AP_L}{L}$.

From this relationship, we can note three related conclusions:

1. If labor's marginal product exceeds its average product, $\frac{\partial(Q/L)}{\partial L} > 0$. That is, labor's average product will be rising.
2. If labor's marginal product is less than its average product, $\frac{\partial(Q/L)}{\partial L} < 0$. That is, labor's average product will be falling.
3. If labor's marginal product equals its average product, $\frac{\partial(Q/L)}{\partial L} = 0$ and the average product will reach its maximum value at that point. (Note that $\frac{\partial(Q/L)}{\partial L} = 0$ is the first-order condition for maximum average product.)