10 Graphs

Introduction

In this chapter, we consider ways in which *Mathematica* can help you explore and understand graphs. In particular, we describe how to visualize and perform computations on graphs.

10.1 Graphs and Graph Models

In the Wolfram Language, a graph is represented by a special raw object with head `Graph`. You should think about `Graph` not as a function, but as a way to describe and represent an object. To explain this distinction, consider a fraction, such as $\frac{5}{9}$. We think about this as a single number, and the Wolfram Language treats it as a simple object, but if you delve down using `FullForm`, you see that it is a bit more complicated.

In[1]:=
```
5/9 //FullForm
```

Out[1]//FullForm=
```
Rational[5,9]
```

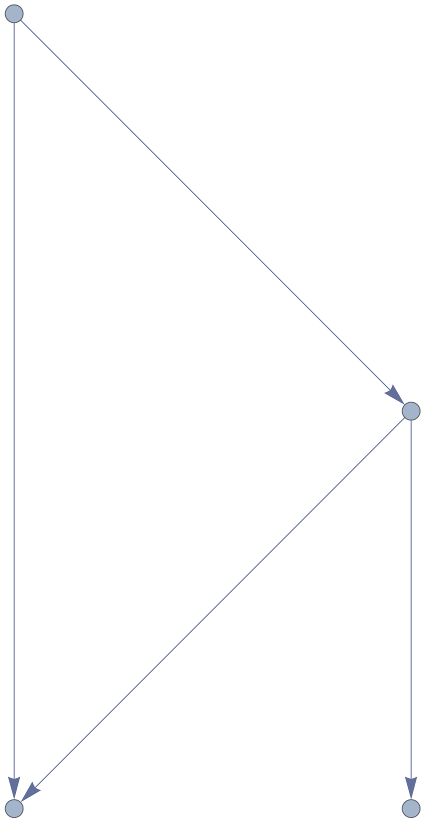
`FullForm` reveals that the fraction 5/9 is actually represented by `Rational` applied to two integers. `Rational` is not a function, in the usual sense. Rather, it is a head that tells *Mathematica* what the contents mean. `Graph` is exactly the same, if a bit more complicated. It is a head that tells *Mathematica* that the contents represent a graph object. Moreover, just like a `Rational` is typically displayed in the usual fraction notation, a `Graph` object is displayed as a drawing of the graph.

The simplest way to specify a `Graph` object in the Wolfram Language is by specifying the edges as a list of rules. You typically use positive integers or strings for the vertices (although other expressions can be used). An edge between the vertex represented by 1 and the vertex represented by “a” is given as the rule `1->"a"`.

As an example, the following defines a directed version of the graph shown in Exercise 3 in Section 10.1.

In[2]:=
```
exercise3directed=
  Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"}]
```

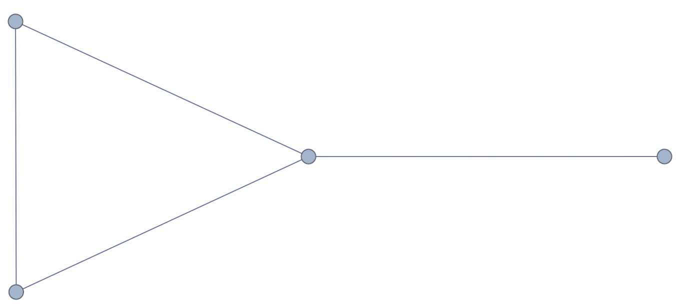
Out[2]=



To produce an undirected simple graph, you can set the option `DirectedEdges` to `False`.

In[3]:=
```
exercise3=Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False]
```

Out[3]=

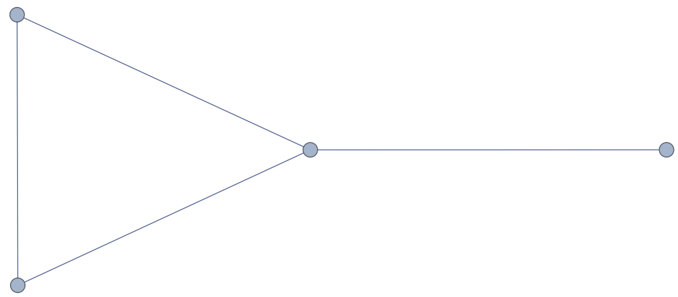


Note that the difference in appearance in the directed and undirected versions is a result of a preference in the Wolfram Language to draw directed graphs with arrows pointing down. In the undirected •-• case, there is no such concern and a more compact image results.

You can also create (undirected) simple graphs using a `TwoWayRule` (<->, ↔) or the symbol •-• in place of `Rule` (->, →). The undirected edge symbol is entered by typing ESC ue ESC.

In[4]:=
```
Graph[{"a"↔"b","a"↔"c","b"↔"c","b"↔"d"}]
```

Out[4]=

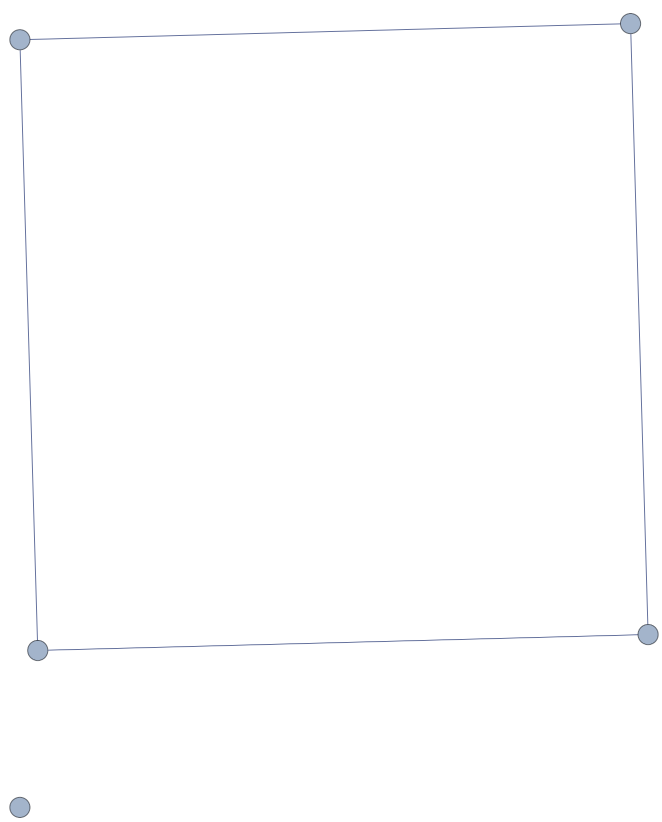


If you prefer, you can use the symbol $\mapsto$, entered as $\boxed{\text{ESC}}$de$\boxed{\text{ESC}}$ in place of `Rule` (->) for a directed graph. In this manual, we will generally use `Rule` (->) to specify all edges together with the `DirectedEdges` option, when needed, as this frequently involves the least typing and typos do not result in mixed graphs. There are, however, some functions that apply to edges and require that you correctly distinguish between directed (using -> or $\boxed{\text{ESC}}$de$\boxed{\text{ESC}}$) and undirected edges (using <-> or $\boxed{\text{ESC}}$ue$\boxed{\text{ESC}}$). Pay special attention to those situations.

If you wish, you may give an explicit list of vertices as the first argument to `Graph`. This is only required in the situation when a graph has a vertex not adjacent to any other, as in the example below.

In[5]:=
```
Graph[{1,2,3,4,5},{1→2,2→3,3→4,4→1},
  DirectedEdges→False]
```

Out[5]=



The VertexList and EdgeList functions applied to a Graph object produce lists of the vertices and edges of the graph.

```
In[6]:= VertexList[exercise3]
Out[6]= {a,b,c,d}
In[7]:= EdgeList[exercise3]
Out[7]= {a⟷b,a⟷c,b⟷c,b⟷d}
```

Observe that the output of EdgeList is a list of undirected edges, using the symbol ⟷, despite the fact that we defined the graph using rules. This is because the effect of setting the DirectedEdges option to False is to transform the rules into undirected edges. Using FullForm, you can see that these are stored using the head UndirectedEdge.

```
In[8]:= EdgeList[exercise3]//FullForm
Out[8]//FullForm=
        List[UndirectedEdge["a","b"],UndirectedEdge["a","c"],
          UndirectedEdge["b","c"],UndirectedEdge["b","d"]]
```

Note that the same is true in the directed case, with the edges represented internally as DirectedEdge.

```
In[9]:= EdgeList[exercise3directed]
Out[9]= {a→b,a→c,b→c,b→d}
In[10]:= EdgeList[exercise3directed]//FullForm
Out[10]//FullForm=
        List[DirectedEdge["a","b"],DirectedEdge["a","c"],
          DirectedEdge["b","c"],DirectedEdge["b","d"]]
```

You can use DirectedEdge and UndirectedEdge yourself in creating graphs.

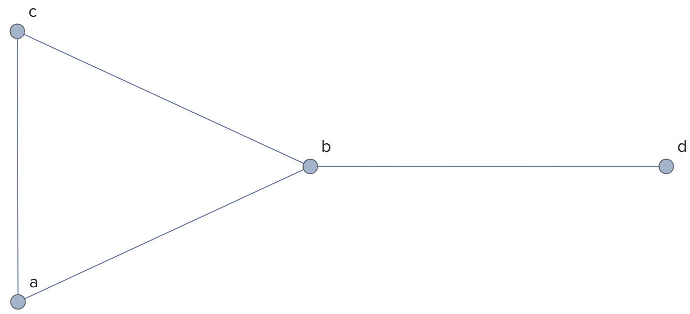
Options Affecting the Visual Output of Graph

In this section, we will explain just a few of the options available for changing the visual appearance of Graph objects. Readers interested in greater control over the display of Graph objects should refer to the documentation pages.

To display the names of vertices, use the VertexLabels option. Probably the most common value for this option is "Name", including the quotation marks, which causes each vertex to be labeled with its name. As an example, we modify the definition given above of the graph from Exercise 3, with this option invoked.

```
In[11]:= Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
           DirectedEdges→False,VertexLabels→"Name"]
```

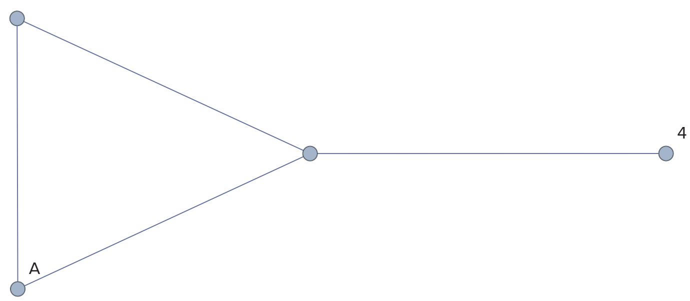
Out[11]=



Another use of the `VertexLabels` option is to specify labels for specific vertices. You do this by identifying the option with a list consisting of rules associating the name of the vertex with the desired label. The following illustrates this by labeling two of the vertices of the graph from Exercise 3.

In[12]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,VertexLabels→{"a"→"A","d"→4}]
```

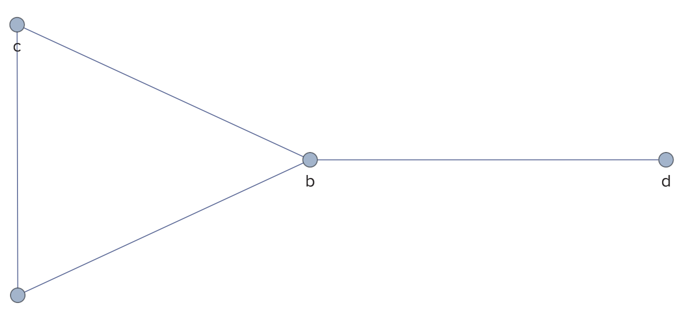
Out[12]=



The `VertexLabels` option can be usefully combined with the `Placed` symbol. The basic syntax of `Placed` is with a first argument being an expression to be displayed and a second argument indicating the position. Common choices for position include `Above`, `Below`, `Before`, `After`, and `Center`. `Placed` can be used with first argument `"Name"` in order to have all vertices labeled with their name or with a specific label for a particular vertex.

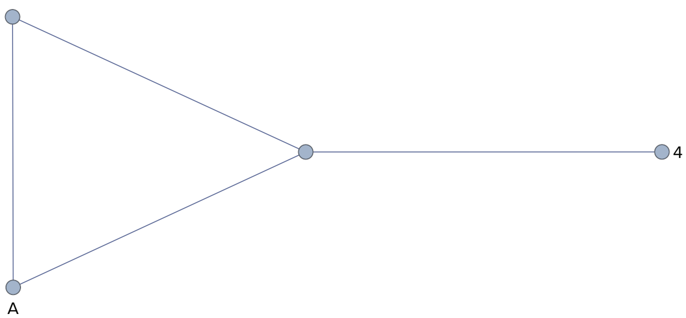
In[13]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,VertexLabels→Placed["Name",Below]]
```

Out[13]=



In[14]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,
  VertexLabels→{"a"->Placed["A",Below],
    "d"→Placed[4,After]}]
```

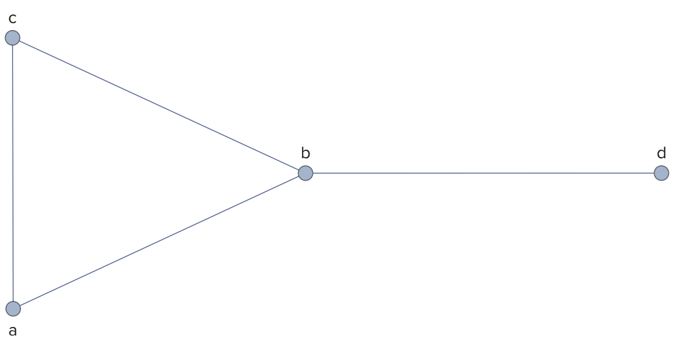
Out[14]=



Observe that you can combine rules specifying labels for individual vertices with a default label specification.

In[15]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,
  VertexLabels→{"a"->Placed["Name",Below],
    Placed["Name",Above]}]
```

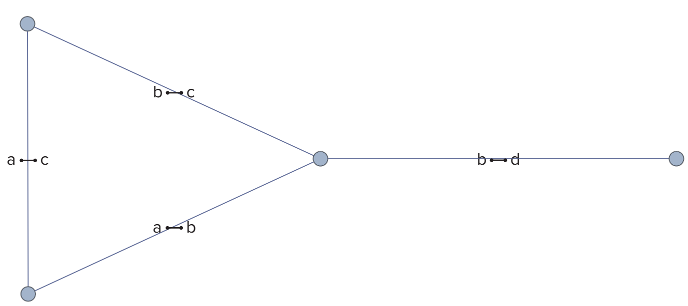
Out[15]=



Similarly, edges in a graph can be labeled using the `EdgeLabels` option. Once again, the value `"Name"` will cause all edges to be labeled.

In[16]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,EdgeLabels→"Name"]
```

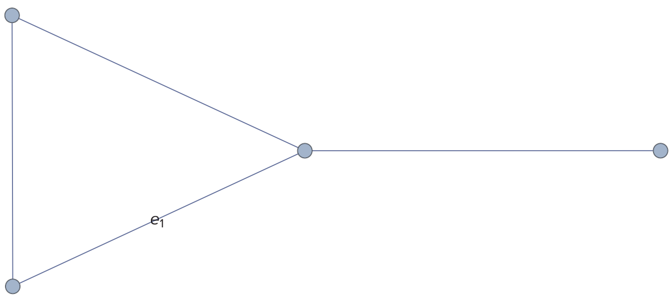
Out[16]=



As with vertices, you can also choose to label specific edges with arbitrary labels by giving a list of rules as the value to `EdgeLabels`. When doing this, the edges should be specified using either •–• ($\boxed{\text{ESC}}$ue$\boxed{\text{ESC}}$) or •→($\boxed{\text{ESC}}$de$\boxed{\text{ESC}}$); alternatively, the edge can be specified with a `Rule` (->) provided that the edge is surrounded by parentheses.

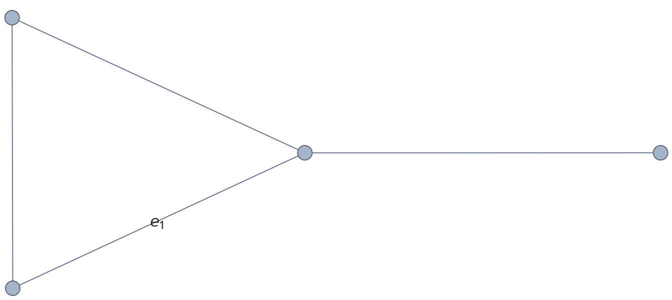
In[17]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,EdgeLabels→{"a"•–•"b"→e1}]
```

Out[17]=



In[18]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,EdgeLabels→{("a"->"b")→e1}]
```

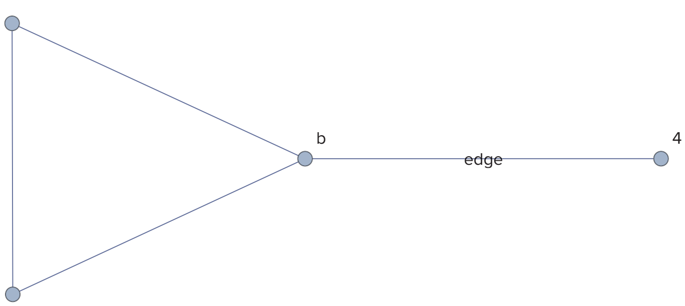
Out[18]=



As an alternative to `VertexLabels` and `EdgeLabels`, you can use the wrapper `Labeled` around an edge in the list of edges or, in conjunction with the optional list of vertices, around a vertex. The first element in `Labeled` is the name of the vertex or the edge definition. The second element is the label to be used.

In[19]:=
```
Graph[{"a",Labeled["b","b"],"c",Labeled["d",4]},
  {"a"→"b","a"→"c","b"→"c",Labeled["b"→"d","edge"]},
  DirectedEdges→False]
```

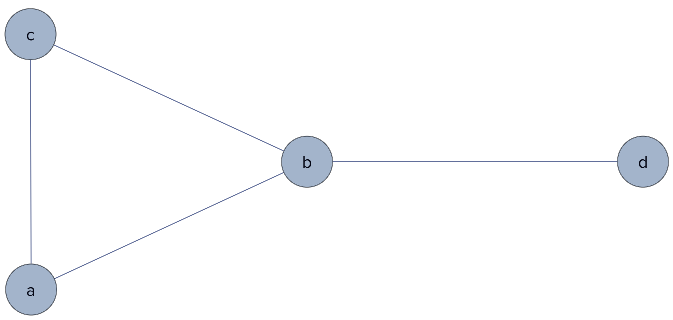
Out[19]=



You can change the size of vertices with the `VertexSize` option. Some possible values include `Tiny`, `Small`, `Medium`, and `Large`, or a number between 0 and 1. A numerical value indicates that the size of the vertex should be that proportion of the distance between the closest two vertices. This can be used in conjunction with `Placed` to have vertex labels appear inside the vertex.

In[20]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,
  VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

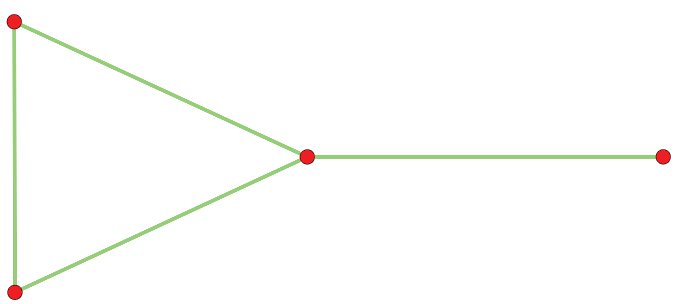
Out[20]=



You can further control the appearance, such as the color, of vertices and edges with the `VertexStyle` and `EdgeStyle` options. Multiple styles can be combined with the `Directive` wrapper. For example, the following illustrates how to create red vertices and thick green edges. Readers interested in exploring the various options available should consult the Graphics Directives guide.

In[21]:=
```
Graph[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→False,VertexStyle→Red,
  EdgeStyle→Directive[Green,Thick]]
```

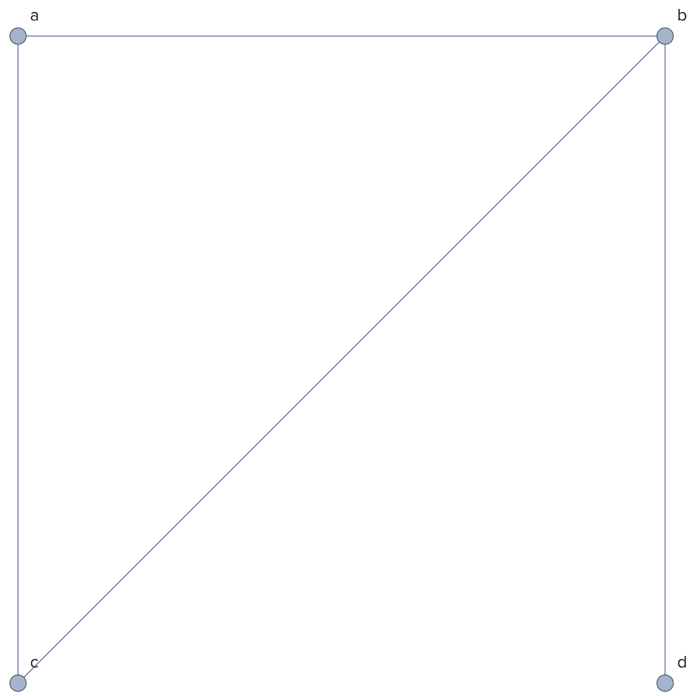
Out[21]=



The `VertexCoordinates` option is used to specify the locations of vertices. The value for the `VertexCoordinates` option is a list of pairs of the form $\{\{x_1,y_1\},\{x_2,y_2\},...\}$, with the pairs specifying the coordinates of a vertex. The order of the locations must correspond to the order of the vertices in the graph. When an explicit list of vertices is given as the optional first element of `Graph`, that list specifies the order. Otherwise, the order is determined by when the vertex is first encountered in the list of edges, and is the same as the output of `VertexList`. Below, we use `VertexCoordinates` to redraw the graph from Exercise 3 with the vertices in the same positions as in the image in the textbook.

```
In[22]:= Graph[{"a","b","c","d"},
           {"a"→"b","a"→"c","b"→"c","b"→"d"},
           DirectedEdges→False,VertexLabels→"Name",
           VertexCoordinates→{{0,1},{1,1},{0,0},{1,0}}]
```

Out[22]=



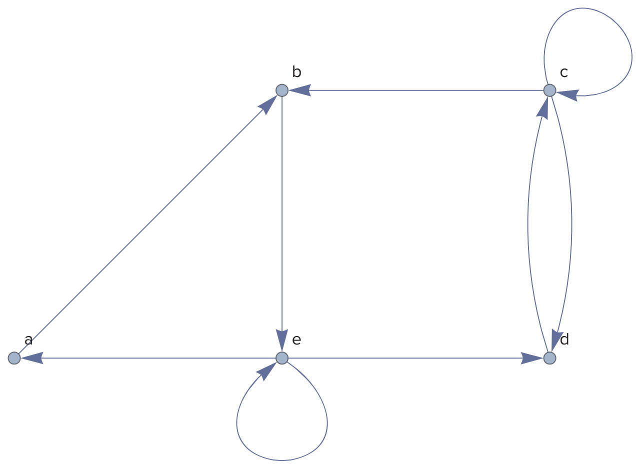
You can also exert control over the layout of the vertices of a graph, but without specifying each coordinate, by using the `GraphLayout` option to specify the algorithm used to choose the vertex location. Common choices include: `"SpringEmbedding"`, which treats edges as springs and minimizes the mechanical energy of the system; `"SpringElectricalEmbedding"`, which treats edges as springs and vertices as electrical charges and minimizes mechanical and electrical energy; `"HighDimensionalEmbedding"`, which is like the spring-electrical method but computes in high dimensions and then projects down to two; `"CircularEmbedding"`, which places the vertices on a circle; and `"LayeredDrawing"`, which places vertices in layers and attempts to reduce the number of edges between non-adjacent layers.

Modifying Graphs

`Graph` objects may contain loops, as illustrated below with a replica of Exercise 7 from Section 10.1. Loops are created simply by including an edge from a vertex to itself.

```
In[23]:= exercise7=Graph[{"a","b","c","d","e"},
           {"a"→"b","b"→"e","c"→"b","c"→"c","c"→"d",
            "d"→"c","e"→"a","e"→"d","e"→"e"},
           VertexLabels→"Name",
           VertexCoordinates→{{1,1},{2,2},{3,2},{3,1},{2,1}}]
```

Out[23]=

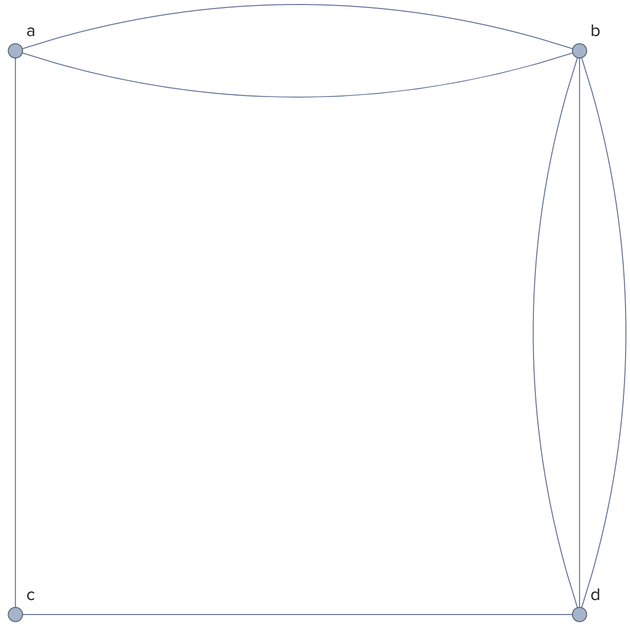


Graph objects may also include multiple edges. The following is the graph from Exercise 4 from Section 10.1.

In[24]:=

```
exercise4=
  Graph[{"a"→"b","a"→"b","a"→"c","b"→"d",
    "b"→"d","b"→"d","c"→"d"},DirectedEdges→False,
    VertexLabels→"Name",
    VertexCoordinates→{{1,2},{2,2},{1,1},{2,1}}]
```

Out[24]=

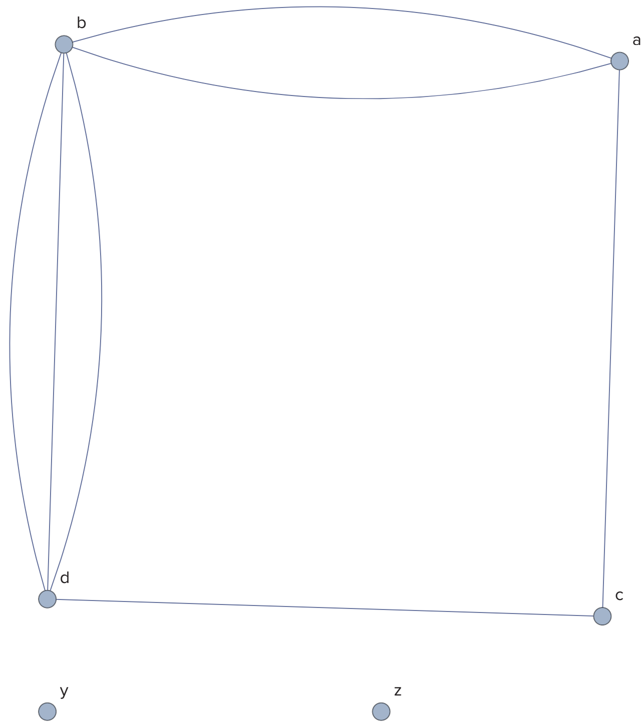


You can modify existing graphs by adding and deleting edges and vertices with the functions VertexAdd, VertexDelete, EdgeAdd, and EdgeDelete. Each of these functions requires a graph as the first argument. The second argument is usually either a vertex, an edge, or a list of vertices or edges. The deletion functions can, in place of a vertex, edge, or list, accept a pattern as the second argument in order to delete all of the vertices or edges that match the pattern.

Beginning with the graph from Exercise 4, we use `VertexAdd` to add two new vertices to the graph.

In[25]:=
```
modifyExercise4=VertexAdd[exercise4,{"y","z"}]
```

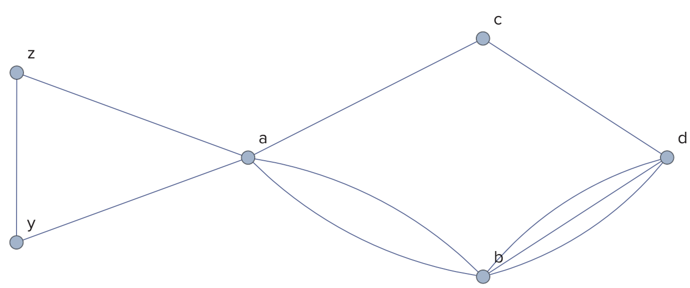
Out[25]=



Now, we add edges to connect the new vertices with the rest of the graph. Note that, in this example, the Wolfram Language interprets rules as undirected edges since the original graph is undirected.

In[26]:=
```
modifyExercise4=EdgeAdd[modifyExercise4,
  {"a"→"z","a"→"y","y"→"z"}]
```

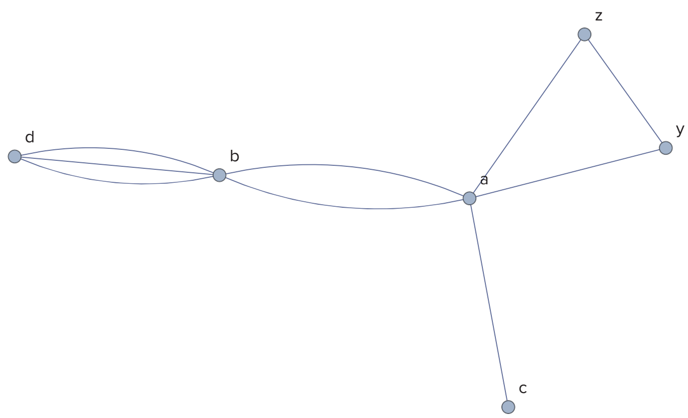
Out[26]=



Next, we delete one of the old edges. Note that to delete an edge from an undirected graph, you must use •–• (ESC ue ESC) or `TwoWayRule` (<->). Likewise, deleting an edge from a directed graph must use •→(ESC de ESC) or `Rule` (->).

In[27]:=
```
modifyExercise4=EdgeDelete[modifyExercise4,"d"↔"c"]
```

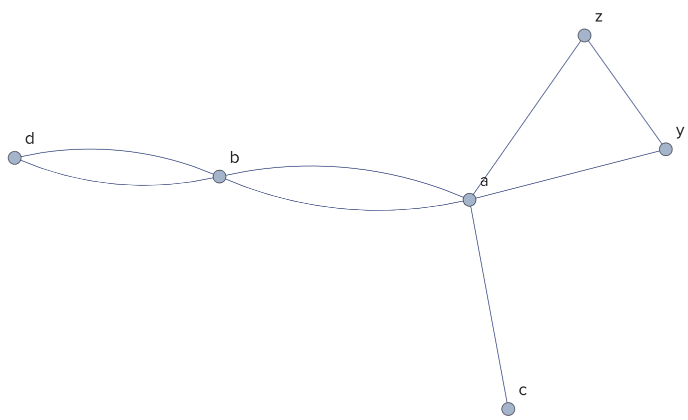
Out[27]=



Note that deleting an edge between two vertices with multiple edges will delete one of the edges.

In[28]:=
```
modifyExercise4=EdgeDelete[modifyExercise4,"d"↔"b"]
```

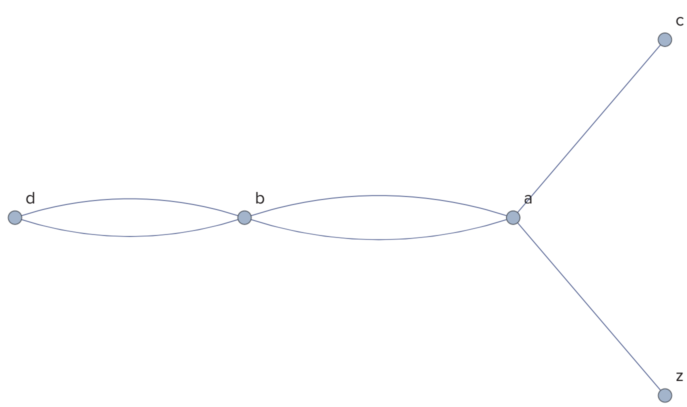
Out[28]=



Finally, note that deleting a vertex also deletes all the edges incident with that vertex.

In[29]:=
```
modifyExercise4=VertexDelete[modifyExercise4,"y"]
```

Out[29]=



Semantic Networks

The textbook defines semantic networks at the end of Section 10.1. *Mathematica*'s capabilities for working with graphs together with the extensive and growing Wolfram Knowledgebase are a powerful combination. We will create a semantic network for the word "program" using the Wolfram Knowledgebase.

The `WordData` function will accept a string representing a word.

```
In[30]:= WordData["program"]

Out[30]= {{program,Noun,Performance},{program,Noun,Idea},
  {program,Noun,System},{program,Noun,WrittenDocument},
  {program,Noun,SoftwareSystem},{program,Noun,Show},
  {program,Noun,Information},{program,Noun,Promulgation},
  {program,Verb,Schedule},{program,Verb,CreateMentally}}
```

The output displayed above is a list of lists, with the first member of each list being the word itself, the second being the part of speech, and the third indicating a possible "sense" of the word.

`WordData` accepts a second argument that allows you to get information about specific properties associated with the word. Some of the properties are `"Antonyms"`, `"Synonyms"`, `"BroaderTerms"`, `"NarrowerTerms"`, `"Definitions"` and `"Examples"`.

```
In[31]:= WordData["program","BroaderTerms"]

Out[31]= {{program,Noun,Performance}→{performance},
  {program,Noun,Idea}→{idea,thought},
  {program,Noun,System}→{system,system of rules},
  {program,Noun,WrittenDocument}→
    {document,papers,written document},
  {program,Noun,SoftwareSystem}→{computer software,
    package,software,software package,
    software program,software system},
  {program,Noun,Show}→{show},
  {program,Noun,Information}→{info,information},
  {program,Noun,Promulgation}→
    {announcement,promulgation},
  {program,Verb,Schedule}→{schedule},
  {program,Verb,CreateMentally}→
    {create by mental act,create mentally}}
```

In the graph of the semantic network for the word "program," the terms on the right-hand side of the rules in the output above will be the neighbors of "program." The first step is to extract all of those terms. Fortunately, `WordData` accepts a third argument to specify the form of the output. If we give `"List"` as that third argument, the output will be a list of all the terms related to "program."

```
In[32]:= WordData["program","BroaderTerms","List"]
```

```
Out[32]= {announcement,computer software,create by mental act,
          create mentally,document,idea,info,information,package,
          papers,performance,promulgation,schedule,show,software,
          software package,software program,software system,
          system,system of rules,thought,written document}
```

We will need to create edges between “program” and each of these words, which we can do with `Map`.

```
In[33]:= Map[DirectedEdge["program",#]&,
           WordData["program","BroaderTerms","List"]]

Out[33]= {program→announcement,program→computer software,
          program→create by mental act,program→create mentally,
          program→document,program→idea,program→info,
          program→information,program→package,
          program→papers,program→performance,
          program→promulgation,program→schedule,program→show,
          program→software,program→software package,
          program→software program,program→software system,
          program→system,program→system of rules,
          program→thought,program→written document}
```

This is not yet a very interesting network, so we take it one step further and find the words related to those related to “program.” We use a `Table` to apply the technique above to all the words that appear in the list of broader terms for “program.” The result is rather long, so to save space, we limit the number of lines of output with `Short`.

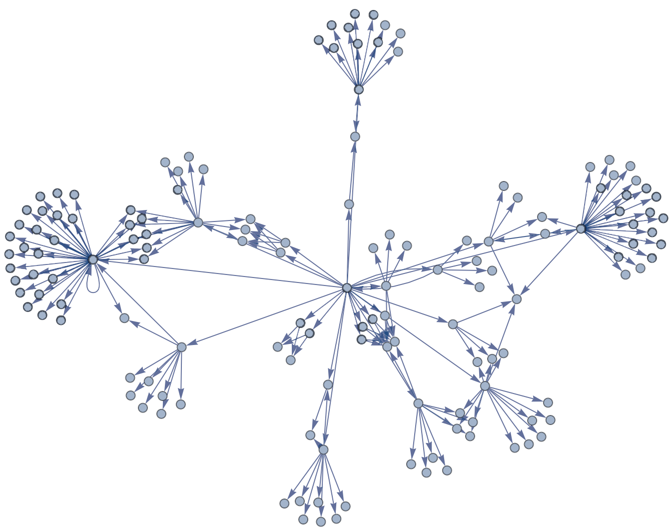
```
In[34]:= Short[
           Table[Map[DirectedEdge[word,#]&,
             WordData[word,"BroaderTerms","List"]],
            {word,WordData["program","BroaderTerms","List"]}],
           10]

Out[34]//Short=
          {{announcement→statement},{computer software→code,
           computer software→computer code},<<18>>,
           {thought→belief,thought→cognitive content,
           thought→content,thought→higher cognitive process,
           thought→mental object},
           {written document→piece of writing,
           written document→writing,
           written document→written material}}
```

Thus, the edges of the graph of this semantic network can be obtained by flattening that result and combining it with the edges incident on “program.” This graph will be too large to display the terms on the graph, so we label the vertices using `Placed` with second argument `Tooltip`, so that if you hover the mouse pointer over a vertex, it will display the term associated with the vertex.

```
In[35]:= Graph[
           Join[Map[DirectedEdge["program",#]&,
             WordData["program","BroaderTerms","List"]],
           Flatten[
              Table[Map[DirectedEdge[word,#]&,
                WordData[word,"BroaderTerms","List"]],
              {word,WordData["program","BroaderTerms","List"]}]]],
          VertexLabels→Placed["Name",Tooltip]]
```

Out[35]=



You might be surprised by the presence of a self-loop in the graph, indicating the presence of a word related to itself. Hovering the mouse over that vertex, you can see that the word in question is "show." Applying `WordData` to "show" with third argument `"Rules"` will give detailed information, including the part of speech and sense of the word on the right-hand side of the rules. Again, this output will be long, so we use `Select` to find those with the word "show" on the right-hand side of the rule.

```
In[36]:= Select[WordData["show","BroaderTerms","Rules"],
           MemberQ[#[[2]],"show",2]&]

Out[36]= {{show,Verb,Register}→
           {{read,Verb,Indicate},{record,Verb,Indicate},
            {register,Verb,Indicate},{show,Verb,Read}},
          {show,Verb,Exhibit}→{{show,Verb,Demonstrate}}}
```

We see that the word "show" is related to itself in two different ways: the sense "Read" is broader than the sense "Register," and the sense "Exhibit" is broader than the sense "Demonstrate."

It is not always clear from the "sense" what specific definition is meant. `WordData` can be applied to a full word specification, that is, the list containing the word, part of speech, and sense. The second argument `"Definition"` or `"Examples"` is useful in this context.

```
In[37]:= WordData[{"show","Verb","Read"},"Definitions"]
```

Out[37]=
```
{{show,Verb,Read}→
  indicate a certain reading; of gauges and instruments}
```

In[38]:=
```
WordData[{"show","Verb","Register"},"Definitions"]
```

Out[38]=
```
{{show,Verb,Register}→give evidence of, as of records}
```

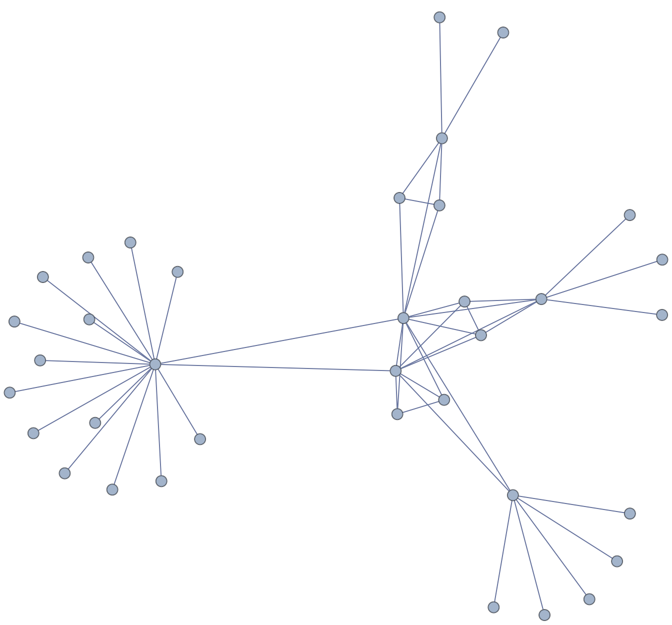
In[39]:=
```
WordData[{"show","Verb","Register"},"Examples"]
```

Out[39]=
```
{{show,Verb,Register}→
  {The diary shows his distress that evening}}
```

Finally, compare the graph produced from broader terms to that consisting of synonyms. Note that, in this case, the edges are undirected, representing the symmetric nature of the synonym relation. Also note that we create the graph with `SimpleGraph` so as to eliminate the multiple edges.

In[40]:=
```
SimpleGraph[
  Join[Map[UndirectedEdge["program",#]&,
    WordData["program","Synonyms","List"]],
   Flatten[
    Table[Map[UndirectedEdge[word,#]&,
      WordData[word,"Synonyms","List"]],
     {word,WordData["program","Synonyms","List"]}]]],
   VertexLabels→Placed["Name",Tooltip]]
```

Out[40]=



GraphPlot

The `GraphPlot` function was introduced in *Mathematica* version 6. This function is older than and mostly superseded by the `Graph` object. However, `GraphPlot` provides some different functionality and is worth a brief discussion. Keep in mind that `GraphPlot` was designed specifically as a way to create images of graphs, before the `Graph` object was introduced, so if your focus is on creating an image of a graph and are having difficulty getting `Graph` to display the graph how you wish, `GraphPlot` is worth considering.

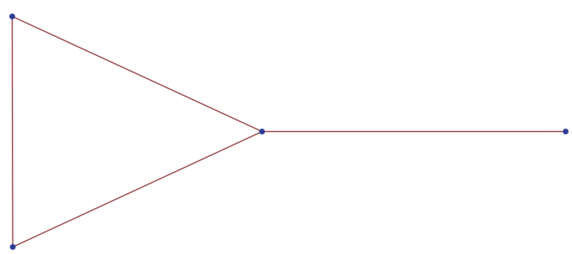
The most important difference between `GraphPlot` and `Graph` is the fact that `Graph` produces a graph object, which can be computed with and manipulated, whereas `GraphPlot` produces a graphics object, which is the same kind of output as `Plot` or `Plot3D` produce.

The basic input for `GraphPlot` is either a list of rules representing the edges of the graph or a `Graph` object. `GraphPlot` does not allow the use of the symbols ⟷ or ↔, nor does it allow the use of `TwoWayRule` (<->).

We will use the graph from Exercise 3 as the example.

In[41]:=
```
GraphPlot[{"a"→"b","a"→"c","b"→"c","b"→"d"}]
```

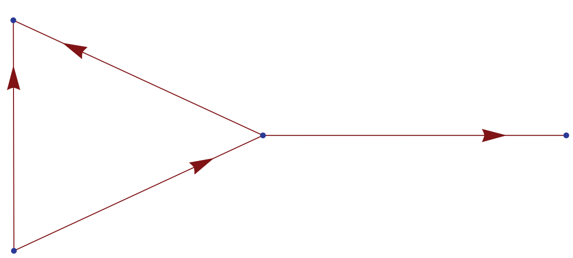
Out[41]=



Right away, you can see an important difference between `GraphPlot` and `Graph`. Specifically, where `Graph`'s default behavior is to interpret rules as directed edges, `GraphPlot` assumes that rules are not directed. To draw a directed graph, you must use the `DirectedEdges` option.

In[42]:=
```
GraphPlot[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  DirectedEdges→True]
```

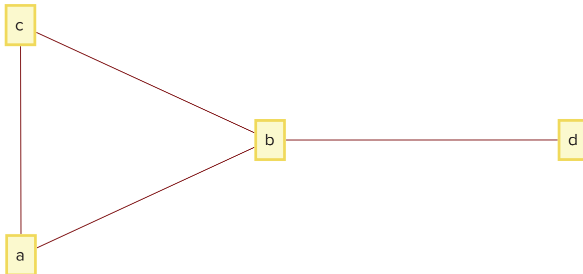
Out[42]=



To display the names of vertices, you use the `VertexLabeling` option. The default behavior, provided the number of vertices is not too large, is for the names of vertices to appear in a tooltip when you move the mouse pointer over the vertex. This option can be explicitly given with value `Tooltip`. By setting the `VertexLabeling` option to `True`, the names will be displayed on the graph itself.

In[43]:=
```
GraphPlot[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  VertexLabeling→True]
```

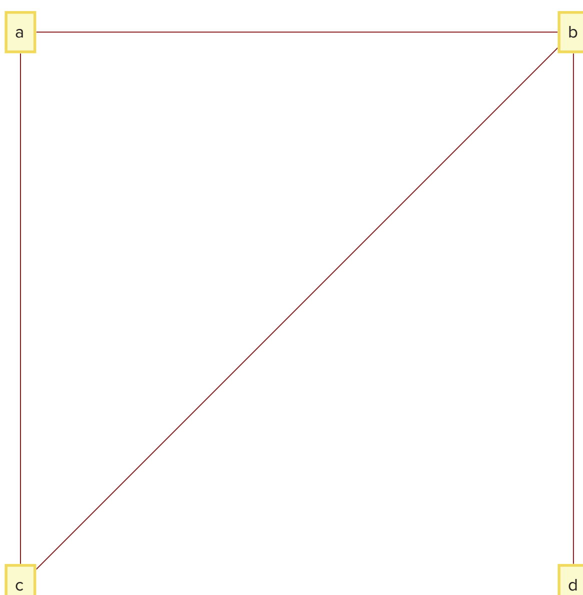
Out[43]=



To explicitly control the placement of vertices, you use the `VertexCoordinateRules` option. The value of this option is a list of rules identifying the name of a vertex with a pair representing *x* and *y* coordinates. You can use the symbol `Automatic` in place of the coordinate pair to indicate that *Mathematica* should determine the location for that vertex automatically. This is the default for vertices that are omitted from the rules. You can also use `Automatic` in place of either the *x* or *y* value within a pair if you wish to specify one value but leave the other to *Mathematica* to determine. We use `VertexCoordinateRules` to rearrange the vertices in the graph from Exercise 3 to match the locations in the textbook.

In[44]:=
```
GraphPlot[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  VertexLabeling→True,
  VertexCoordinateRules→
    {"a"→{0,1},"b"→{1,1},"c"→{0,0},"d"→{1,0}}]
```

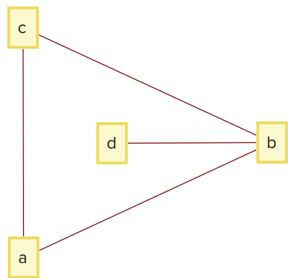
Out[44]=



One of the benefits of `GraphPlot` is that the displayed graphic can be dynamically manipulated. The code below produces default display of the graph from Exercise 3. If you double-click on the image and then double-click on a vertex (so that the vertex is highlighted), you can adjust the position of the vertex by dragging it. Below, we have moved vertex “d” into the center of the graph. Observe that the edge automatically adjusts.

In[45]:=
```
GraphPlot[{"a"→"b","a"→"c","b"→"c","b"→"d"},
  VertexLabeling→True]
```

Out[45]=



10.2 Graph Terminology and Special Types of Graphs

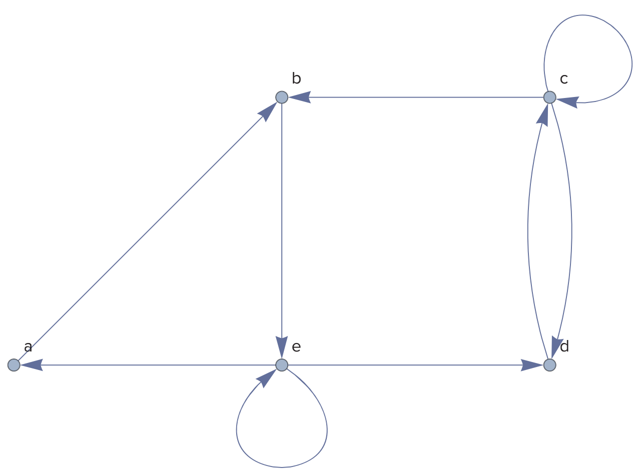
In this section, we will see how to use *Mathematica* to perform computations related to some of the basic terminology of graphs, such as calculating degree. We will also look at some of the special families of graphs included in the Wolfram Language. Finally, we will discuss subgraphs and unions of graphs.

Degree

For a `Graph` object, the Wolfram Language includes the function `VertexDegree` for determining the degree of a vertex. Given a `Graph` object and one of the graph's vertices, the function returns the number of edges incident to that vertex. For example, we can check the degrees of vertices *a* and *e* of `exercise7` from the previous section.

In[47]:=
```
exercise7
```

Out[47]=



In[48]:=
```
VertexDegree[exercise7,"a"]
```

Out[48]= 2

In[49]:=
```
VertexDegree[exercise7,"e"]
```

Out[49]= 5

Observe that the loop at the vertex e counts as 2 towards the degree of that vertex. Moreover, note that with this directed graph, `VertexDegree` calculates the number of edges incident to the given vertex without regard for their direction. The Wolfram Language includes `VertexInDegree` and `VertexOutDegree` for calculating the directed values. As an example, consider vertex d from above.

In[50]:=
```
VertexInDegree[exercise7,"d"]
```
Out[50]=
```
2
```
In[51]:=
```
VertexOutDegree[exercise7,"d"]
```
Out[51]=
```
1
```

All three of these functions can be used without a second argument. If they are passed only the graph as the sole argument, they will return a list of the degrees of the vertices. Note that the output is in the same order as the output from `VertexList`.

In[52]:=
```
VertexList[exercise7]
```
Out[52]=
```
{a,b,c,d,e}
```
In[53]:=
```
VertexDegree[exercise7]
```
Out[53]=
```
{2,3,5,3,5}
```
In[54]:=
```
VertexInDegree[exercise7]
```
Out[54]=
```
{1,2,2,2,2}
```
In[55]:=
```
VertexOutDegree[exercise7]
```
Out[55]=
```
{1,1,3,1,3}
```

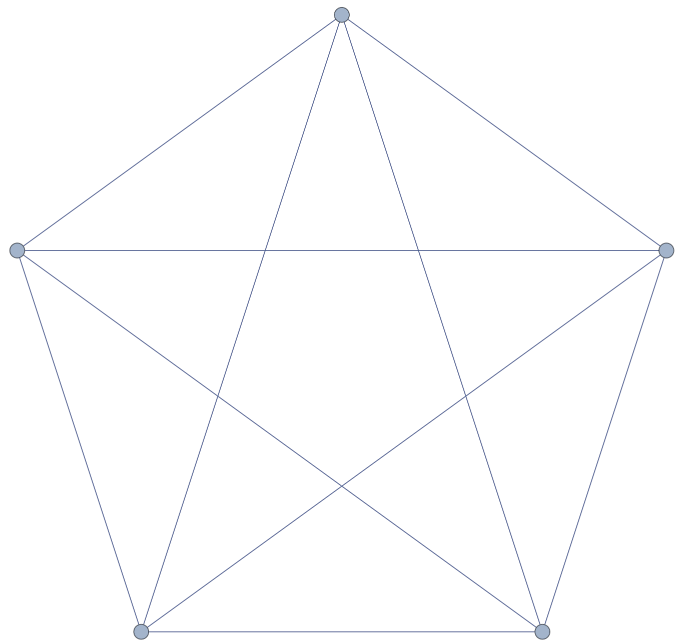
Some Special Simple Graphs

The textbook discusses several families of graphs, including complete graphs, cycles, and wheels. The Wolfram Language provides functions for easily creating these and other special graphs.

We begin with complete graphs. Recall that a complete graph is a simple, undirected graph on a given number of vertices that has all possible edges between those vertices. The complete graph on n vertices is denoted K_n. The complete graph on n vertices can be obtained with the function `CompleteGraph` applied to n. For example, we can generate and display K_5, the complete graph on 5 vertices.

In[56]:=
```
CompleteGraph[5]
```

Out[56]=

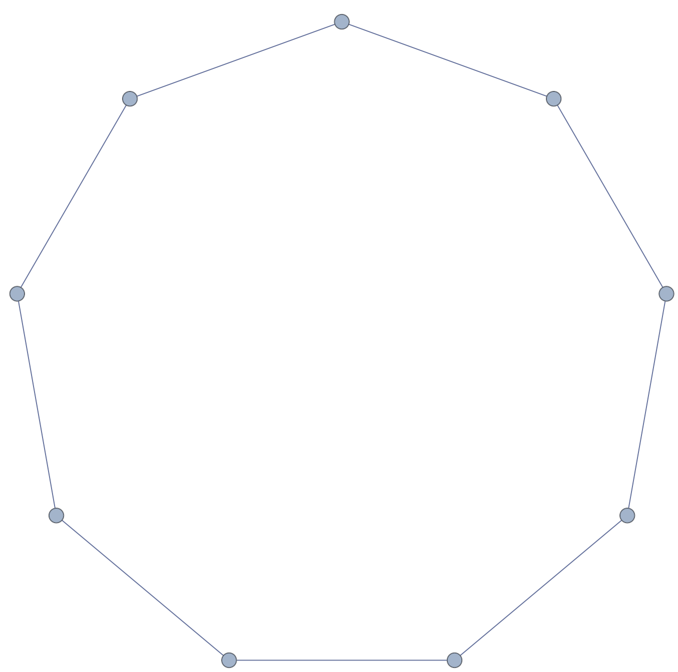


Similarly, the cycle C_n is obtained with the function `CycleGraph`.

In[57]:=

```
CycleGraph[9]
```

Out[57]=

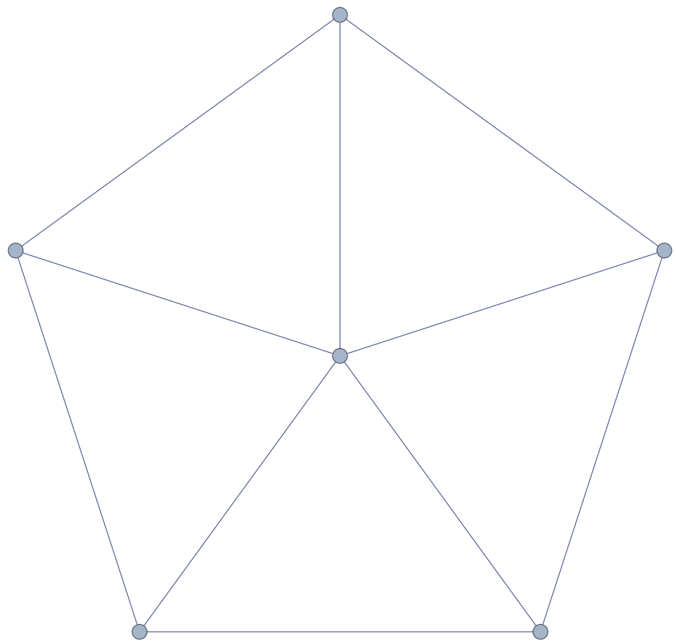


A wheel W_n is obtained from the cycle graph C_n by adding one additional vertex adjacent to all n of the original vertices. In the Wolfram Language, wheel graphs are obtained by `WheelGraph` applied to the value $n + 1$, the total number of vertices in the wheel, not just the outside ring.

In[58]:=

```
WheelGraph[6]
```

Out[58]=

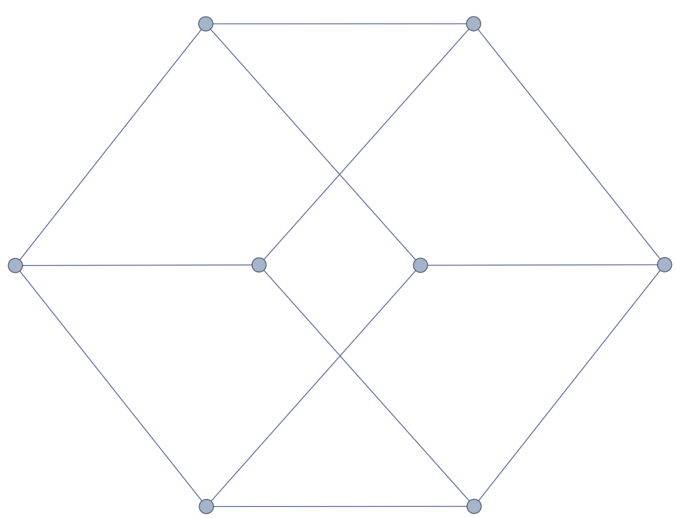


Hypercubes

To construct the n-cube Q_n, we use the `HypercubeGraph` function applied to the dimension n. Recall the definition of the hypercube graph given in Example 8 of Section 10.2. There are 2^n vertices labeled with the binary representations of the numbers 0 through $2^n - 1$. Two vertices are adjacent if their binary representations differ in only one digit. Here is the presentation of the three-dimensional cube.

In[59]:=
```
HypercubeGraph[3]
```

Out[59]=

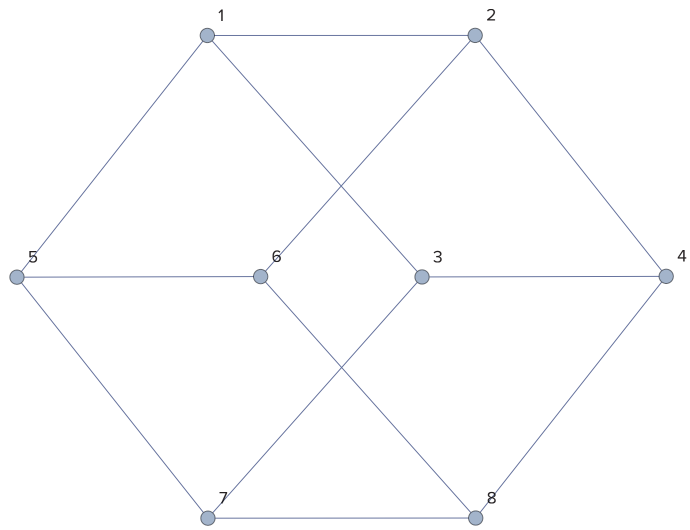


By default, the vertices are not labeled. To have label the vertices, we can use the `VertexLabels` option. While the named parametric graphs generally accept all the same options that `Graph` does, `HypercubeGraph` in particular is more robust passed to `Graph`.

In[61]:=

```
Graph[HypercubeGraph[3],VertexLabels→"Name"]
```

Out[61]=



One more modification will allow us to see the connection between this image and the definition. Instead of using `"Name"` as the argument to `VertexLabels`, we can specify the labels explicitly by setting the option to a list of rules identifying the integers with the binary expression.

The definition of Q_n tells us that the vertices should be considered to be the binary representations of the integers from 0 to 7. To display this, we will apply labels by subtracting 1 from the integer vertex names and using the `IntegerString` function to obtain the binary representation. The `IntegerString` function requires two arguments, an integer and a base, and produces a string representing the integer in that base. A third optional argument allows you to specify a minimal length for the string.

The following `Table` produces a list of rules identifying the vertex names with the appropriate label.

In[61]:=

```
Table[v→IntegerString[v-1,2,3],{v,8}]
```

Out[61]=

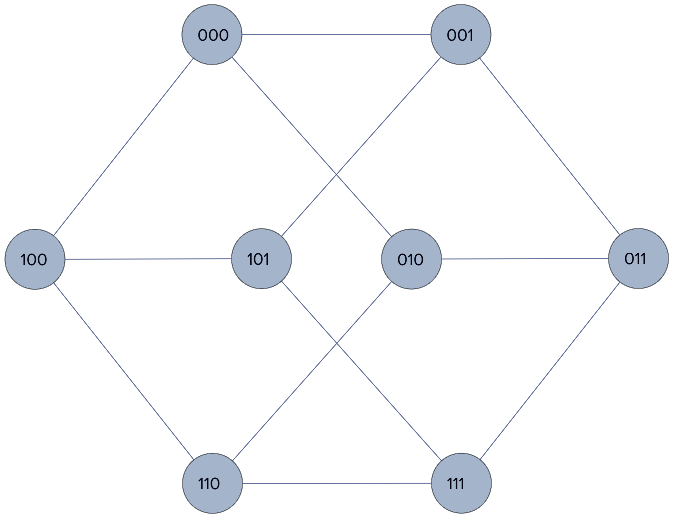
```
{1→000,2→001,3→010,
  4→011,5→100,6→101,7→110,8→111}
```

We can combined this with the `Placed` symbol to move the labels inside the vertices. Be warned that, while this is an effective method in general, it can cause the kernel to crash when passed as an option directly to `HypercubeGraph`. If this happens, passing the `HypercubeGraph` as an option to the more general `Graph` can fix the problem.

In[62]:=

```
Graph[HypercubeGraph[3],
  VertexLabels→
    Table[v→Placed[IntegerString[v-1,2,3],Center],
      {v,8}],VertexSize→Large]
```

Out[62]=



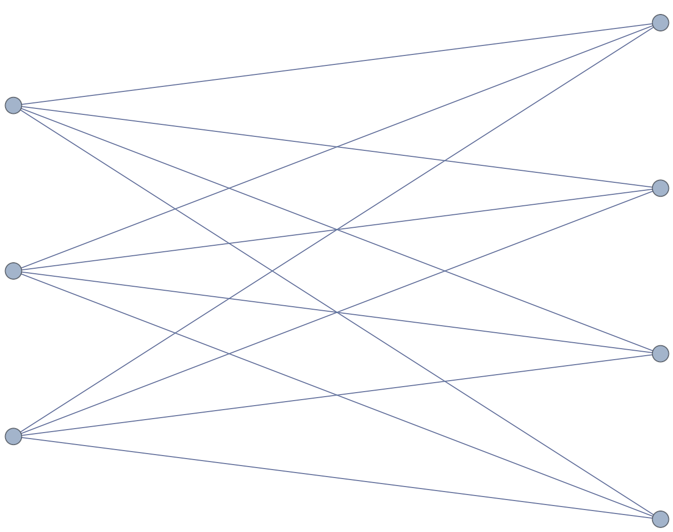
Bipartite Graphs

Another important class of graphs is the bipartite graphs. A bipartite graph is one whose vertex set can be partitioned into two disjoint sets such that every edge has one vertex in each of the partitioning sets. In other words, no two vertices in the same partitioning set are adjacent. We write $V = (A, B)$ to indicate that the vertex set V is partitioned into the sets A and B.

The complete bipartite graph $K_{m,n}$ is a bipartite graph with bipartition $V = (A, B)$ such that there are m vertices in A and n in B and such that there is an edge for every pair of vertices $a \in A$ and $b \in B$. The `CompleteGraph` function can be used to create complete bipartite graphs. The argument is the list consisting of the pair of m and n.

In[63]:=
```
CompleteGraph[{3,4}]
```

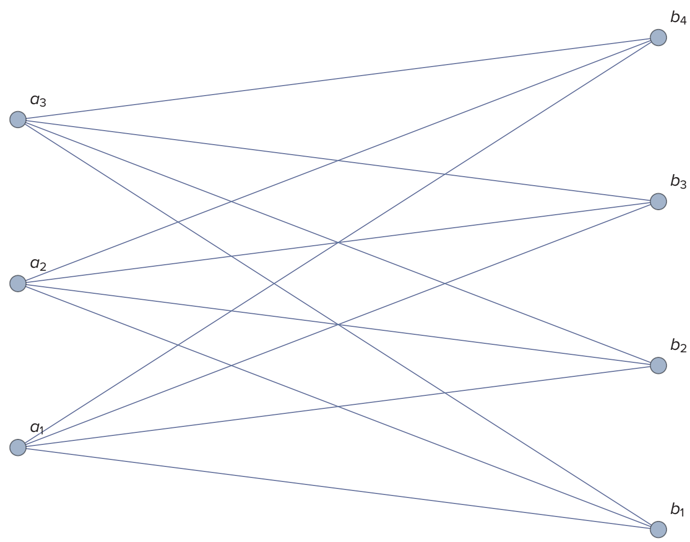
Out[63]=



Notice that *Mathematica* draws the complete bipartite graph with the two partitioning sets along the left and right to make the partition visually clear. As with the other functions in this section, the usual options for `Graph` apply. We illustrate how to label the vertices in a meaningful way. To enter a subscript in *Mathematica*, press CTRL and -.

In[64]:=
```
CompleteGraph[{3,4},
  VertexLabels→Union[Table[i→a_i,{i,3}],
    Table[i→b_{i-3},{i,4,7}]]]
```

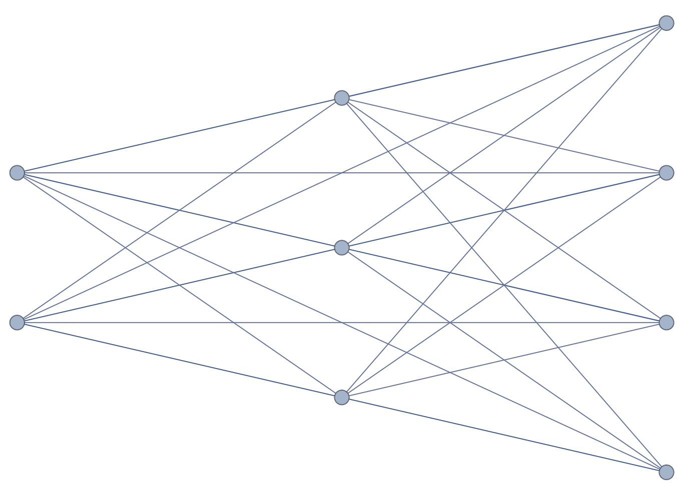
Out[64]=



The Wolfram Language also supports complete multipartite graphs. A *k*-partite graph is a graph in which the vertices can be partitioned into *k* disjoint sets so that no two vertices in any one of the partitioning sets are adjacent.

In[65]:=
```
CompleteGraph[{2,3,4}]
```

Out[65]=



The function `BipartiteGraphQ` determines whether a given graph is bipartite. This function accepts a graph as its sole argument and returns `True` if the graph is bipartite.

In[66]:=
```
BipartiteGraphQ[HypercubeGraph[3]]
```

Out[66]=
```
True
```

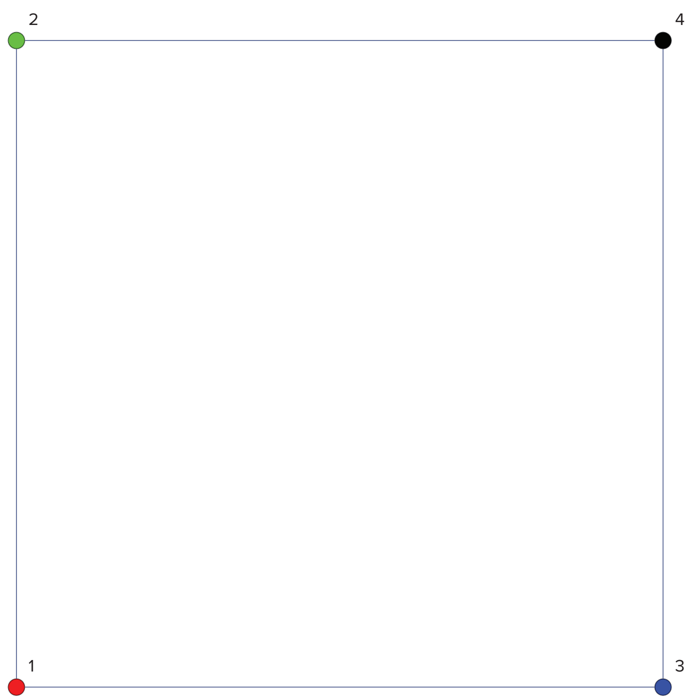
Bipartition

It is worthwhile, however, to create a version of `BipartiteGraphQ` from scratch in order to better understand both the representation of graphs in the Wolfram Language and an algorithm that determines whether the graph is bipartite and finds a bipartition. Our function will apply to a `Graph` object. Moreover, instead of just returning true, our function will, if the graph is bipartite, display the graph with the vertices colored red and green to represent the partitioning. Of course, if the graph is not bipartite, the function will return false.

For `Graph` objects, the colors of vertices can be changed by setting the `VertexStyle` option to a single style for global changes or to a list of rules to set options for individual vertices as is shown below.

```
In[67]:= Graph[HypercubeGraph[2],VertexLabels→"Name",
           VertexStyle→{1→Red,2→Green,3→Blue,4→Black}]
```

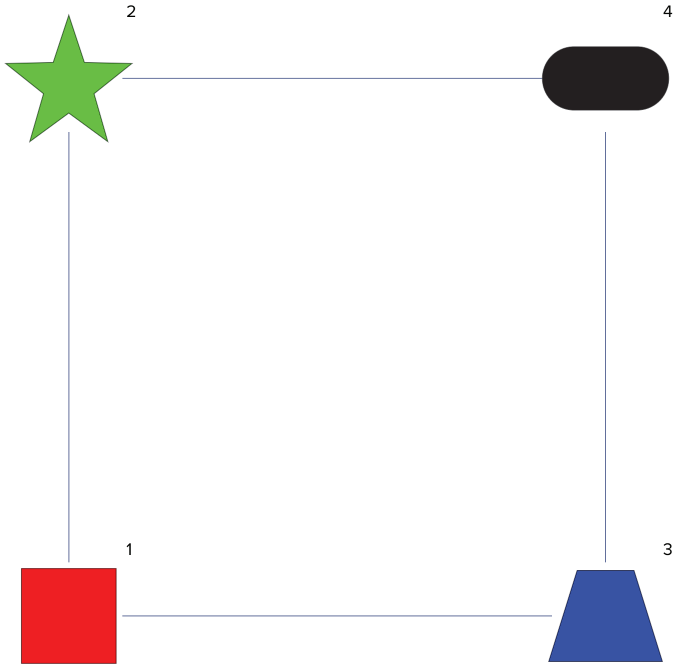
Out[67]=



In addition to the color, the shape of the vertices can be changed using the `VertexShapeFunction` option. This option can be used to take full control over the drawing of vertices by defining a function that builds the vertex out of graphics primitives. Fortunately, there are also a variety of predefined shape functions, as illustrated below.

```
In[68]:= Graph[HypercubeGraph[2],VertexLabels→"Name",
           VertexStyle→{1→Red,2→Green,3→Blue,4→Black},
           VertexShapeFunction→
             {1→"Square",2->"Star",3->"UpTrapezoid",
             4→"Capsule"},VertexSize→Medium]
```

Out[68]=



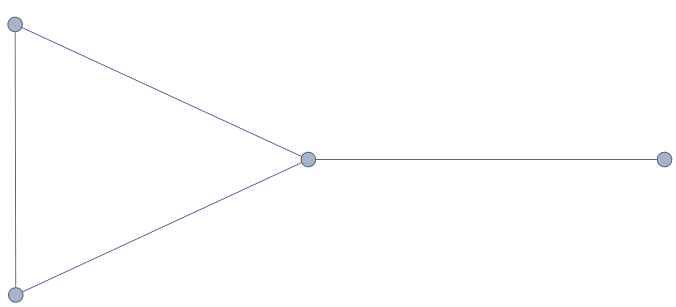
Our version of `BipartiteGraphQ` will need to determine the list of neighbors of a given vertex. For a `Graph` object, the built-in function `AdjacencyList` applied to a graph and a vertex will return the list of vertices adjacent to it. However, we will create our own version of this to better illustrate how to manipulate graphs on a basic level.

To find the neighbors of a vertex in a `Graph` object, we need to find all of the edges containing the given vertex and output the other vertex in the edge. The `Cases` function can be used for this. The first argument to `Cases` is a list of expressions. The second argument is a pattern expressing which elements of the first argument should be output. In our case, we want to pick out those edges, one of whose elements is the desired vertex.

Before applying `Cases` we need to extract the edges from the `Graph` and determine what patterns to look for. The `EdgeList` function returns a list of the edges of a graph. If we apply `FullForm` as well, we can see the internal structure. We use the graph from Exercise 3 in Section 10.1, which we defined above, as an example.

In[69]:=
```
exercise3
```

Out[69]=



In[70]:=
```
EdgeList[exercise3]//FullForm
```

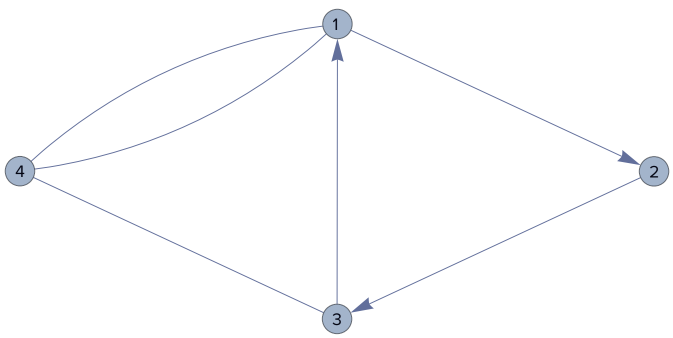
Out[70]//FullForm=
```
List[UndirectedEdge["a","b"],UndirectedEdge["a","c"],
  UndirectedEdge["b","c"],UndirectedEdge["b","d"]]
```

Observe that this is a list of pairs of vertices with the head `UndirectedEdge`. If the edges were directed, the head would be `DirectedEdge`.

Consider the following graph, with both directed and undirected edges.

In[71]:=
```
partitionExample=
  Graph[{1→2,2→3,3→1,3↔4,1↔4,1↔4},
   VertexLabels→Placed["Name",Center],VertexSize→Small]
```

Out[71]=



Suppose we are looking for edges involving the vertex 3. Those edges will either be of the form ***Head***`[3,_]` or ***Head***`[_,3]`, where ***Head*** is either `DirectedEdge` or `UndirectedEdge`. In a pattern, we use the `Alternatives` (`|`) operator to combined two or more possibilities in a single pattern. The following picks out all of the edges incident to vertex 3 in `partitionExample`.

In[72]:=
```
EdgeList[partitionExample]
```

Out[72]=
```
{1→2,2→3,3→1,3–4,1–4,1–4}
```

In[73]:=
```
Cases[EdgeList[partitionExample],
  DirectedEdge[3,_]|DirectedEdge[_,3]|
  UndirectedEdge[3,_]|UndirectedEdge[_,3]]
```

Out[73]=
```
{2→3,3→1,3–4}
```

Note that these edges form the neighborhood graph of vertex 3, that is, the subgraph consisting of vertex 3 and all of its neighbors. However, our interest is the vertices, not the edges. To obtain the vertices, we will take advantage of another feature of the `Cases` function: the second argument can be given as a rule. The left-hand side of the rule is the pattern expressing which elements of the original list should match and the right-hand side describes what to include in the output for that matching element. Since we want the other vertex in the output, not the entire rule, we will name the blanks in the rule and output the vertex.

In[74]:=
```
Cases[EdgeList[partitionExample],
  DirectedEdge[3,x_]|DirectedEdge[x_,3]|
   UndirectedEdge[3,x_]|UndirectedEdge[x_,3]:→x]
```

Out[74]= {2,1,4}

Note that we use RuleDelayed (:>, :→) to protect against the symbol x having been assigned a value elsewhere. It is a good idea to use SetDelayed (:=) and RuleDelayed (:>, :→) when named blanks are involved.

Applying Union will remove duplicates and sort the results. Replacing the example data with arguments allows us to create a function.

In[75]:=
```
neighbors[g_Graph,v_]:=
  Union[Cases[EdgeList[g],
    DirectedEdge[v,x_]|DirectedEdge[x_,v]|
      UndirectedEdge[v,x_]|UndirectedEdge[x_,v]:→x]]
```

We now turn to our version of BipartiteGraphQ. The idea of the function is as follows. (Note that this method is based on forming a spanning tree of the graph, a concept discussed in Section 11.4 of the textbook).

1. Pick a vertex v from the vertex set and place it in the set A.
2. Place all of v's neighbors in set B.
3. For each vertex w in the set B that has not already been processed, place all of w's neighbors that are not already in either set into the set A.
4. Repeat step 3, reversing A and B until no more vertices remain to be processed.
5. Once step 4 is complete, we have formed a disjoint partition of the vertices. We then examine each edge of the graph and ensure that no edge has both ends in A or both ends in B. If some edge fails that test, then the graph is not bipartite. If all of the edges do pass the test, then the graph is bipartite and (A, B) is a bipartition.

In the implementation, rather than using two sets A and B, we use the symbol AB as an indexed variable, with AB[0] representing one set and AB[1] the other. This allows us to refer to them both as AB[i] and vary i.

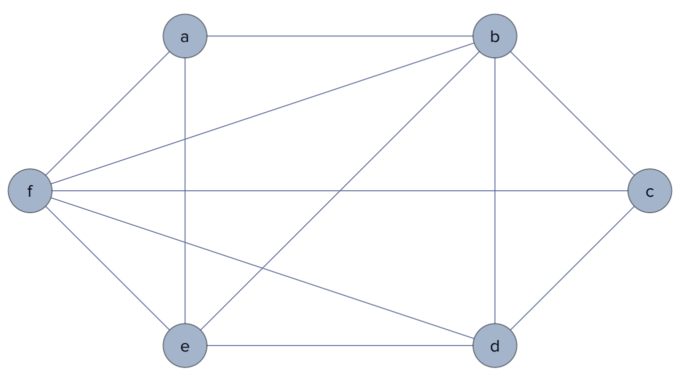
In[76]:=
```
drawBipartite[g_Graph]:=Module[{V,AB,i,T,w,e},
  V=VertexList[g];
  w=V[[1]];
  AB[0]={w};
  AB[1]={};
  i=0;
  While[V≠{},
    T=Intersection[V,AB[i]];
    i=Mod[i+1,2];
    Do[
      AB[i]=Union[AB[i],Complement[neighbors[g,w],
        Union[AB[0],AB[1]]]]
      ,{w,T}];
    V=Complement[V,T]
  ];
  Catch[
    Do[
```

```
      If[(MemberQ[AB[0],e[[1]]]&&MemberQ[AB[0],e[[2]]])||
        (MemberQ[AB[1],e[[1]]]&&MemberQ[AB[1],e[[2]]]),
       Throw[False]]
      ,{e,EdgeList[g]}];
    Graph[g,
      VertexStyle→
        Union[Table[i→Green,{i,AB[0]}],
          Table[i→Yellow,{i,AB[1]}]],
      VertexShapeFunction→Table[i→"Square",{i,AB[0]}]]
  ]
]
```

Note that if the graph is not bipartite, such as the graph *H* from Figure 8 in Section 10.3 of the textbook, the function will output false.

In[77]:=
```
figure8H=Graph[{"a","b","c","d","e","f"},
  {"a"→"b","a"→"e","a"→"f","b"→"c","b"→"d",
  "b"→"e","b"→"f","c"→"d","c"→"f","d"→"e",
  "d"→"f","e"→"f"},DirectedEdges→False,
  VertexCoordinates→
    {{1,2},{3,2},{4,1},{3,0},{1,0},{0,1}},
  VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

Out[77]=



In[78]:=
```
drawBipartite[figure8H]
```

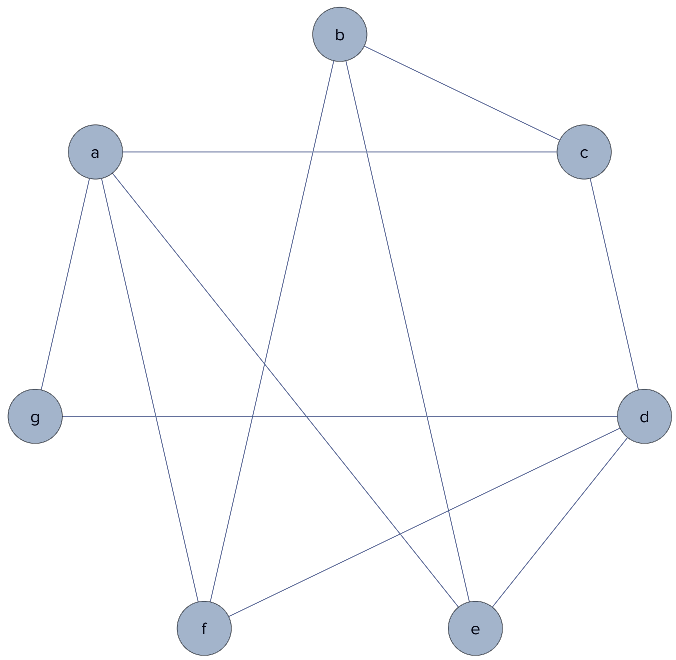
Out[78]=
```
False
```

On the other hand, if the graph is bipartite, such as the graph *G* shown in Figure 8 from Section 10.3, the function will draw the graph with the vertices colored to illustrate the bipartition. Note the use of `GraphLayout` in the initial creation of graph *G* below. The help page lists possible layout options. Of course, you can always specify vertex locations manually with `VertexCoordinates`, as above, or omit both options and *Mathematica* will choose a placement for you.

In[79]:=
```
figure8G=Graph[{"a","g","f","e","d","c","b"},
  {"a"→"c","a"→"e","a"→"f","a"→"g","b"→"c",
  "b"→"e","b"→"f","c"→"d","d"→"e","d"→"f",
```

```
"d"→"g"},DirectedEdges→False,
GraphLayout→"CircularEmbedding",
VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

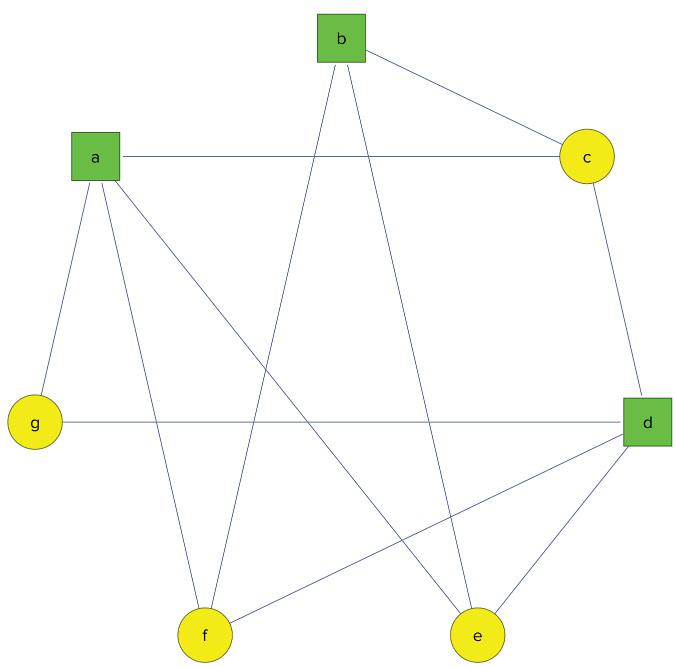
Out[79]=



In[80]:=

```
drawBipartite[figure8G]
```

Out[80]=



Observe that applying our function has preserved the layout of the original. This is typical for Wolfram Language functions that apply to `Graph` objects. Because the options are part of the data that make up the `Graph`, the options are passed along to any functions that act on the `Graph`.

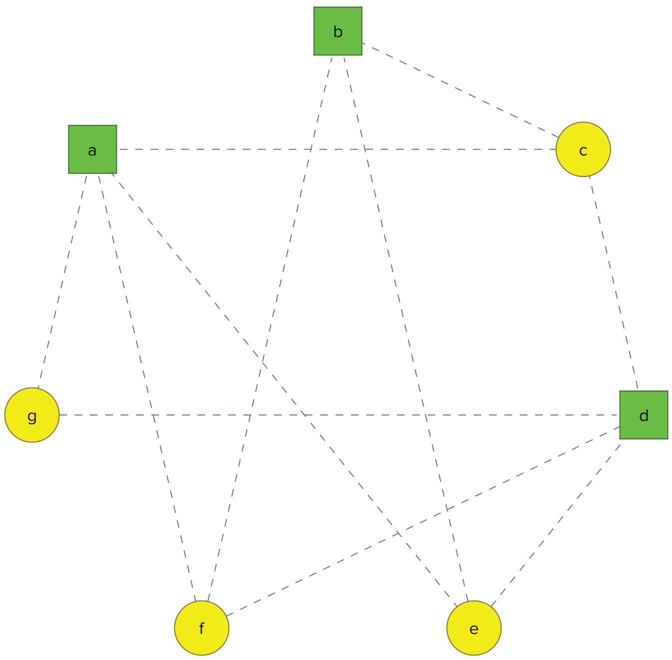
For convenience, we define a version of `drawBipartite` that can accept additional `Graph` options and pass them through our function to where we create the new `Graph`. The easiest way to do this is to add a `BlankNullSequence` (`___`) to the arguments of our function. The result is that the function requires a `Graph` as the first argument, but can accept any number of additional arguments or no additional arguments. We then just pass these to `Graph`.

In[81]:=
```
drawBipartite[g_Graph,opts___]:=
  Module[{V,AB,i,T,w,e},
    V=VertexList[g];
    w=V[[1]];
    AB[0]={w};
    AB[1]={};
    i=0;
    While[V≠{},
      T=Intersection[V,AB[i]];
      i=Mod[i+1,2];
      Do[
        AB[i]=Union[AB[i],Complement[neighbors[g,w],
          Union[AB[0],AB[1]]]]
        ,{w,T}];
      V=Complement[V,T]
    ];
    Catch[
      Do[
        If[(MemberQ[AB[0],e[[1]]]&&MemberQ[AB[0],e[[2]]])||
            (MemberQ[AB[1],e[[1]]]&&MemberQ[AB[1],e[[2]]]),
          Throw[False]]
        ,{e,EdgeList[g]}];
      Graph[g,VertexStyle→
        Union[Table[i→Green,{i,AB[0]}],
          Table[i→Yellow,{i,AB[1]}]],
        VertexShapeFunction→Table[i→"Square",{i,AB[0]}],
        opts]
    ]
  ]
```

For example, we might want to draw the edges dashed.

In[82]:=
```
drawBipartite[figure8G,EdgeStyle→Dashed]
```

Out[82]=



Bipartite Graphs and Matchings

Mathematica can help us find maximal matchings in a bipartite graph. We will use Figure 10a from Section 10.2 of the text as an example. To improve readability, we have abbreviated the names to their first letter and shortened the descriptions of the jobs.

In[83]:=
```
figure10aEdges={"A"→"req","A"→"test","B"→"arch",
  "B"→"imp","B"→"test","C"→"req","C"→"arch",
  "C"→"imp","D"→"req"}
```

Out[83]=
```
{A→req,A→test,B→arch,B→imp,
  B→test,C→req,C→arch,C→imp,D→req}
```

In order to draw the graph meaningfully, in the same fashion as in the text, we will specify the coordinates of each vertex. This is done by setting the `VertexCoordinates` option to a list of coordinates, with the order of the list matching the order of the vertices. To ensure that our order is correct, we will specify a vertex list when we create the `Graph`.

In[84]:=
```
figure10aVertices=
  {"A","B","C","D","req","test","arch","imp"}
```

Out[84]=
```
{A,B,C,D,req,test,arch,imp}
```

To create the list of coordinates, we use `Table` twice and specify that the names should have y-coordinate 1 and the jobs should have y-coordinate 0. `Join` is used to combine the two lists.

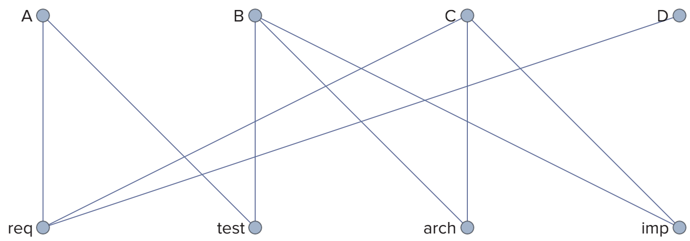
In[85]:=
```
figure10aCoordinates=
  Join[Table[{i,1},{i,4}],Table[{i,0},{i,4}]]
```

Out[85]=
```
{{1,1},{2,1},{3,1},{4,1},{1,0},{2,0},{3,0},{4,0}}
```

We now create the graph.

```
In[86]:= figure10a=Graph[figure10aVertices,figure10aEdges,
           DirectedEdges→False,
           VertexLabels→Placed["Name",Before],
           VertexCoordinates→figure10aCoordinates]
```

Out[86]=



To find a maximal matching, we use the function `FindIndependentEdgeSet`. The term *independent edge set* is synonymous with matching. The only allowed argument to this function is the graph. It returns a list of edges in a matching.

```
In[87]:= figure10aMatching=FindIndependentEdgeSet[figure10a]
```

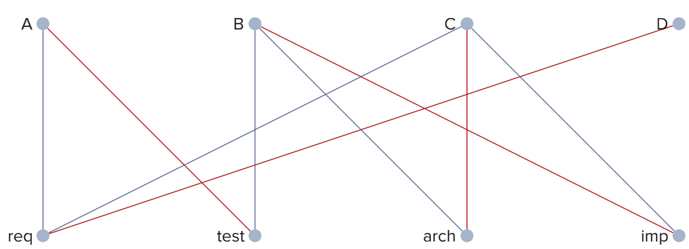
```
Out[87]= {A⟷test,B⟷imp,C⟷arch,D⟷req}
```

The output indicates that one maximal matching has Alvarez assigned to testing, Berkowitz to implementation, Chen to architecture, and Davis to requirements.

We can visualize this matching by highlighting the edges that form the matching using the function `HighlightGraph`. This function requires two arguments. The first is a graph. The second is a list of the elements to highlight. In this case, the second argument will be the output from `FindIndependentEdgeSet`.

```
In[88]:= HighlightGraph[figure10a,figure10aMatching]
```

Out[88]=

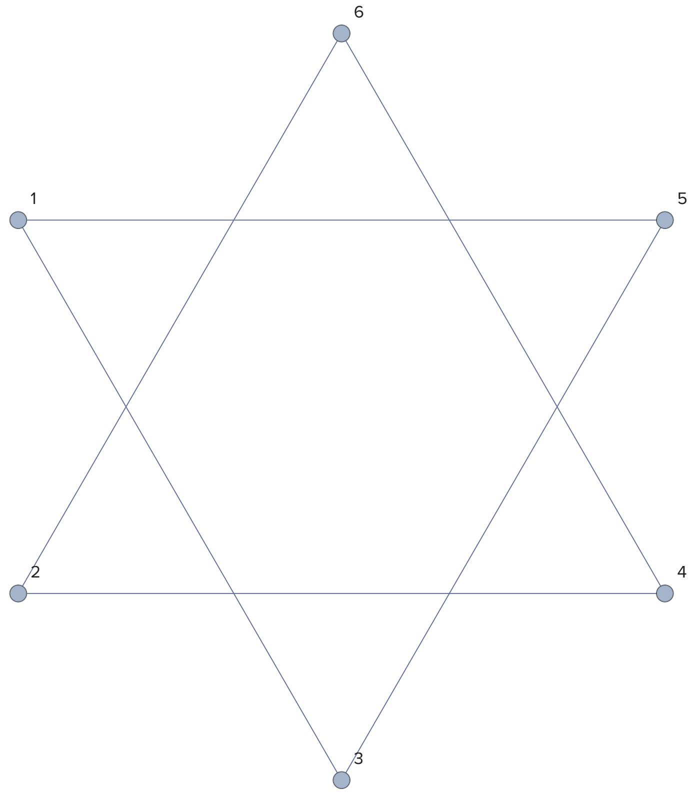


Subgraphs and Induced Subgraphs

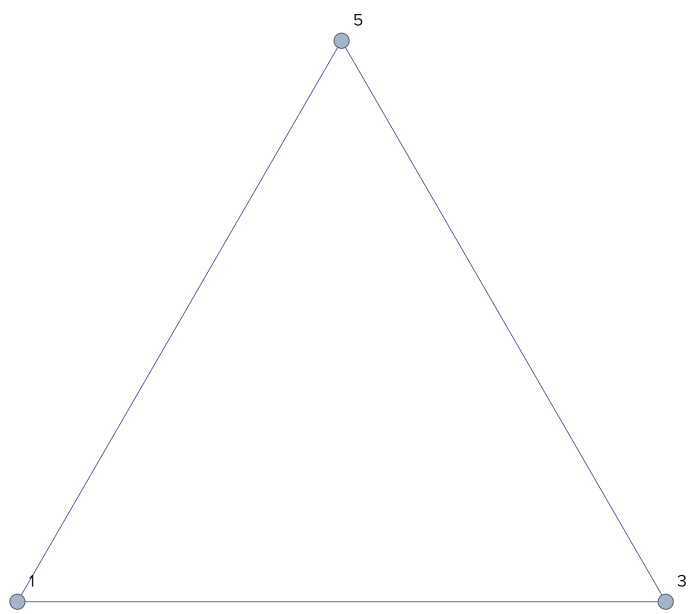
The Wolfram Language provides the `Subgraph` function for producing a subgraph from an existing graph. Given a graph and a list of edges, `Subgraph` produces the `Graph` consisting of the edges and all the vertices that are an endpoint of one of the given edges.

For example, we create the circulant graph with 6 vertices and jump 2, and then create the subgraph consisting of one of triangles. A circulant graph on n vertices with jump j is a graph with vertices $\{1, 2, \ldots, n\}$, typically arranged in a circle, and such that vertex u is connected to vertex v whenever $u \neq v$ and $u \equiv v \bmod j$. Visually, each vertex is connected to the vertex j further around the circle.

In[89]:=
```
star=CirculantGraph[6,2,VertexLabels→"Name"]
```
Out[89]=

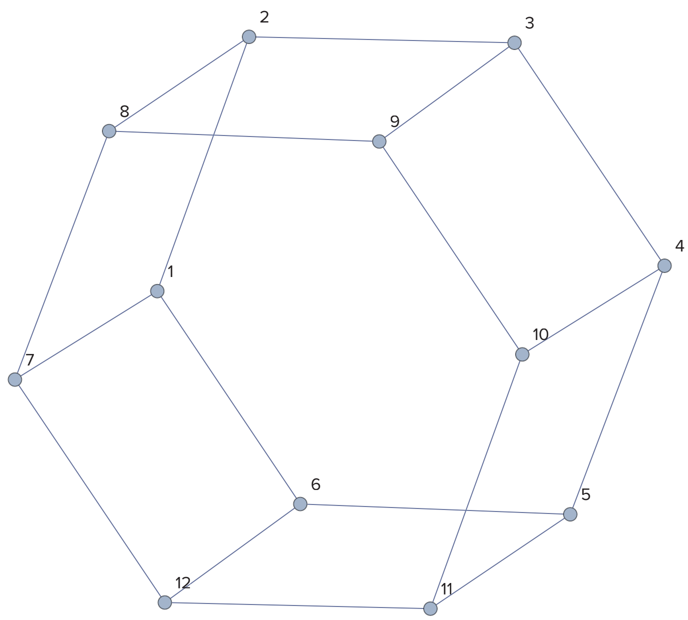


In[90]:=
```
substar=Subgraph[star,{1→3,3→5,5→1},
  VertexLabels→"Name"]
```
Out[90]=

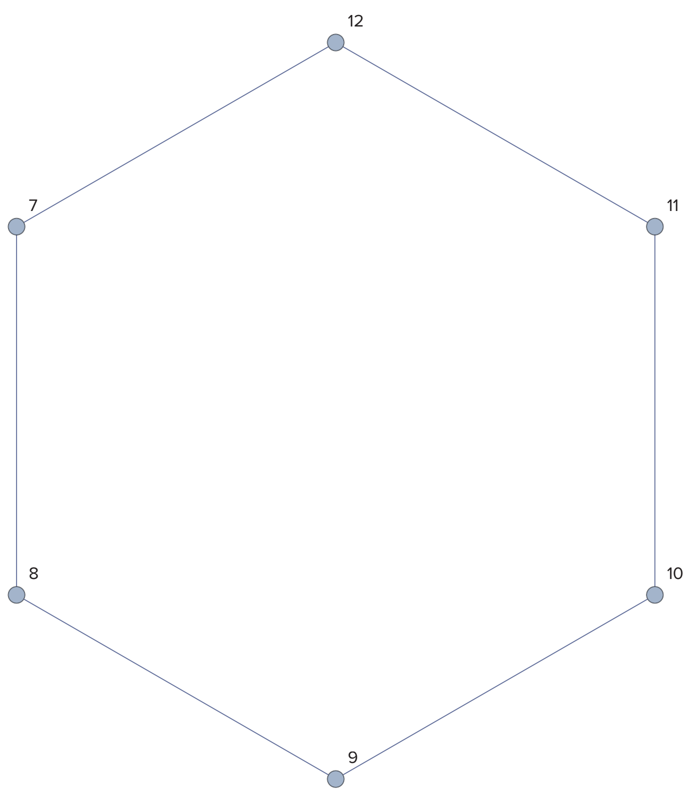


Alternatively, you can give a list of vertices as the second argument to `Subgraph`. The result is the graph induced by the given vertices, that is, the graph consisting of the vertices and all the edges from the original graph with both endpoints in the set of vertices. Below we consider a prism graph and one of its layers.

In[91]:=
```
prism=
  Graph[{1→2,2→3,3→4,4→5,5→6,6→1,7→8,
    8→9,9→10,10→11,11→12,12→7,1→7,2→8,
    3→9,4→10,5→11,6→12},DirectedEdges→False,
    VertexLabels→"Name"]
```
Out[91]=



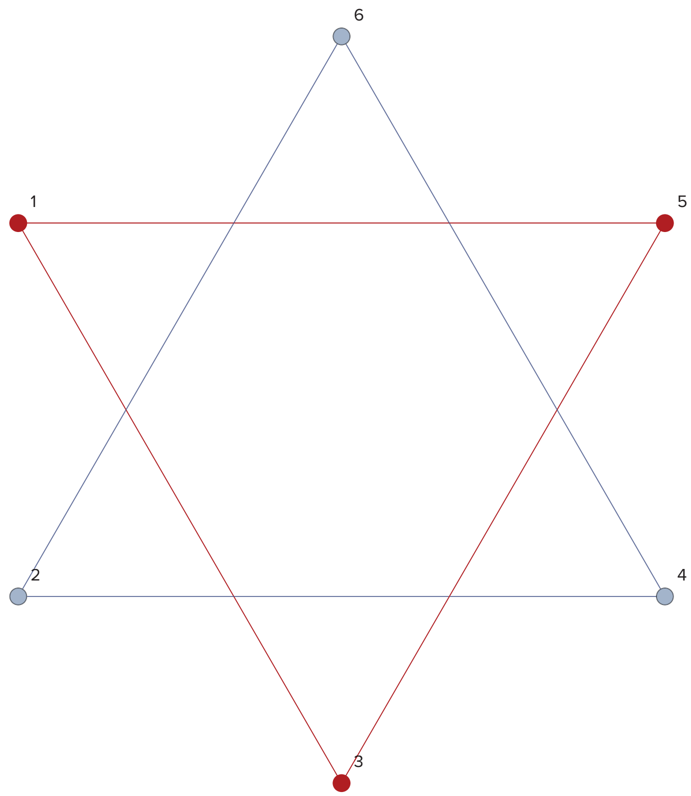
In[92]:=
```
subprism=Subgraph[prism,{7,8,9,10,11,12},
  VertexLabels→"Name"]
```
Out[92]=



The `HighlightGraph` function can be used to display a subgraph relative to the original by passing the original graph as the first argument and the subgraph as the second.

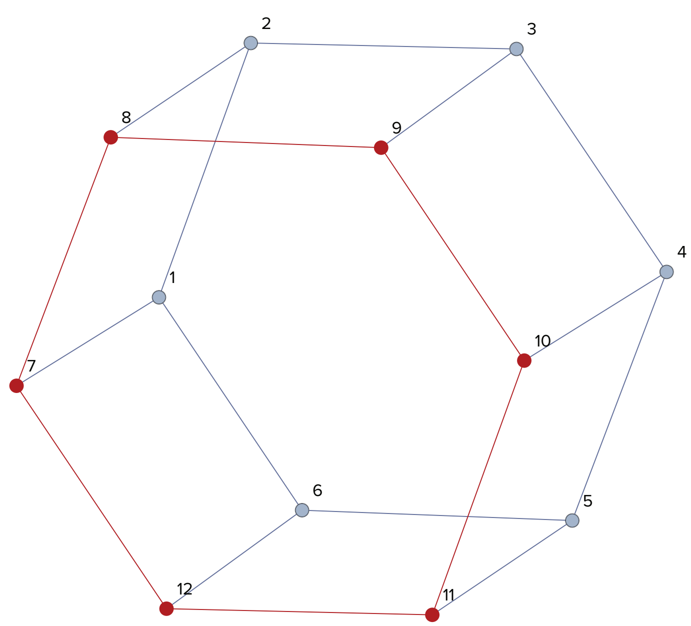
In[93]:=
```
HighlightGraph[star,substar]
```

Out[93]=



In[94]:=
```
HighlightGraph[prism,subprism]
```

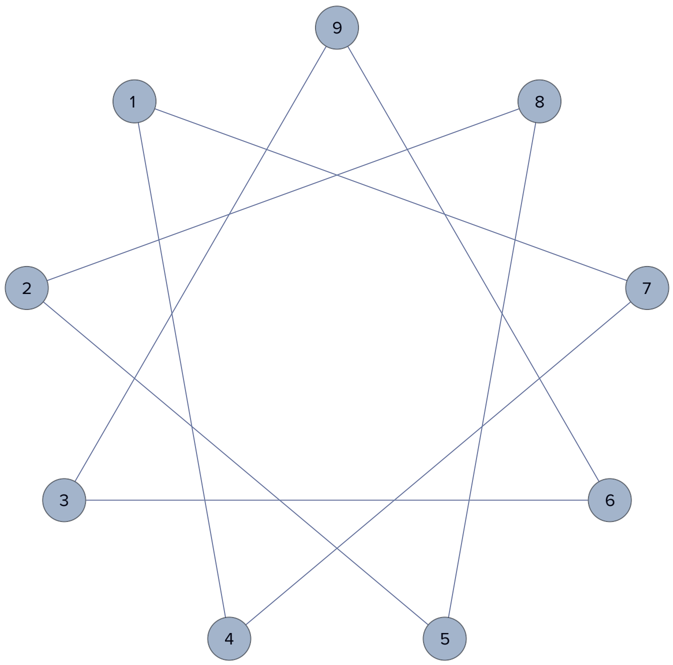
Out[94]=



Note that the second argument of `HighlightGraph` can be a list of subgraphs. Below, we highlight the three triangles in the circulant graph on 9 vertices with jump 3.

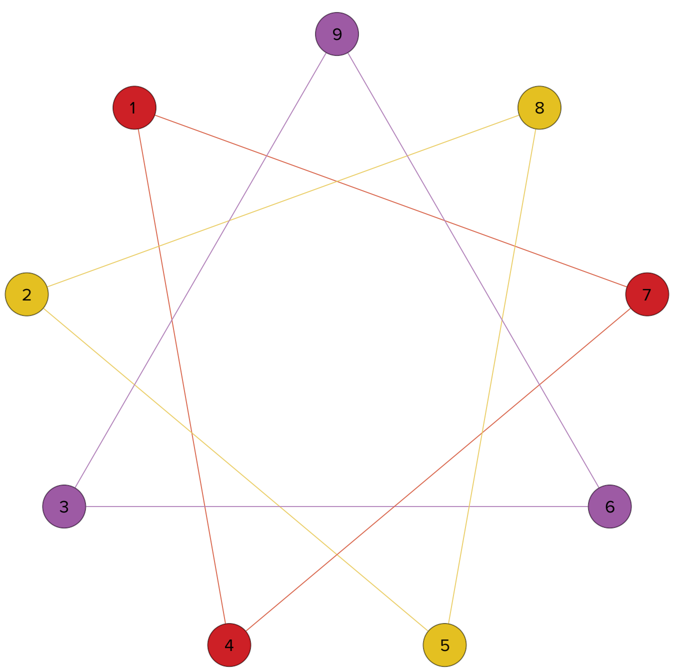
In[95]:=
```
circulant9=CirculantGraph[9,3,
  VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

Out[95]=



In[96]:=
```
HighlightGraph[circulant9,
  Table[Subgraph[circulant9,{i,i+3,i+6}],{i,3}]]
```

Out[96]=



Deleting Vertices and Edges

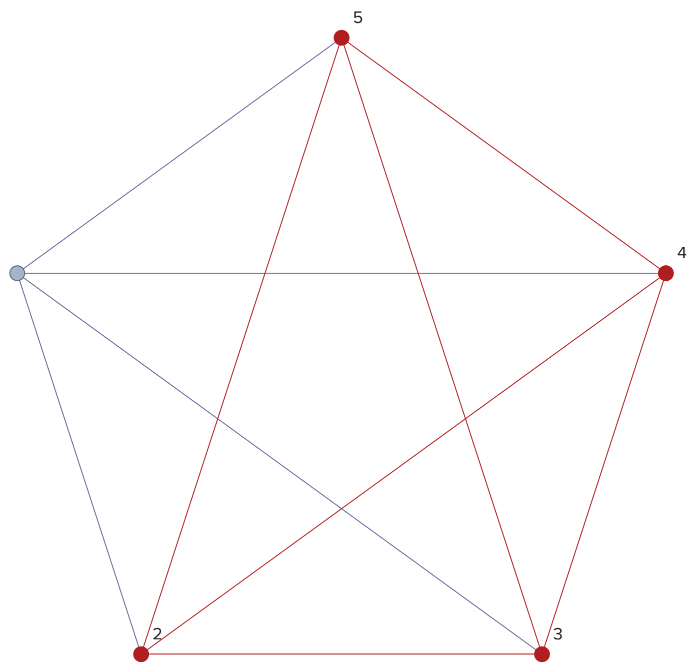
Subgraphs can also be produced by deleting vertices or edges. The `VertexDelete` and `EdgeDelete` functions were described in Section 10.1 of this manual, but are worth revisiting. `VertexDelete` takes two arguments: a graph and a vertex or list of vertices. The function returns a new graph with the vertex or vertices and all incident edges removed. Here we highlight the subgraph of the complete graph K_4 that is obtained by deleting a vertex.

In[97]:=
```
deleteVExStart=CompleteGraph[5];
```

In[98]:=
```
deleteVExEnd=VertexDelete[deleteVExStart,1];
```

In[99]:=
```
HighlightGraph[deleteVExStart,deleteVExEnd,
  VertexLabels→"Name"]
```

Out[99]=



`EdgeDelete` also takes two arguments, a graph and an edge or a list of edges. For example, we can remove the outer ring of K_5 as follows:

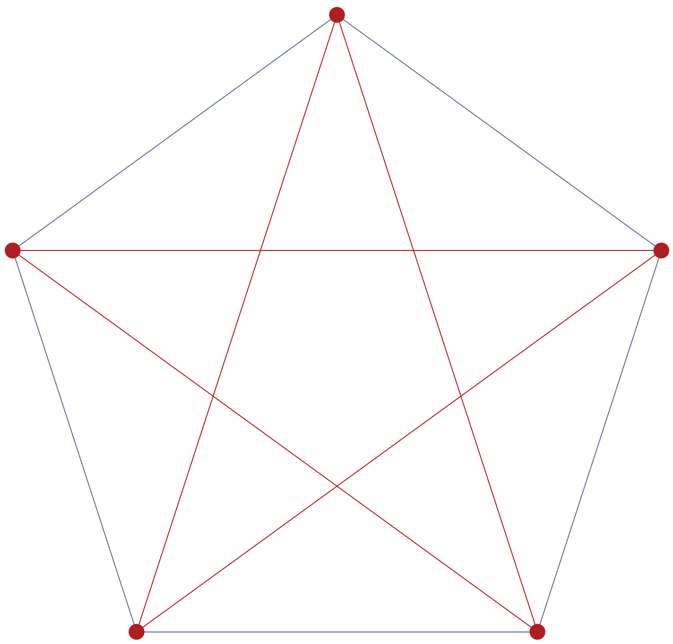
In[100]:=
```
deleteEexStart=CompleteGraph[5];
```

In[101]:=
```
deleteEexEdges=Join[Table[i→i+1,{i,1,4}],{1→5}];
```

In[102]:=
```
deleteEexEnd=EdgeDelete[deleteEexStart,deleteEexEdges];
```

In[103]:=
```
HighlightGraph[deleteEexStart,deleteEexEnd]
```

Out[103]=

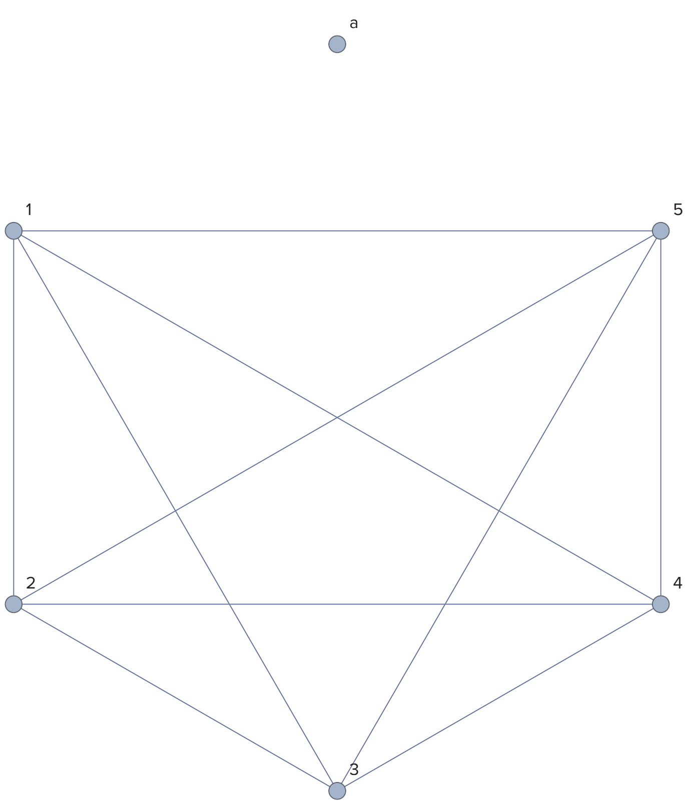


Adding Vertices and Edges

The functions for adding vertices and edges are very similar. `VertexAdd` accepts a graph and either a vertex or list of vertices to add to the graph.

In[104]:=
```
VertexAdd[CompleteGraph[5,VertexLabels→"Name"],"a"]
```

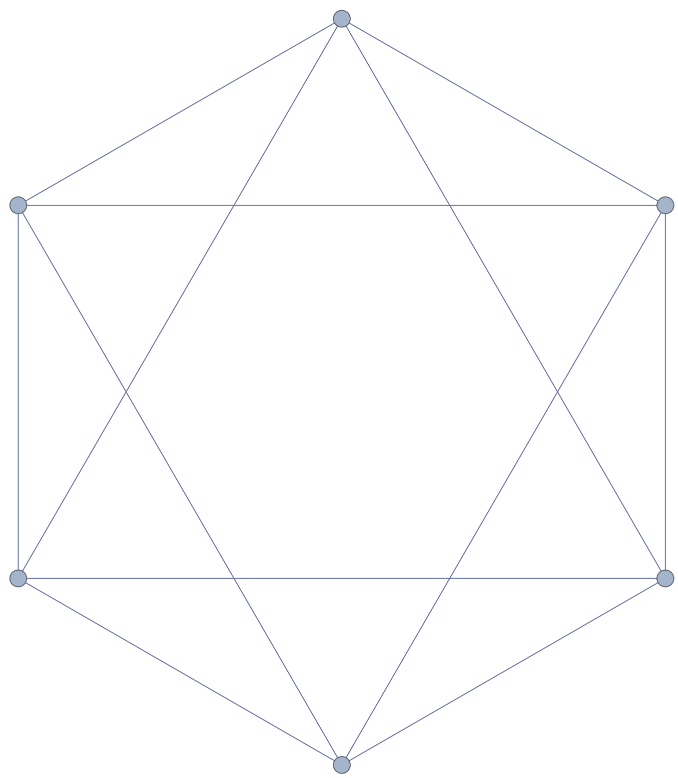
Out[104]=



`EdgeAdd` acts on a graph and adds an edge or a list of edges. Note that you can use rules to describe the edges and *Mathematica* will interpret them as directed or not depending on whether the original graph is directed.

In[105]:=
```
EdgeAdd[CycleGraph[6],{1→3,2→4,3→5,4→6,5→1,6→2}]
```

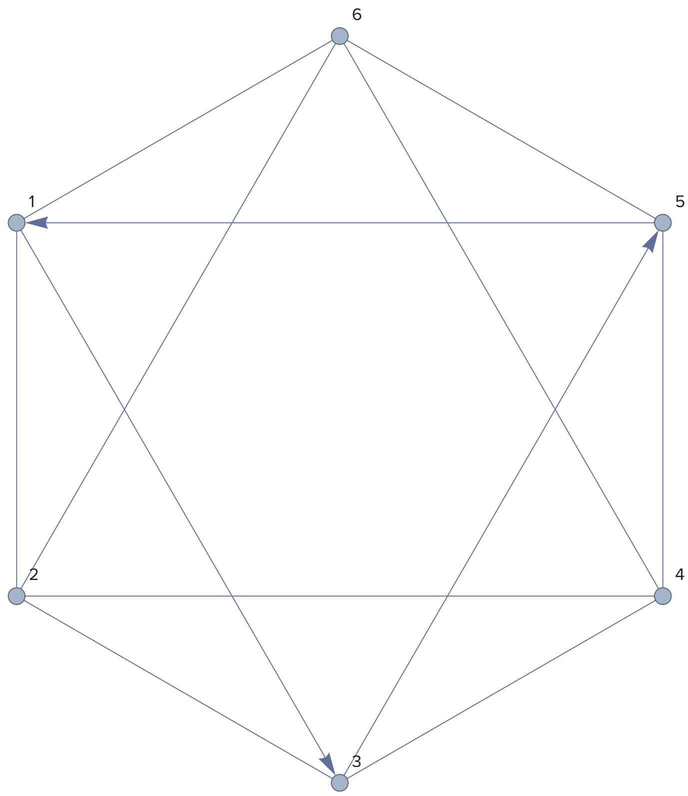
Out[105]=



You can specify directed edges using `DirectedEdge` or ↦ (ESC ue ESC); you can specify undirected edges by `UndirectedEdge`, ↔ (ESC de ESC), or `TwoWayRule` (<->). When a graph includes any directed edges, `Rule` (->) will be interpreted as a directed edge.

In[106]:=
```
EdgeAdd[CycleGraph[6,VertexLabels→"Name"],
  {DirectedEdge[1,3],3↦5,5→1,UndirectedEdge[4,6],
  2↦4,6↔2}]
```

Out[106]=



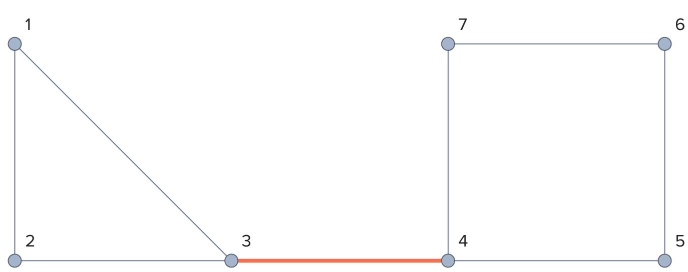
Edge Contraction

Recall that an edge contraction for an edge *e* with endpoints *u* and *v* consists of deleting the edge, merging *u* and *v* into a new vertex *w*, and preserving all edges (other than *e*) which had *u* or *v* as an endpoint by setting *w* as the new endpoint. As an illustration, consider the graph shown below.

In[107]:=

```
exampleContraction=
  Graph[{1,2,3,4,5,6,7},
    {1→2,1→3,2→3,Style[3→4,{Thick,Red}],4→5,
     4→7,5→6,6→7},DirectedEdges→False,
    VertexLabels→"Name",
    VertexCoordinates→
      {{0,1},{0,0},{1,0},{2,0},{3,0},{3,1},{2,1}}]
```

Out[107]=

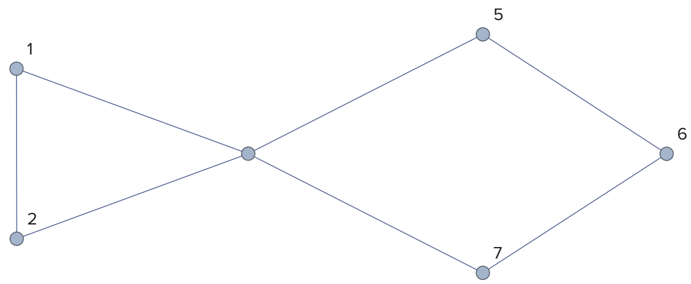


Observe the use of the `Style` wrapper on the edge between vertices 3 and 4 to highlight that edge.

The function `EdgeContract` can be used to contract an edge. It takes two arguments: a graph and an edge or list of edges.

In[108]:=
```
EdgeContract[exampleContraction,3→4]
```

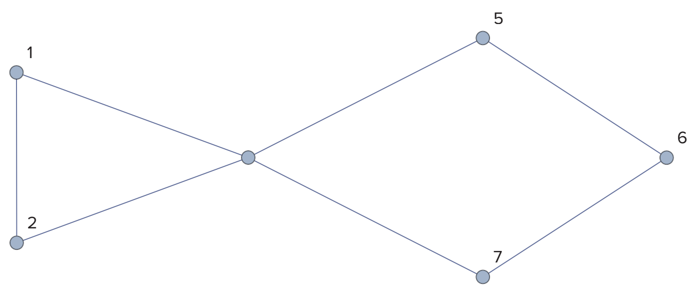
Out[108]=



The Wolfram Language also includes a similar function, `VertexContract`. The second argument of `VertexContract` is a list of vertices, and the result is the graph obtained by combining all the vertices in the list into a single vertex and removing all edges between the merged vertices. When the vertices are the endpoints of an edge, the effect is the same as `EdgeContract`.

In[109]:=
```
VertexContract[exampleContraction,{3,4}]
```

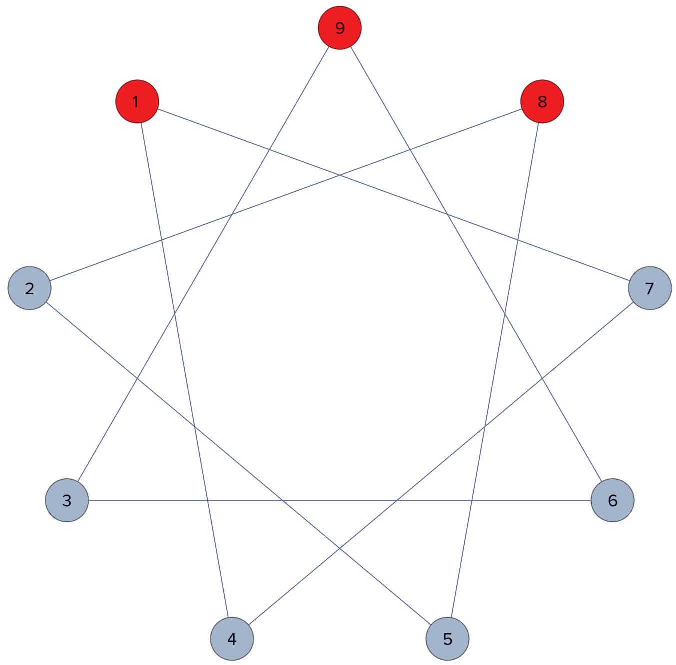
Out[109]=



The second argument, however, is not limited to two vertices, nor do they need to be adjacent, as the example below illustrates.

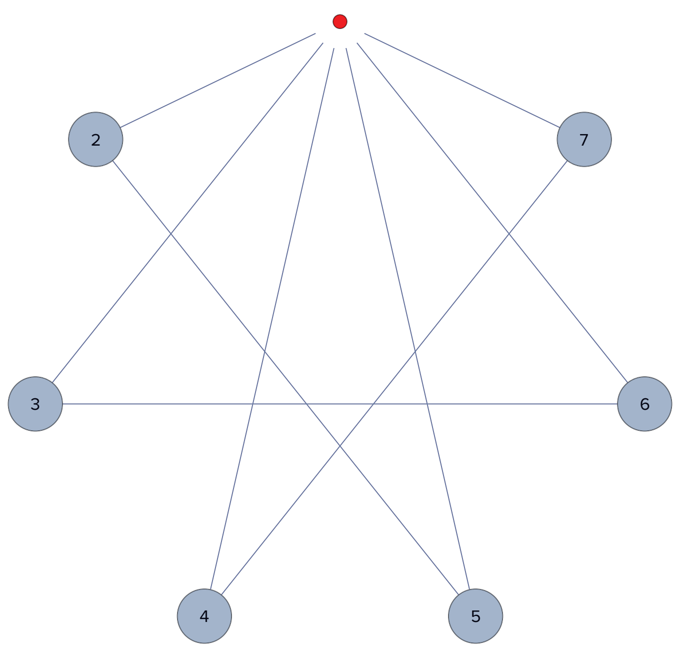
In[110]:=
```
exampleContraction2=
  CirculantGraph[9,3,VertexLabels→Placed["Name",Center],
   VertexSize→Medium,
   VertexStyle→Table[i→Red,{i,{1,9,8}}]]
```

Out[110]=



In[111]:=
```
Graph[VertexContract[exampleContraction2,{9,1,8}],
  VertexStyle→{9→Red}]
```

Out[111]=



Note that when the vertices are merged, the new vertex is given the name of the vertex that appears first in the list. Observe above that we listed 9 first in the set of vertices, and so it is 9 that we style as red. That the new vertex is unnumbered and smaller than before illustrates that the result of the contraction function is to delete all the original vertices, along with all options attached to them. Also observe that the contraction functions do not accept the options available to `Graph`, so we had to apply `Graph` to the result of the contraction.

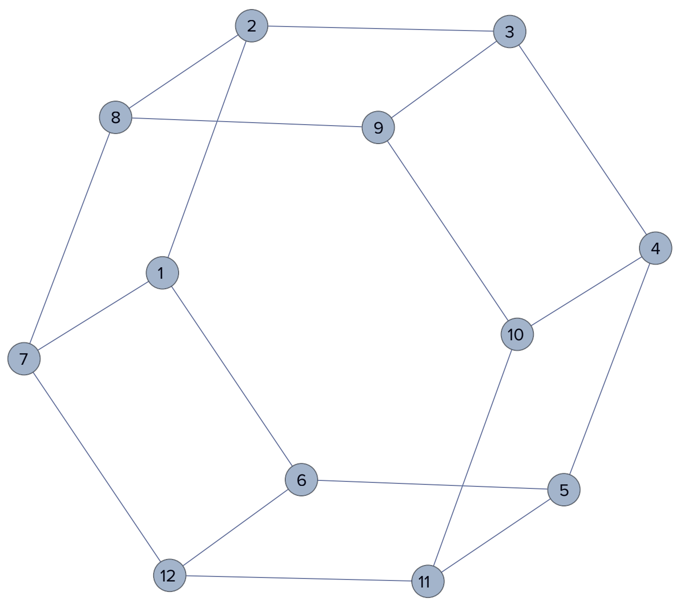
Unions and Complements of Graphs

Recall that the union of two graphs is the graph obtained by taking the union of the sets of vertices and the sets of edges from the two graphs.

As an example, we will “fill in” a prism graph by computing the union of the prism with the complete graph on the vertices in one ring. We begin with a version of the prism we created above.

In[112]:=
```
unionExampleA=
  Graph[{1→2,2→3,3→4,4→5,5→6,6→1,7→8,
  8→9,9→10,10→11,11→12,12→7,1→7,2→8,
  3→9,4→10,5→11,6→12},DirectedEdges→False,
  VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

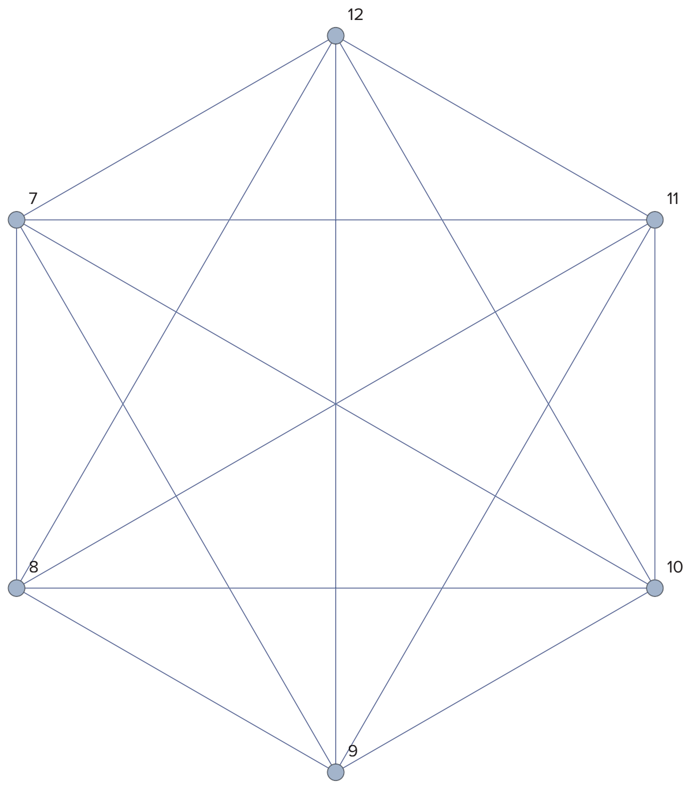
Out[112]=



We use the complete graph on 6 vertices as the second graph that will form part of the union. By default, `CompleteGraph` with argument 6 will form the complete graph on the vertices from 1 through 6. Suppose instead that we want the complete graph on the vertices from 7 through 12. To do this, apply the `VertexReplace` function. This function accepts two arguments: a graph and a list of rules specifying how to modify the names of the vertices. In this instance, we will replace vertex 1 with 7, 2 with 8, and so on, so we need to use the list of rules $\{1 \to 7, 2 \to 6, \ldots, 6 \to 12\}$, which we will create with a `Table`.

In[113]:=
```
unionExampleB=
  VertexReplace[CompleteGraph[6,VertexLabels→"Name"],
    Table[i→i+6,{i,6}]]
```

Out[113]=



Note that the same effect can be obtained with the function `IndexGraph`, with first argument a graph and second argument the smallest integer to be used (defaulting to 1 if the second argument is omitted). `VertexReplace` is the more general function.

Also note that this is not the same as setting different labels for the vertices. The result of `VertexReplace` is a graph with the internal names of the vertices changed.

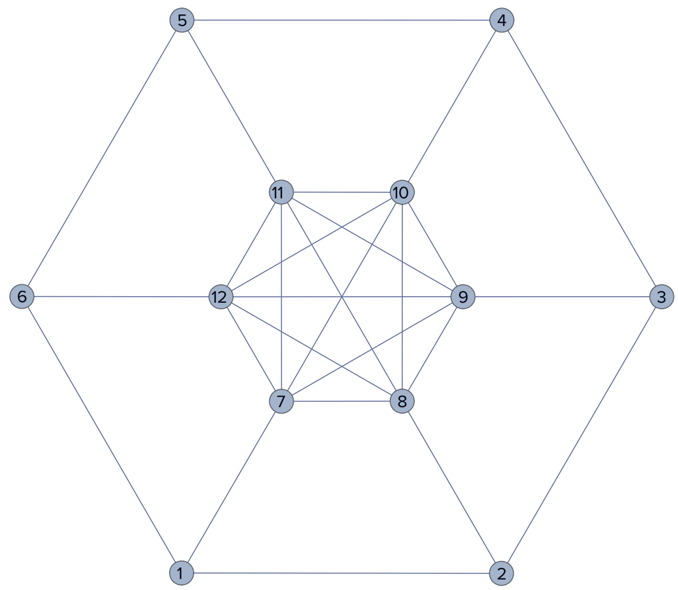
In[114]:=
```
VertexList[unionExampleB]
```

Out[114]=
```
{7,8,9,10,11,12}
```

To obtain the union of the graphs, we apply the `GraphUnion` function, which simply takes the two (or more) graphs as arguments, along with the usual options.

In[115]:=
```
unionExample=GraphUnion[unionExampleA,unionExampleB,
  VertexLabels→Placed["Name",Center],VertexSize→Medium]
```

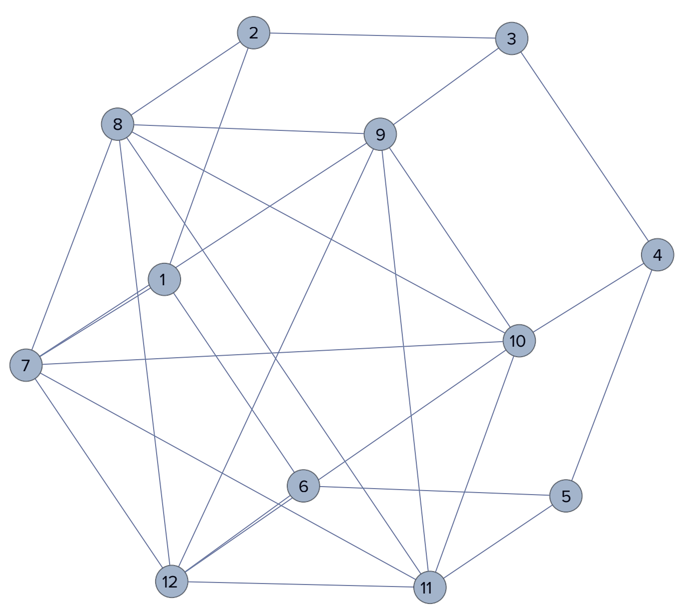
Out[115]=



Note that *Mathematica* has rearranged the locations of the vertices. If you prefer the three-dimensional appearance of the prism, you can impose those locations by setting the `VertexCoordinates` property when performing the union. The `GraphEmbedding` function applied to a graph returns a list of all the coordinates of the vertices. Since we did not add any new vertices here, we can simply apply `GraphEmbedding` to the original graph.

In[116]:=
```
unionExample=GraphUnion[unionExampleA,unionExampleB,
  VertexLabels→Placed["Name",Center],VertexSize→Medium,
  VertexCoordinates→GraphEmbedding[unionExampleA]]
```

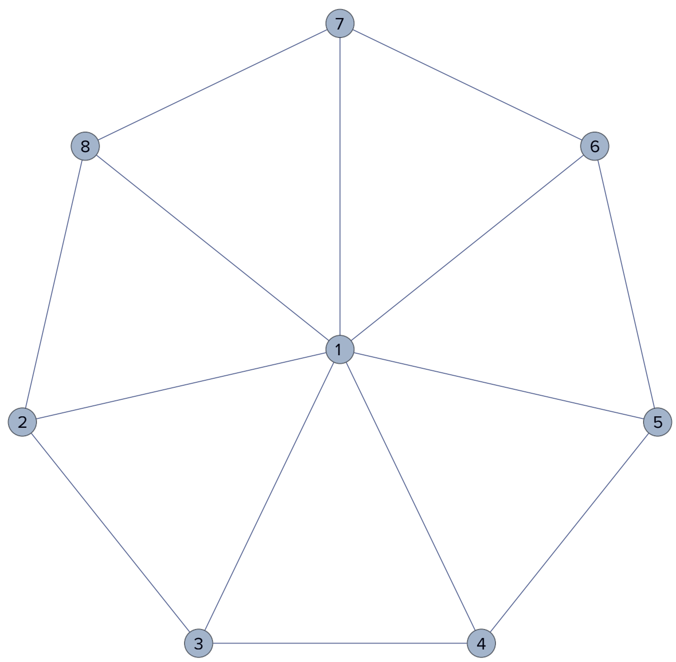
Out[116]=



Finally, we consider graph complements, described in Exercise 61 of Section 10.2. The complement, $\overline{G}$, of a graph G is the graph whose vertex set is the same as that of G, but whose edge set is the set of all pairs of G that have no edge between them. In other words, if G has n vertices, then the edge set of G is the complement of the edge set of G relative to K_n, the complete graph on n vertices. The Wolfram Language has a function to compute the complement of a graph: `GraphComplement`.

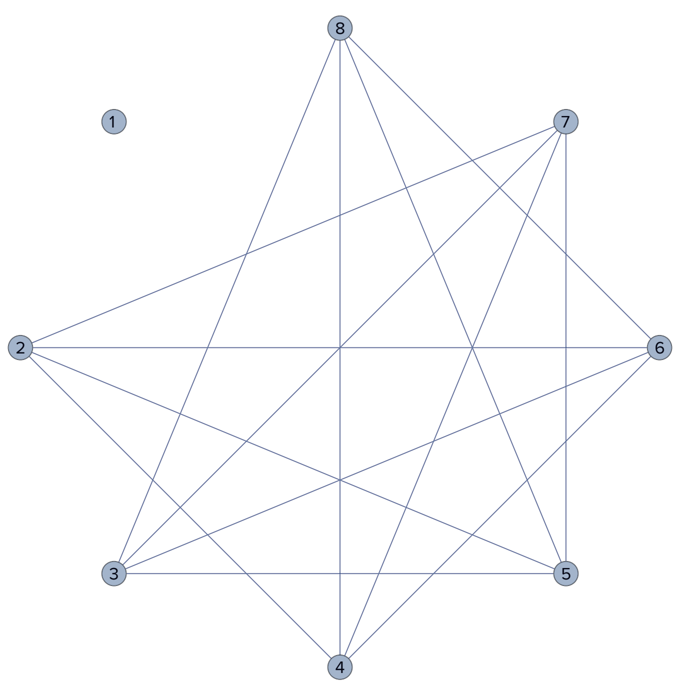
In[117]:=
```
WheelGraph[8,VertexLabels→Placed["Name",Center],
  VertexSize→Small]
```

Out[117]=



In[118]:=
```
GraphComplement[WheelGraph[8],
  VertexLabels→Placed["Name",Center], VertexSize→Small]
```

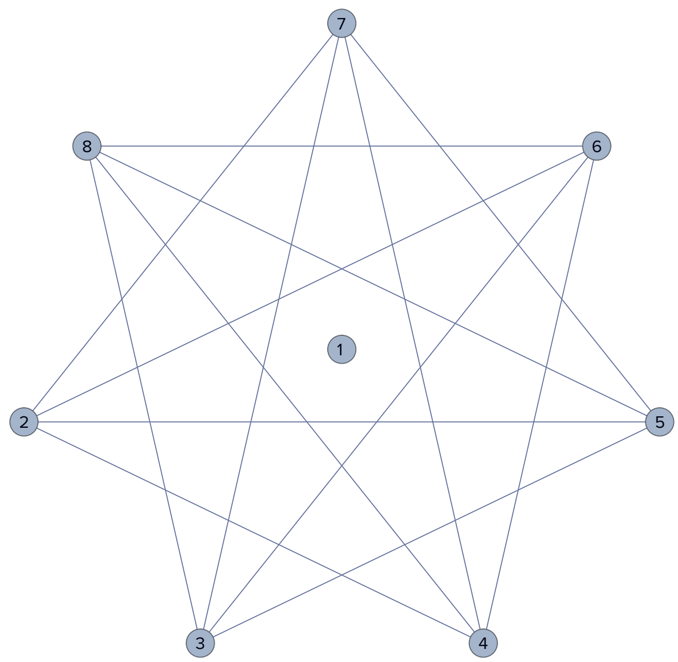
Out[118]=



Again, *Mathematica* has rearranged the vertices. We can impose the original locations as follows.

In[119]:=
```
complementExample=GraphComplement[WheelGraph[8],
  VertexLabels→Placed["Name",Center],VertexSize→Small,
  VertexCoordinates→GraphEmbedding[WheelGraph[8]]]
```

Out[119]=



10.3 Representing Graphs and Graph Isomorphism

In this section, we will see how to represent graphs in terms of adjacency matrices, adjacency lists, and incidence matrices. We will then use the adjacency matrix representation to help determine whether two graphs are isomorphic.

Adjacency Matrices

The adjacency matrix of a graph G with n vertices is the $n \times n$ matrix whose (i,j) entry is 1 if there is an edge from vertex i to vertex j and 0 if not. You can define a graph by passing an adjacency matrix, represented as a list of lists, to the function `AdjacencyGraph`.

As an example, we reproduce Example 4 from Section 10.3.

In[120]:=
```
exampleAdjM={{0,1,1,0},{1,0,0,1},{1,0,0,1},{0,1,1,0}};
exampleAdjM//MatrixForm
```

Out[121]//MatrixForm=

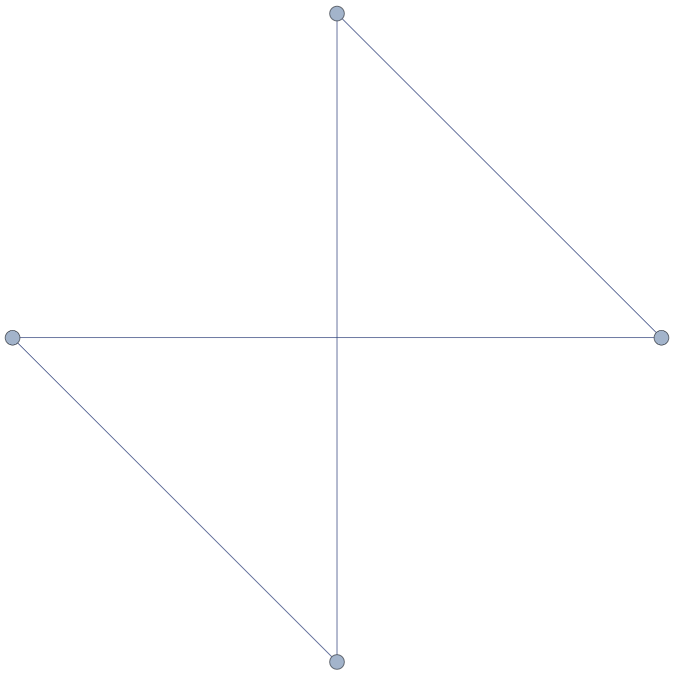
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Recall that we must invoke `MatrixForm` in an expression separate from the definition of the symbol in order to avoid having the `MatrixForm` head permanently stored in the symbol.

We now invoke the AdjacencyGraph function with this matrix.

```
In[122]:=  AdjacencyGraph[exampleAdjM]
```

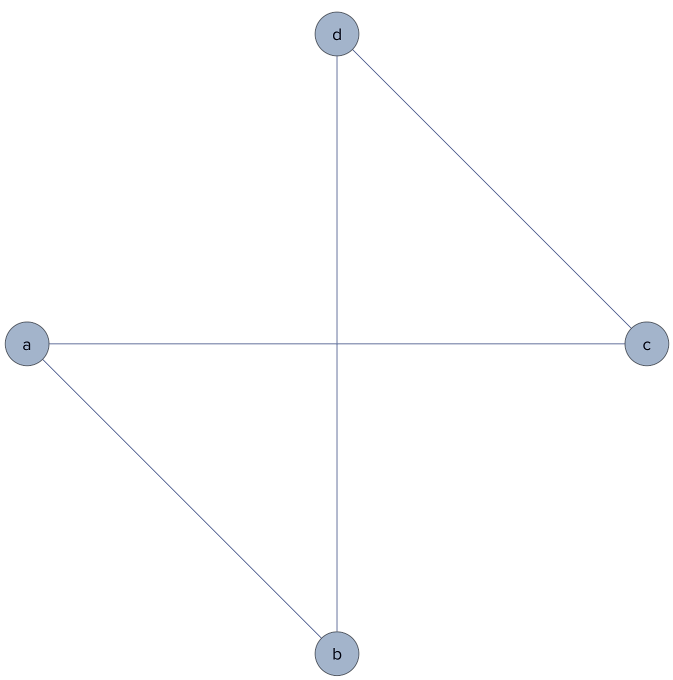
Out[122]=



In the textbook, the vertices for this graph were labeled as letters rather than numbers. The AdjacencyGraph function accepts a list of names for the vertices as an optional first argument. It also accepts the usual graph options.

```
In[123]:=  AdjacencyGraph[{"a","b","c","d"},exampleAdjM,
             VertexLabels→Placed["Name",Center],VertexSize→Small]
```

Out[123]=



Notice that this is the same graph as is produced in the textbook, with the exception of the locations of the vertices.

The Wolfram Language also provides a function, `AdjacencyMatrix`, for computing the adjacency matrix of a graph. The output of this function is always a `SparseArray` object, which is more efficient at storing large arrays with many entries 0. You can display a `SparseArray` with `MatrixForm`, as usual, and you can convert it into the usual list of lists representation with the function `Normal`.

```
In[124]:= wheelAdjacency=AdjacencyMatrix[WheelGraph[7]]

Out[124]=
SparseArray [ Specified elements: 24
              Dimensions: {7, 7} ]
```

```
In[125]:= wheelAdjacency//MatrixForm
```

Out[125]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

```
In[126]:= wheelAdjacency//Normal

Out[126]= {{0,1,1,1,1,1,1},{1,0,1,0,0,0,1},{1,1,0,1,0,0,0},
          {1,0,1,0,1,0,0},{1,0,0,1,0,1,0},{1,0,0,0,1,0,1},
          {1,1,0,0,0,1,0}}
```

Adjacency Lists

Recall that a representation of a graph as an adjacency list consists of the lists of neighbors of each vertex.

Earlier, we saw the function `AdjacencyList` used to determine the list of vertices adjacent to a given vertex. For example, the following determines the vertices adjacent to vertex 2 in the wheel graph on 7 vertices.

```
In[127]:= AdjacencyList[WheelGraph[7],2]

Out[127]= {1,3,7}
```

By looping over the vertex set, we create a function that returns an `Association` containing the adjacency lists. Note that applying `Dataset` to a single `Association`, displays a table with the rows labeled by the keys.

```
In[128]:= adjacencyList[G_Graph]:=
            Association@@Table[v→AdjacencyList[G,v],
              {v,VertexList[G]}]

In[129]:= adjacencyList[WheelGraph[7]]//Dataset
```

Out[129]=

1	{2, 3, 4, 5, 6, 7}
2	{1, 3, 7}
3	{1, 2, 4}
4	{1, 3, 5}
5	{1, 4, 6}
6	{1, 5, 7}
7	{1, 2, 6}

The Wolfram Language does not include a function to create a graph from an adjacency list. However, it is not difficult to create a function that transforms an adjacency list into a graph object. We will use the adjacency matrix as an intermediate in order to illustrate how to build a matrix as a `SparseArray`. We will create a function `adjacencyListGraph` that accepts as its argument an `Association` whose keys are the vertices and with the lists of adjacent vertices as the corresponding values. We will assume that the graph does not contain multiple edges.

Our first step is to define a predicate that checks that the rules making up the association are of the correct form. Specifically, each rule in an association representing an adjacency list for a graph can have any object as the left-hand element, but the right-hand element must be a list.

In[130]:=
```
adjacencyRuleQ[r_Rule]:=ListQ[r[[2]]]
```

With this predicate in place, the signature of our function will be:

```
adjacencyListGraph[A:Association[__?adjacencyRuleQ]]:=
```

This will ensure that the argument is an `Association` representing an adjacency list of a graph.

The main work of the function will be to transform the list into a matrix with 1s in the locations specified by the adjacency list. We mentioned above that a `SparseArray` is particularly suitable for matrices with few non-zero entries. To further explore this type of object, we will have our `adjacencyListGraph` function create a `SparseArray` as part of its operation. There are several different syntax options for creating a `SparseArray`, but we will use the most descriptive: a list of rules identifying positions with values. For example, to create a 4×4 matrix with 1s in positions $(1,3)$, $(2,4)$, and $(4,1)$, you enter the following.

In[131]:=
```
SparseArray[{{1,3}→1,{2,4}→1,{4,1}→1},{4,4}]
  //MatrixForm
```

Out[131]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the first argument is a list of rules with the positions within the matrix given as lists on the left hand of each rule, and the value that belongs in the position on the right. The second argument to `SparseArray` is a list specifying the dimensions of the matrix. If the second argument is omitted, the Wolfram Language will attempt to deduce the size of the matrix from the given entries.

The `adjacencyListGraph` function will begin by determining the vertex set of the graph. Consider the example adjacency list shown below.

```
In[132]:= exampleList=<|"a"→{"b","c",3},"b"→{"c"},"c"→{3}|>

Out[132]= <|a→{b,c,3},b→{c},c→{3}|>
```

Note that the vertices are not of all the same type—three are strings and another an integer. Also note that one of the vertices does not appear as a key. While a properly formed adjacency list would include the rule `3→{}`, we will design our function to be sufficiently robust to handle such input. We use `Keys` and `Values` to extract the keys and values. The result of `Values` is a list of lists and so will need to be flattened.

```
In[133]:= exampleListVertices=
            Union[Keys[exampleList],Flatten@Values[exampleList]]

Out[133]= {3,a,b,c}
```

In order to create an adjacency matrix, we need to translate the names of the vertices into positions. We will create an `Association` with the vertex names as keys and their positions in the list of vertices as the values. The `MapThread` function will accept a function of multiple arguments and a list of lists. The result will be a list in which each element will be the function applied to the elements in the corresponding positions. This is illustrated below with a undefined symbol in the first argument.

```
In[134]:= MapThread[f,{{1,2,3},{"a","b","c"}}]

Out[134]= {f[1,a],f[2,b],f[3,c]}
```

This allows us to create a list of rules identifying the names of vertices with their position in the list.

```
In[135]:= exampleDictionary=
            Association[
              MapThread[Rule,{exampleListVertices,Range[4]}]]

Out[135]= <|3→1,a→2,b→3,c→4|>
```

With this association in place, we can access the position of a vertex as follows:

```
In[136]:= exampleDictionary["a"]

Out[136]= 2
```

To build the rules that define the `SparseArray`, we use `Table` with two table variables—first looping over the keys of the association and then over the lists of vertices adjacent to them.

```
In[137]:= Table[{exampleDictionary[v],exampleDictionary[w]}→1,
            {v,Keys[exampleList]},{w,exampleList[v]}]

Out[137]= {{{2,3}→1,{2,4}→1,{2,1}→1},{{3,4}→1},{{4,1}→1}}
```

This will need to be flattened in order to serve as the argument of SparseArray.

In[138]:=
```
SparseArray[
  Flatten[
    Table[{exampleDictionary[v],exampleDictionary[w]}→1,
      {v,Keys[exampleList]},{w,exampleList[v]}]]]
  //MatrixForm
```

Out[138]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

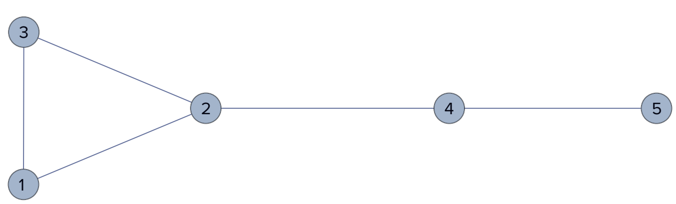
We now build adjacencyListGraph. We use a BlankNullSequence (___), which matches any number of argument, including 0, to pass options from this function to AdjacencyGraph.

In[139]:=
```
adjacencyListGraph[A:Association[__?adjacencyRuleQ],
   opts___]:=Module[{V,n,D,rules,v,w},
  V=Union[Keys[A],Flatten@Values[A]];
  n=Length[V];
  D=Association[MapThread[Rule,{V,Range[n]}]];
  rules=Flatten[Table[{D[v],D[w]}→1,{v,Keys[A]},
    {w,A[v]}]];
  AdjacencyGraph[SparseArray[rules,{n,n}],opts]
]
```

We apply this function to an example.

In[140]:=
```
exampleAL=
  adjacencyListGraph[<|1→{2,3},2→{1,3,4},
    3→{1,2},4→{2,5},5→{4}|>,VertexSize→Medium,
    VertexLabels→Placed["Name",Center]]
```

Out[140]=



Incidence Matrices

The third representation of graphs that we are considering is the incidence matrix. For a graph G with n vertices and m edges, the associated incidence matrix is the $n \times m$ matrix whose (i, j) entry is 1 if vertex i is an endpoint of edge j, and 0 otherwise.

The Wolfram Language includes functions for working with incidence matrices. Given an incidence matrix M, the function IncidenceGraph will produce the associated graph. As an example, we reverse

Example 6 from Section 10.3 and use the incidence matrix given in the solution in order to reproduce the graph.

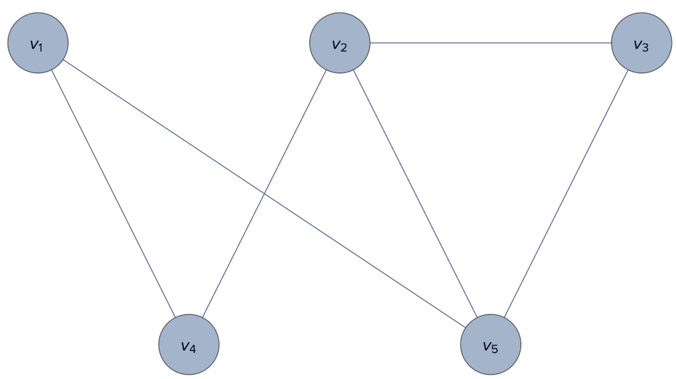
In[141]:=
```
exampleIncidenceM={{1,1,0,0,0,0},{0,0,1,1,0,1},
  {0,0,0,0,1,1},{1,0,1,0,0,0},{0,1,0,1,1,0}};
exampleIncidenceM//MatrixForm
```

Out[142]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

In[143]:=
```
exampleIncidenceG=
  IncidenceGraph[exampleIncidenceM,
   VertexLabels→Placed["Name",Center,v_#&],
   VertexSize→Medium,
   VertexCoordinates→
    {{0,1},{1,1},{2,1},{0.5,0},{1.5,0}}]
```

Out[143]=



Note that, in order to label the vertices with subscripted v's, we use `Placed` with a function as the third argument. This function applies to the label determined by the first argument to produce the displayed label. So these vertices are named 1 through 6 and those names become subscripts in their labels.

For the reverse, the `IncidenceMatrix` function will produce the incidence matrix for a `Graph` object. We apply this function to the previous graph. The output from this function is a `SparseArray` object, so we apply `MatrixForm` to view it.

In[144]:=
```
IncidenceMatrix[exampleIncidenceG]//MatrixForm
```

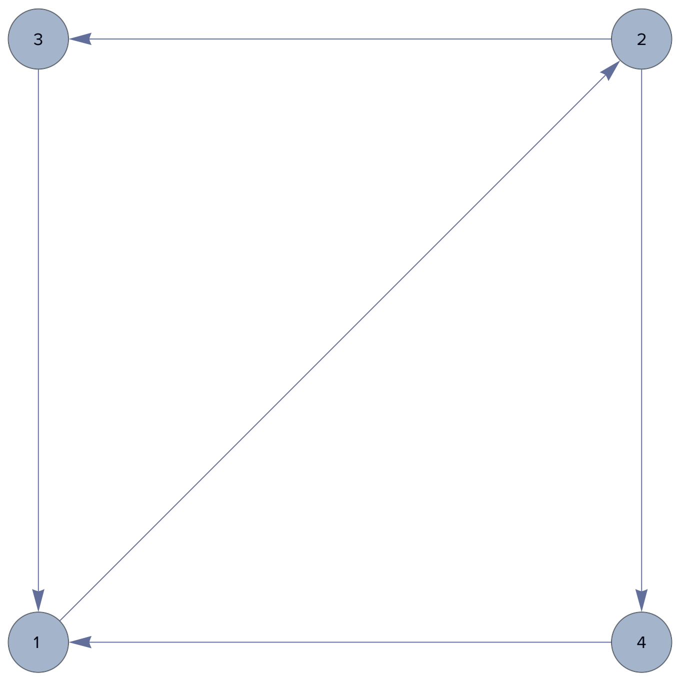
Out[144]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

For a directed graph, the `IncidenceMatrix` function returns a matrix with a 1 in position (i, j) indicating that the vertex i is the head of edge j and an entry of -1 indicating that the vertex is the tail of the edge.

In[145]:=
```
directedIncidenceG=
  Graph[{1→2,2→3,3→1,2→4,4→1},DirectedEdges→True,
    VertexLabels→Placed["Name",Center],VertexSize→Small,
    VertexCoordinates→{{0,0},{1,1},{0,1},{1,0}}]
```

Out[145]=



In[146]:=
```
IncidenceMatrix[directedIncidenceG]//MatrixForm
```

Out[146]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Isomorphism of Graphs

We conclude Section 10.3 with a brief discussion of isomorphism of graphs and graph invariants. Determining whether two graphs are isomorphic is a difficult problem. The naive approach (exhaustively checking each possible mapping) can require exponential time.

Graph invariants are useful tools for confirming that two graphs are not isomorphic. While there is no complete collection of graph invariants that will definitively conclude whether two graphs are or are not isomorphic, they can, for many pairs of graphs, quickly demonstrate the impossibility of an isomorphism. We will create a function that will check some of the basic invariants: number of vertices, number of edges, whether the graph is directed, and whether it is bipartite. We also introduce another invariant: the degree sequence.

For a graph G, the *degree sequence* is the list of the degrees of the vertices of the graph sorted in ascending order. The Wolfram Language function `VertexDegree` applied to a graph and a vertex returns the degree of the vertex. Applied to the graph with no second argument, it produces a list of the degrees of the vertices of a graph, listed in order of the vertices. Since this depends on the order in which *Mathematica* stores the vertices, it is not an invariant. However, applying the `Sort` function to the result of `VertexDegree` returns the degree sequence for the graph, which is an invariant.

The function defined below checks, one at a time, the invariants we have mentioned. If any of the invariants indicate that the graphs are not isomorphic, the procedure prints a statement to that effect.

In[147]:=
```
checkInvariants[G1_Graph,G2_Graph]:=
 Module[{notIsomorphic=False},
  If[VertexCount[G1]≠VertexCount[G2],
   notIsomorphic=True;
   Print["Different numbers of vertices."]
  ];
  If[EdgeCount[G1]≠EdgeCount[G2],
   notIsomorphic=True;
   Print["Different numbers of edges."]
  ];
  If[!Equivalent[DirectedGraphQ[G1],DirectedGraphQ[G2]],
   notIsomorphic=True;
   Print["One is directed, one is undirected."]
  ];
  If[!Equivalent[BipartiteGraphQ[G1],
     BipartiteGraphQ[G2]],
   notIsomorphic=True;
   Print["One is bipartite, one is not."]
  ];
  If[Sort[VertexDegree[G1]]≠Sort[VertexDegree[G2]],
   notIsomorphic=True;
   Print["Degree sequences do not match."]
  ];
  If[notIsomorphic,
   Print["The graphs are not isomorphic."],
   Print["The graphs MAY be isomorphic."]
  ]
 ]
```

In[148]:=
```
checkInvariants[directedIncidenceG,exampleIncidenceG]
```

```
Different numbers of vertices.

Different numbers of edges.

One is directed, one is undirected.
```

```
Degree sequences do not match.

The graphs are not isomorphic.
```

In[149]:=
```
checkInvariants[CompleteGraph[3],CycleGraph[3]]
```

```
The graphs MAY be isomorphic.
```

The Wolfram Language provides a function, `IsomorphicGraphQ`, for determining whether or not two graphs are isomorphic. This function applies to any `Graph` object. The `IsomorphicGraphQ` function accepts two graphs as its arguments. It returns `True` if the graphs are isomorphic.

In[150]:=
```
IsomorphicGraphQ[CompleteGraph[3],CycleGraph[3]]
```

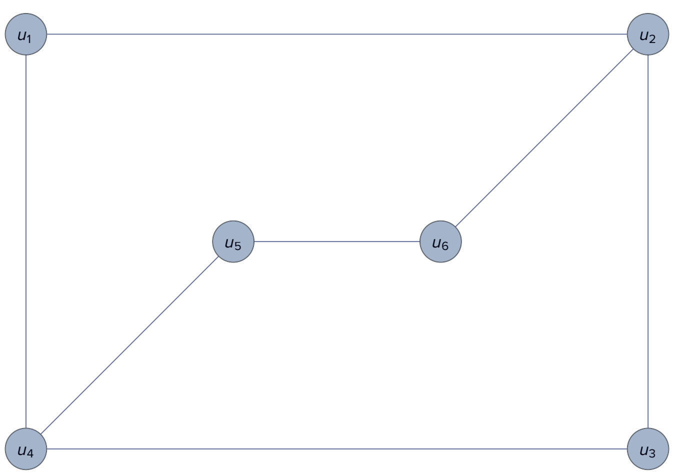
Out[150]=
```
True
```

The `FindGraphIsomorphism` function can be used to determine an explicit isomorphism for a pair of graphs that are in fact isomorphic. Like `IsomorphicGraphQ`, the only required arguments are the two graphs, although you can give a positive integer or the symbol `All` as the third argument in order to attempt to produce multiple isomorphisms. If the two graphs are in fact isomorphic, the output is a list of associations composed of rules of the form $v \to w$ indicating that vertex v in the first graph is mapped to vertex w in the second graph. If the graphs are not isomorphic, `IsomorphicGraphQ` returns an empty list.

We illustrate by reproducing the graphs in Figure 12 of Section 10.3 of the textbook.

In[151]:=
```
figure12G=Graph[{1,2,3,4,5,6},
  {1→2,1→4,2→3,2→6,3→4,4→5,5→6},
  DirectedEdges→False,
  VertexLabels→Placed["Name",Center,u#&],
  VertexSize→Medium,
  VertexCoordinates→
    {{0,2},{3,2},{3,0},{0,0},{1,1},{2,1}}]
```

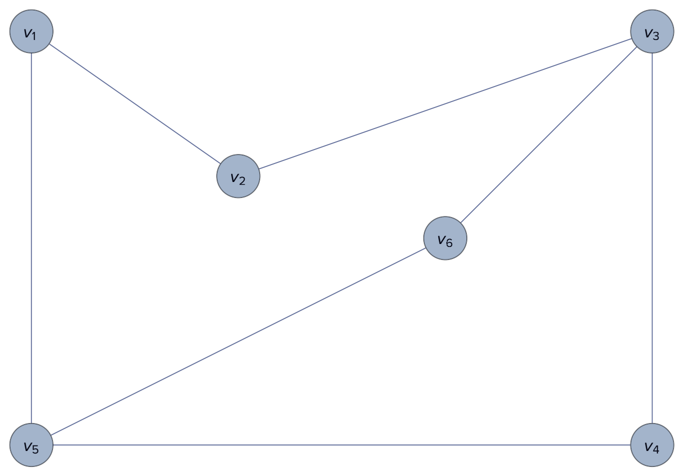
Out[151]=



In[152]:=
```
figure12H=Graph[{1,2,3,4,5,6},
  {1→2,1→5,2→3,3→4,3→6,4→5,5→6},
    DirectedEdges→False,
```

```
        VertexLabels→Placed["Name",Center,v#&],
        VertexSize→Medium,
        VertexCoordinates→
          {{0,2},{1,1.3},{3,2},{3,0},{0,0},{2,1}}]
```

Out[152]=



Applying `IsomorphicGraphQ` confirms that the graphs are isomorphic.

In[153]:=
```
IsomorphicGraphQ[figure12G,figure12H]
```

Out[153]=
```
True
```

`FindGraphIsomorphism` determines the isomorphism.

In[154]:=
```
FindGraphIsomorphism[figure12G,figure12H]
```

Out[154]=
```
{<|1→4,2→3,3→6,4→5,5→1,6→2|>}
```

Observe that there are four different isomorphisms.

In[155]:=
```
FindGraphIsomorphism[figure12G,figure12H,All]
```

Out[155]=
```
{<|1→4,2→3,3→6,4→5,5→1,6→2|>,
  <|1→4,2→5,3→6,4→3,5→2,6→1|>,
  <|1→6,2→3,3→4,4→5,5→1,6→2|>,
  <|1→6,2→5,3→4,4→3,5→2,6→1|>}
```

We can make an isomorphism visible by coloring each vertex in the second graph and using the composition to then determine colors for the first graph. To choose the colors, we'll make use of the Wolfram Language built-in color schemes. There are a number of different ways to use `ColorData`; we will simply use the indexed color schemes. If you apply `ColorData` to a positive integer up to 113, the result is a `ColorDataFunction`, which is a function that returns a color.

In[156]:=
```
ColorData[10][3]
```

Out[156]=

Passing `ColorData` the second argument `"Panel"` (including the quotation marks) will display the colors produced by the particular `ColorDataFunction`.

In[157]:=
```
ColorData[87,"Panel"]
```

Out[157]=

Provided the `ColorDataFunction` has enough colors, we can shade all of the vertices of a graph with different colors by forming a list of rules.

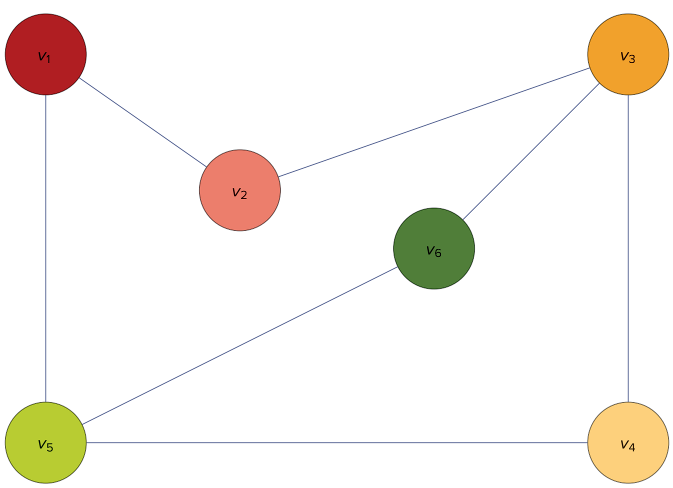
In[158]:=
```
Table[v->ColorData[10][v],{v,6}]
```

Out[158]=
{1 → ■, 2 → ■, 3 → ■, 4 → ■, 5 → ■, 6 → ■}

This list will be the value of `VertexStyle`. Here is the graph H with colors added

In[159]:=
```
figure12Hcolor=Graph[{1,2,3,4,5,6},
  {1→2,1→5,2→3,3→4,3→6,4→5,5→6},
  DirectedEdges→False,
  VertexLabels→Placed["Name",Center,v#&],
  VertexSize→Large,
  VertexStyle→Table[v->ColorData[10][v],{v,6}],
  VertexCoordinates→
  {{0,2},{1,1.3},{3,2},{3,0},{0,0},{2,1}}]
```

Out[159]=



We then color G by applying one of the isomorphisms before accessing the color.

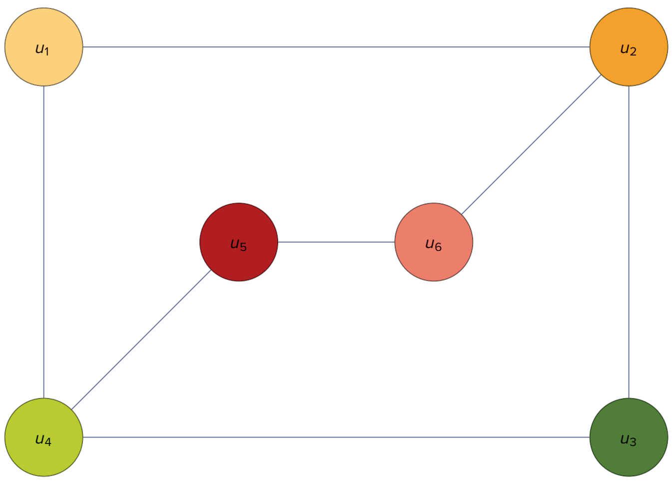
In[160]:=
```
figure12isomorphism=
  FindGraphIsomorphism[figure12G,figure12H,All][[1]]
```

Out[160]=
```
<|1→4,2→3,3→6,4→5,5→1,6→2|>
```

In[161]:=
```
figure12Gcolor=
  Graph[{1,2,3,4,5,6},
```

```
{1→2,1→4,2→3,2→6,3→4,4→5,5→6},
DirectedEdges→False,
VertexLabels→Placed["Name",Center,u#&],
VertexSize→Large,
VertexStyle→
  Table[v->ColorData[10][figure12isomorphism[v]],
  {v,6}],VertexCoordinates→
  {{0,2},{3,2},{3,0},{0,0},{1,1},{2,1}}]
```

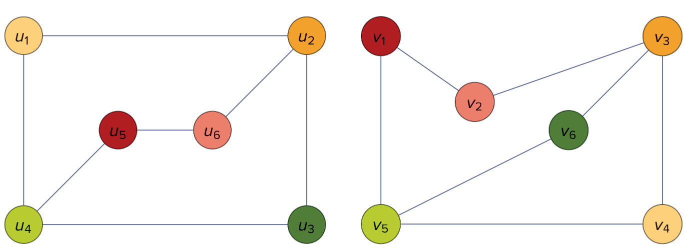
Out[161]=



We put them side-by-side to compare with `GraphicsRow`, which like `Row` for more general display, takes a list of objects and displays them in a row.

In[162]:=
```
GraphicsRow[{figure12Gcolor,figure12Hcolor}]
```

Out[162]=



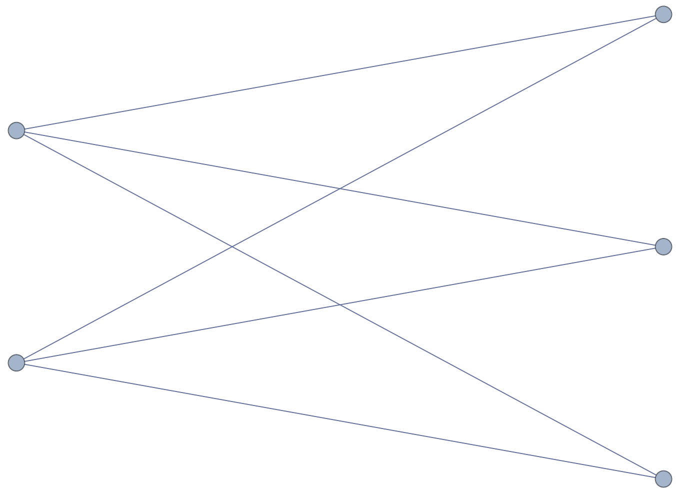
10.4 Connectivity

The Wolfram Language provides a number of functions related to connectivity of graphs.

The first such function that we consider is `ConnectedGraphQ`. This function takes one argument, the name of the graph, and returns true or false. As an example, consider the complete bipartite graph $K_{2,3}$ and its complement.

In[163]:=
```
CompleteGraph[{2,3}]
```

Out[163]=

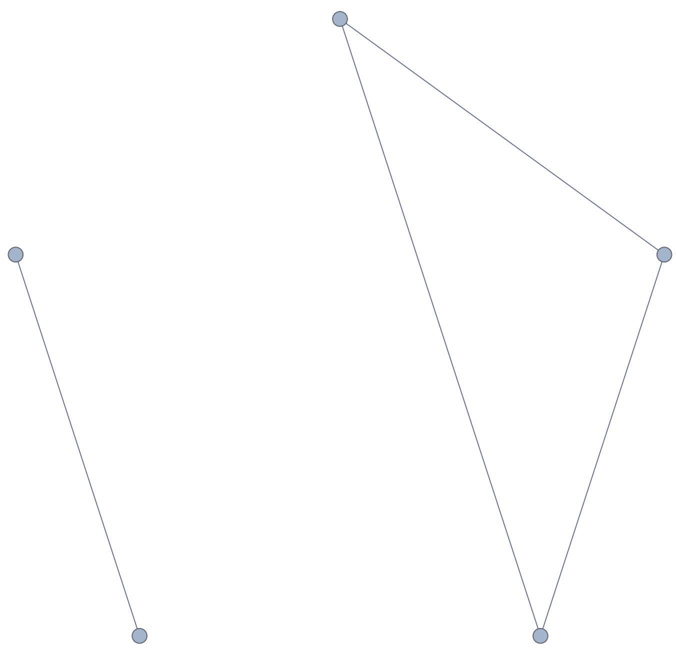


In[164]:= **ConnectedGraphQ[CompleteGraph[{2,3}]]**

Out[164]= True

In[165]:= **GraphComplement[CompleteGraph[{2,3}]]**

Out[165]=



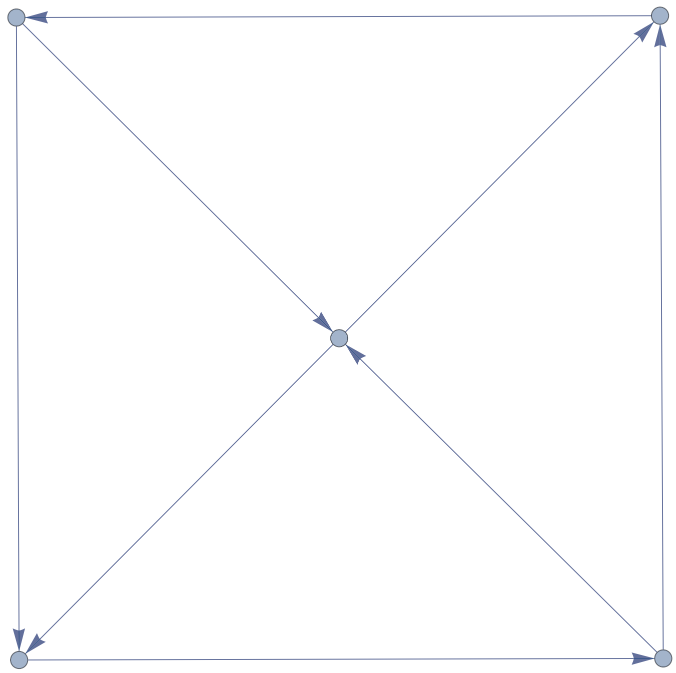
In[166]:= **ConnectedGraphQ[GraphComplement[CompleteGraph[{2,3}]]]**

Out[166]= False

When used with a directed graph, ConnectedGraphQ returns True if the directed graph is strongly connected.

In[167]:= **stronglyConnectedEx=**
Graph[{1→2,2→3,3→4,4→1,1→5,5→2,3→5,5→4}]

Out[167]=



Applying `ConnectedGraphQ` reveals that the above graph is connected.

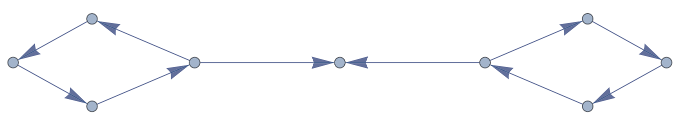
In[168]:=
```
ConnectedGraphQ[stronglyConnectedEx]
```

Out[168]=
```
True
```

The following example, while weakly connected, is not strongly connected and thus `ConnectedGraphQ` will return `False`.

In[169]:=
```
weaklyConnectedEx=Graph[{4→2,2→1,1→3,3→4,4→5,6→8,8→9,9→7,
7→6,6→5}]
```

Out[169]=



In[170]:=
```
ConnectedGraphQ[weaklyConnectedEx]
```

Out[170]=
```
False
```

The `WeaklyConnectedGraphQ` function will determine whether a directed graph is weakly connected. A graph is weakly connected if it is connected as an undirected graph.

In[171]:=
```
WeaklyConnectedGraphQ[weaklyConnectedEx]
```

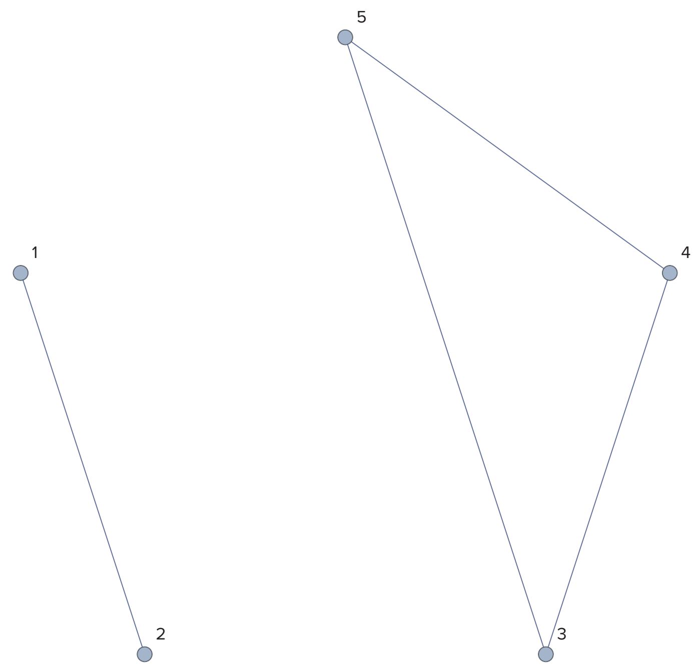
Out[171]=
```
True
```

The Wolfram Language also has functions to extract the connected components of a graph that is not connected. The `ConnectedComponents` function takes a graph as input and returns a list of lists of vertices.

As an example, consider the complement of the graph $K_{2,3}$.

```
In[172]:= GraphComplement[CompleteGraph[{2,3}],VertexLabels→"Name"]
```

Out[172]=




```
In[173]:= ConnectedComponents[GraphComplement[CompleteGraph[{2,3}]]]

Out[173]= {{3,4,5},{1,2}}
```

This output indicates that the complement of $K_{2,3}$ has two connected components, one with vertex set $\{1,2\}$ and the other with vertex set $\{3,4,5\}$.

For directed graphs, `ConnectedComponents` will produce the strongly connected components. The `WeaklyConnectedComponents` function will output the weakly connected components.

```
In[174]:= ConnectedComponents[weaklyConnectedEx]

Out[174]= {{5},{4,2,1,3},{6,8,9,7}}

In[175]:= WeaklyConnectedComponents[weaklyConnectedEx]

Out[175]= {{4,2,1,3,5,6,8,9,7}}
```

Coloring the Components

We now present a function that will color code the connected components in a graph. (This function will color up to ten components before repeating colors.)

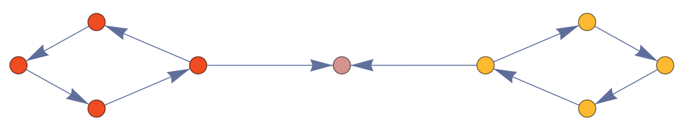
As in the previous section, we use `ColorData` to obtain the colors. Also note the use of `PropertyValue` to set the `VertexStyle` property. We first saw `PropertyValue` in the subsection on edge contraction in Section 10.2 of this manual. Finally, note the use of `Mod` with third argument 1, which causes the result of `Mod` to have minimum value 1. This allows us to cycle through the list of ten colors provided by the particular `ColorDataFunction` we chose.

In[176]:=
```
highlightComponents[G_Graph,opts___]:=
  Module[{components,c,i,v,H=G},
    components=ConnectedComponents[G];
    c=0;
    For[i=1,i≤Length[components],i++,
      c=Mod[c+1,10,1];
      Do[PropertyValue[{H,v},VertexStyle]=
        ColorData[14][c]
       ,{v,components[[i]]}]
    ];
    Graph[H,opts]
  ]
```

We apply this function to the weakly connected graph above.

In[177]:=
```
highlightComponents[weaklyConnectedEx,VertexSize→Medium]
```

Out[177]=



Counting Paths Between Vertices

The last topic that we consider in this section is determining the number of paths between two vertices of a given length. As described in the textbook, if A is the adjacency matrix for a graph (which may be undirected or directed and may include loops and multiple edges), then the (i, j) entry of the matrix A^r is the number of paths of length r from vertex i to vertex j.

As an example, consider the `stronglyConnectedEx` graph from above. We can obtain its adjacency matrix by applying the `AdjacencyMatrix` function to the name of the graph.

In[178]:=
```
aMatrix=AdjacencyMatrix[stronglyConnectedEx];
aMatrix//MatrixForm
```

Out[179]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Recall that `AdjacencyMatrix` produces a `SparseArray`. You could use `Normal` to transform the `SparseArray` object into a usual list of lists representation of the matrix. However, *Mathematica* can compute more efficiently with the `SparseArray`.

Next, compute some powers of the adjacency matrix.

In[180]:=
```
Table[MatrixForm[MatrixPower[aMatrix,i]],{i,2,7}]
```

```
Out[180]= {(0   1         1         1         0
          0       0         0         1         1
          1       1         0         1         0
          0       1         0         0         1
          1       0         1         0         0),
          (1      0         1         1         1
          1       1         0         1         0
          1       1         1         0         1
          0       1         1         1         0
          0       1         0         1         2),
          (1      2         0         2         2
          1       1         1         0         1
          0       2         1         2         2
          1       0         1         1         1
          1       2         1         2         0),
          (2      3         2         2         1
          0       2         1         2         2
          2       2         2         3         1
          1       2         0         2         2
          2       1         2         1         2),
          (2      3         3         3         4
          2       2         2         3         1
          3       3         2         3         4
          2       3         2         2         1
          1       4         1         4         4),
          (3      6         3         7         5
          3       3         2         3         4
          3       7         3         6         5
          2       3         3         3         4
          4       5         4         5         2)}
```

Note that the `MatrixPower` function is used to compute powers of matrices. Using the `Power` (ˆ) operator on a matrix computes element-wise.

We see that there are 4 paths of length 6 from vertex 3 to vertex 5, since the $(3, 5)$ entry in the 6th power of the adjacency matrix is 4. We also see that there are cycles of length 3 for every vertex and there are no cycles of length less than 3. Finally, we know that the shortest path from vertex 2 to vertex 1 is of length 3, since the $(2, 1)$ entry is 0 for the first and second powers of the matrix.

10.5 Euler and Hamilton Paths

In this section, we will show how to use *Mathematica* to solve two problems that seem closely related, but which are quite different in computational complexity. The two problems that will be analyzed are the problem of finding a simple circuit that contains every edge exactly once (an Euler circuit) and the problem of finding a simple circuit that contains every vertex exactly once (a Hamilton circuit). (Note that the textbook uses the term circuit while the Wolfram Language uses the word cycle. These two terms are synonymous.)

Euler Circuits

The Wolfram Language comes equipped with a function to determine if a given simple graph has an Euler circuit or not. This function, `EulerianGraphQ`, takes one argument, a `Graph` object. As an example, we have *Mathematica* check to see if K_5 is Eulerian, that is, has an Euler circuit.

```
In[181]:=  EulerianGraphQ[CompleteGraph[5]]

Out[181]=  True
```

To explicitly find an Euler circuit, we use the function `FindEulerianCycle`. The function accepts one or two arguments. The first argument should be a graph, although you can provide a list of rules specifying the edges rather than a `Graph` object. If the graph is the only argument, the function returns a list containing a single list of edges representing an Euler circuit. For example, the following identifies an Euler circuit on the complete graph K_5.

```
In[182]:=  FindEulerianCycle[CompleteGraph[5]]

Out[182]=  {{1<->5, 5<->4, 4<->3, 3<->5,
             5<->2, 2<->4, 4<->1, 1<->3, 3<->2, 2<->1}}
```

If you provide `FindEulerianCycle` with a positive integer as a second argument, *Mathematica* will attempt to find more than one Euler circuit, with the integer serving as a maximum number of cycles to return, provided they exist. In this case, the output will be a list of lists of edges, with each sublist representing a distinct circuit.

```
In[183]:=  FindEulerianCycle[CompleteGraph[5],3]

Out[183]=  {{1<->2, 2<->5, 5<->4, 4<->3,
             3<->5, 5<->1, 1<->4, 4<->2, 2<->3, 3<->1},
            {1<->2, 2<->5, 5<->4, 4<->3, 3<->5, 5<->1, 1<->3, 3<->2,
             2<->4, 4<->1}, {1<->2, 2<->5, 5<->4, 4<->3,
             3<->2, 2<->4, 4<->1, 1<->5, 5<->3, 3<->1}}
```

With the symbol `All` as the second argument, *Mathematica* will determine all of the Euler circuits. We see below that the complete graph on 5 vertices has 132 Euler circuits.

```
In[184]:=  Length[FindEulerianCycle[CompleteGraph[5],All]]

Out[184]=  132
```

Now, we will visualize this path by creating an animation that successively highlights the edges in the path. To do this we will use the `Animate` function. The `Animate` function takes two arguments, similar to `Table`. The first argument is a Wolfram Language expression, typically one that generates an image and that is dependent on a variable. The second argument is a list describing the range of the variable. The structure of this list is similar to the second argument in a `Table`.

We will also be making use of two options. Setting the `AnimationRunning` option to `False` will prevent the animation from beginning until you explicitly click on the play button. Without this option, the animation would run as soon as *Mathematica* has finished generating it. Setting the `AnimationRepetitions` option to 1 will cause the animation to stop once it has played through. This option defaults to `Infinity`, meaning it will automatically restart every time it reaches the end.

To create the animation, we need a graph and a circuit. We will use K_5 as the graph and we store the circuit as `exampleCircuit`. Note that we apply `First` since `FindEulerianCycle` returns a list of lists.

In[185]:=
```
exampleCircuit=First[FindEulerianCycle[CompleteGraph[5]]]
```

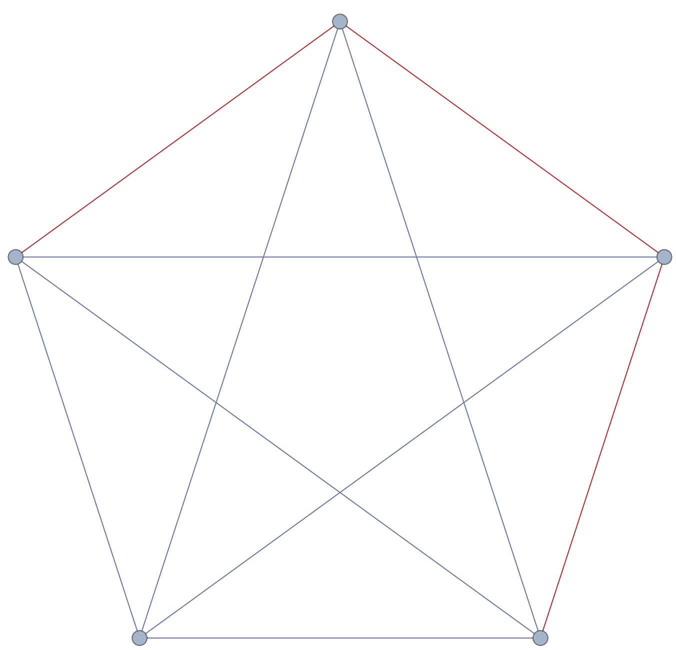
Out[185]=
```
{1⟷5,5⟷4,4⟷3,3⟷5,
  5⟷2,2⟷4,4⟷1,1⟷3,3⟷2,2⟷1}
```

To display the path, we will apply the `HighlightGraph` function. This function was first described in Section 10.2 in the section on bipartite graphs. It takes two arguments: a graph and a list containing vertices, edges, or subgraphs.

To draw the successive stages in the circuit, we will apply `HighlightGraph` to the graph and to a sublist of `exampleCircuit`. We will obtain the sublist by applying `Part` (`[[...]]`) to a `Span` (`;;`) from 1 to the current stage. For example, to display the path after three steps, we enter the following:

In[186]:=
```
HighlightGraph[CompleteGraph[5],exampleCircuit[[1;;3]]]
```

Out[186]=



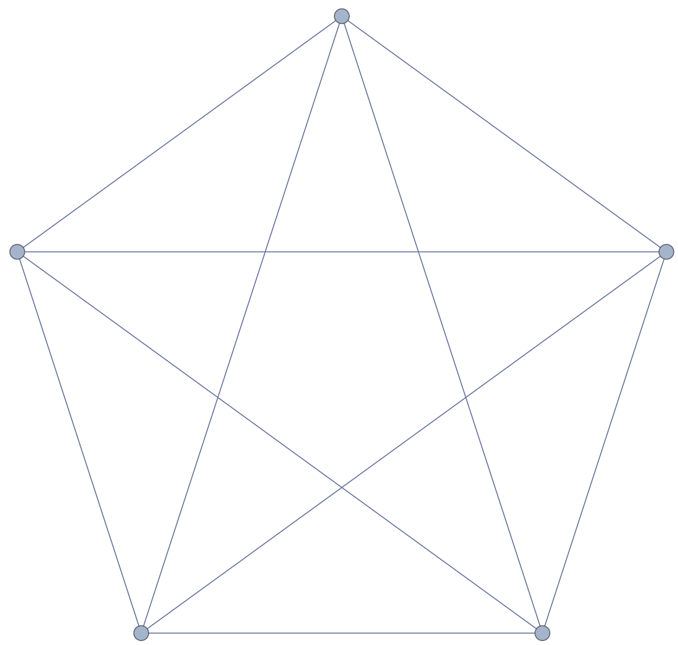
Note that the span from 1 to 0 will result in the empty list and thus nothing highlighted.

In[187]:=
```
exampleCircuit[[1;;0]]
```

Out[187]=
```
{ }
```

In[188]:=
```
HighlightGraph[CompleteGraph[5],exampleCircuit[[1;;0]]]
```

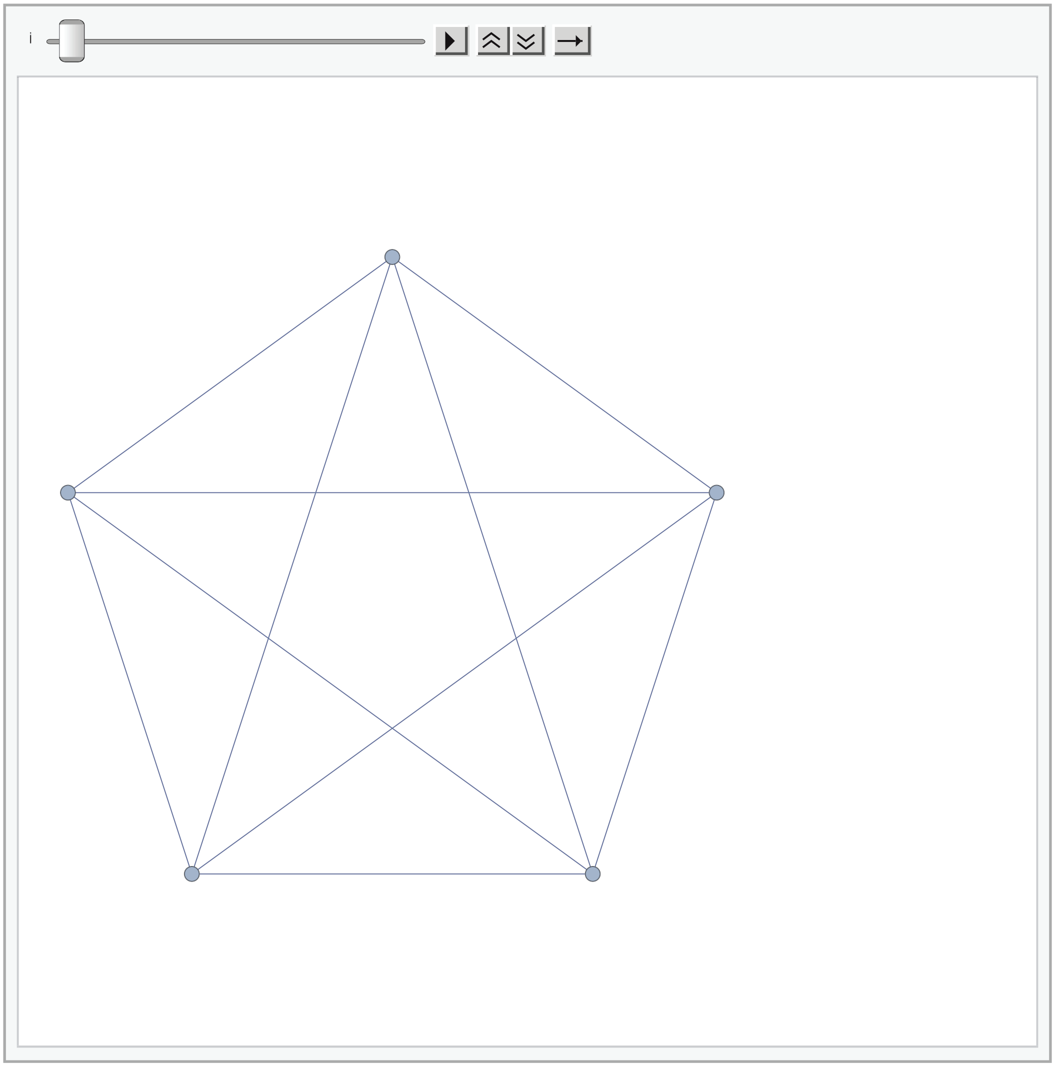
Out[188]=



We produce the animation using `HighlightGraph` as above, with the second argument to the `Span` (`;;`) as a variable. In the second argument of `Animate`, this variable will be set to run from 0 to the `Length` of the path. Note that, unlike `Table`, the variables in an `Animate` are not assumed to be restricted to integers, so we must specify a step value of 1 in the variable specification.

In[189]:=
```
Animate[HighlightGraph[CompleteGraph[5],
    exampleCircuit[[1;;i]]],
  {i,0,Length[exampleCircuit],1},AnimationRunning→False,
  AnimationRepetitions→1]
```

Out[189]=

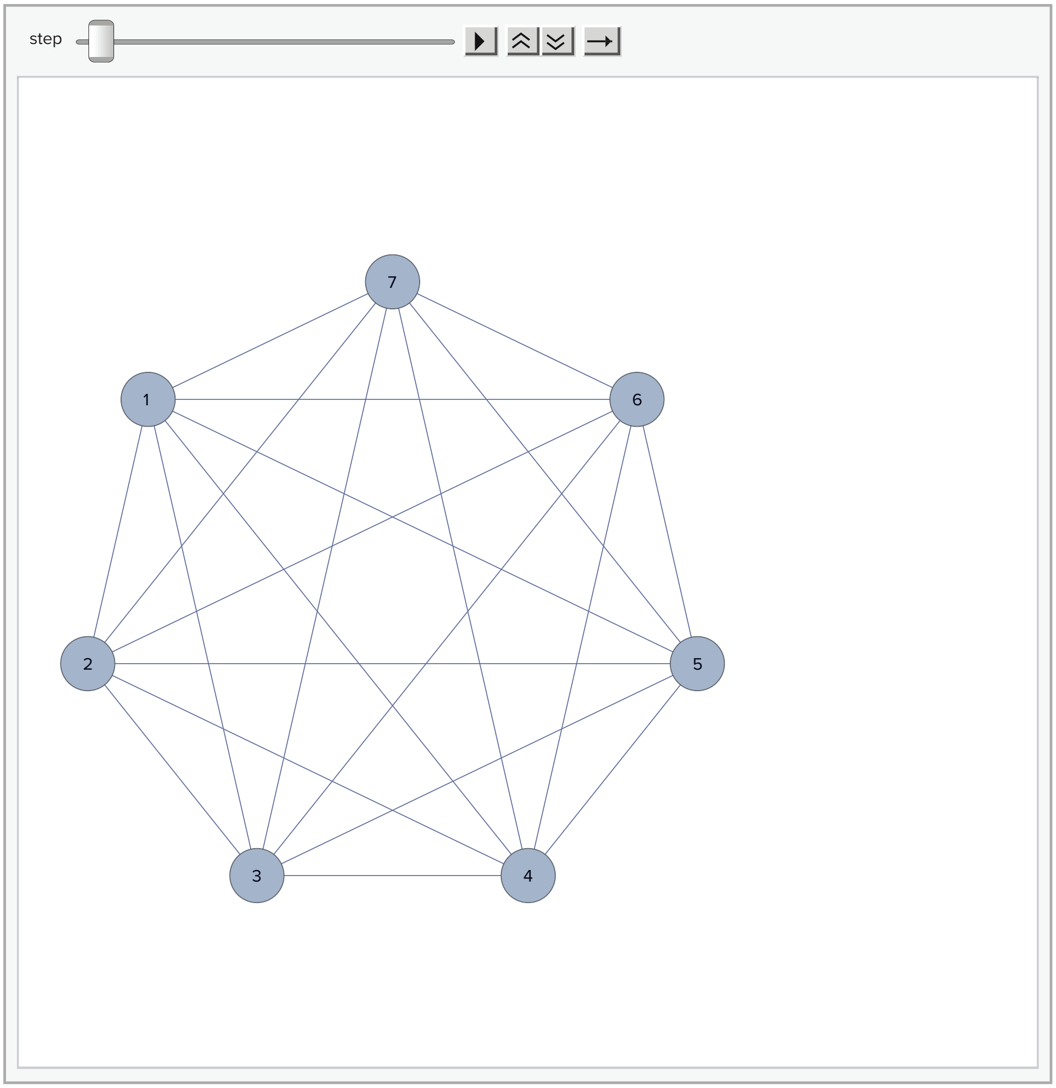


We now turn this into a function. The only difference between the `animatePath` function below and the example from above is that we replace the variable in the loop specification with the list `{i,0,"step"}`. This syntax gives the variable `i` the initial value 0 and labels it as "step" in the animation controller.

In[190]:=
```
animatePath[g_Graph,p_List]:=Module[{i,len},
  len=Length[p];
  Animate[HighlightGraph[g,p[[1;;i]]],
    {{i,0,"step"},0,len,1},AnimationRunning→False,
    AnimationRepetitions→1]
]
```

To use this function, we just pass it an Eulerian graph, a circuit, and any options for drawing the graph.

In[191]:=
```
animatePath[CompleteGraph[7,VertexSize→Medium,
  VertexLabels→Placed["Name",Center]],
  First[FindEulerianCycle[CompleteGraph[7]]]]
```

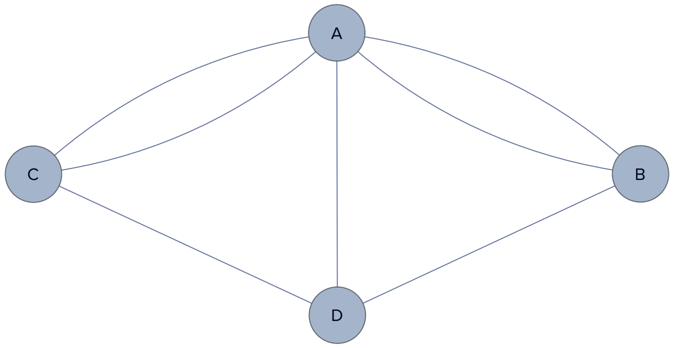


You can see the Euler circuit traced out by clicking on the play button.

Note that while our examples have all been undirected, the functions described here also apply to directed graphs, multigraphs, etc. Here, for example, is the bridges of Königsberg graph.

In[192]:=
```
konigsberg=
  Graph[{"A"→"B","A"→"B","A"→"C","A"→"C",
    "A"→"D","B"→"D","C"→"D"},DirectedEdges→False,
    VertexSize→Medium,VertexLabels→Placed["Name",Center]]
```

Out[192]=



In[193]:=
```
EulerianGraphQ[konigsberg]
```

Out[193]=
```
False
```

Euler Paths

While the Wolfram Language's includes a built-in function for finding Eulerian circuits, it does not have a function for finding Eulerian paths that are not circuits. Moreover, building a function for finding Euler paths from scratch can be helpful in understanding the algorithm for finding both paths and circuits. We know, from Theorem 1 of Section 10.5, that a connected multigraph with at least two vertices has an Euler circuit if and only if the degree of every vertex is even. We also know, from Theorem 2, that if the graph has exactly two vertices of odd degree, then there is a Euler path which is not a circuit.

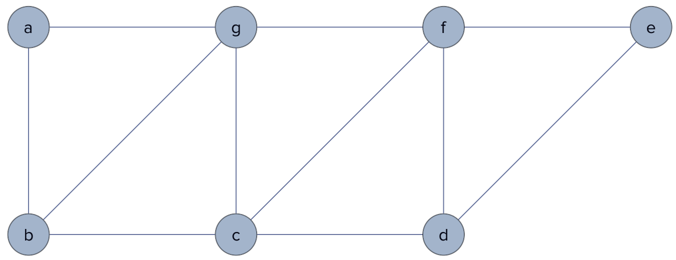
Using this fact, we can write a simple function for determining whether or not an undirected pseudograph (multiple edges and loops allowed) has an Euler path. The function first uses the Wolfram Language built-in functions to confirm that the graph given as the argument is undirected and connected. It uses `VertexDegree` to obtain the list of the degrees of the vertices in the graph. We then `Map` (`/@`) the `EvenQ` predicate over the list to transform the list of degrees into a list of true and false. Finally, we `Count` those that are false and check whether that number, that is, the number of vertices of odd degree, is 0 or 2. We include 0, since an Euler circuit is a path. Note that `MemberQ` determines whether the second argument is a member of the first argument.

In[194]:=
```
eulerPathQ[g_Graph]:=UndirectedGraphQ[g]&&ConnectedGraphQ[g]&&
  MemberQ[{0,2},Count[EvenQ/@VertexDegree[g],False]]
```

Consider the graph G_2 from Example 4, which has an Euler path, but not a circuit.

In[195]:=
```
figure7G2=Graph[{"a","g","f","e","b","c","d"},
  {"a"→"g","a"→"b","b"→"c","b"→"g","c"→"d",
   "c"→"g","c"→"f","d"→"e","d"→"f","e"→"f",
   "f"→"g"},DirectedEdges→False,
  VertexLabels→Placed["Name",Center],
  VertexSize→Medium,
  VertexCoordinates→
   {{0,1},{1,1},{2,1},{3,1},{0,0},{1,0},{2,0}}]
```

Out[195]=



In[196]:=
```
EulerianGraphQ[figure7G2]
```

Out[196]=
```
False
```

In[197]:=
```
eulerPathQ[figure7G2]
```

Out[197]=
```
True
```

Now that we have a test that tells us if a path exists, we modify Algorithm 1 from Section 10.5 in order to find an Euler path, if it exists. The difference between this function and Algorithm 1 is that, if there is not an Euler circuit, we need to be sure to begin at one of the vertices of odd degree and end at the other. To do this, we begin by creating a new graph with an additional vertex adjacent to the two vertices of odd degree. We then follow Algorithm 1 to build an Euler circuit, and then cut out the temporary vertex.

In[198]:=
```
findEulerianPath[g_Graph]:=
  Module[{H,pathQ,oddvertices,newvertex,circuit,subC,
    i,n,v,insertPoint,w,buildingSub,oldC,
    newVlocations},
  If[!eulerPathQ[g],Return[$Failed]];
  circuit={};
  H=g;
  If[Count[EvenQ/@VertexDegree[g],False]==2,
    pathQ=True;
    oddvertices=Select[VertexList[g],
      OddQ[VertexDegree[g,#]]&];
    H=VertexAdd[H,newvertex];
    H=EdgeAdd[H,{newvertex→oddvertices[[1]],
      newvertex→oddvertices[[2]]}],
    pathQ=False
  ];
  While[EdgeList[H]≠{},
    (* find a starting point *)
    If[circuit=={},
      subC={EdgeList[H][[1]]};
      H=EdgeDelete[H,subC[[1]]];
      insertPoint=0,
      For[i=1,i≤Length[circuit],i++,
        v=circuit[[i,2]];
        n=neighbors[H,v];
```

```
        If[n≠{},
          w=n[[1]];
          subC={v→w};
          insertPoint=i;
          H=EdgeDelete[H,UndirectedEdge[v,w]];
          Break[]
        ]
      ]
    ];
    (* build a subcircuit *)
    buildingSub=True;
    While[buildingSub&&EdgeList[H]≠{},
      v=subC[[-1,2]];
      w=First[neighbors[H,v]];
      H=EdgeDelete[H,UndirectedEdge[v,w]];
      AppendTo[subC,UndirectedEdge[v,w]];
      If[w==subC[[1,1]],buildingSub=False]
    ];
    (* splice the subcircuit into the main circuit *)
    If[circuit=={},
      circuit=subC,
      circuit=
        Flatten[Insert[circuit,subC,insertPoint+1]]
    ]
  ];
  If[pathQ,
    newVlocations=Position[circuit,newvertex][[All,1]];
    Join[circuit[[Max[newVlocations]+1;;-1]],
      circuit[[1;;Min[newVlocations]-1]]],
    circuit
    ]
  ]
```

The function begins with a use of `eulerPathQ` in order to avoid searching for a path that cannot exist. It then assigns to the symbol `H` a copy of the graph. It is this copy that is used throughout the rest of the function, rather than the input that was passed to the algorithm. The benefit of using a copy is that the function will be able to manipulate it as the algorithm proceeds, for example, by deleting edges of `H` once they are included in the circuit so that those edges are not reused.

After copying the input graph, the function tests to see whether it has two vertices of odd degree. If it does, then the graph `H` is modified to include a new vertex and edges between this new vertex and the two with odd degree. The result is that `H` has vertices of all even degree, and thus an Euler circuit must exist. Note that the symbol used for the name of the new vertex is localized to the module. This means that, in practice, the symbol actually being used is something along the lines of `newvertex$371`, so it is unlikely to already exist in the graph. The function also sets a flag, `pathQ`, to true or false, depending on whether the graph has a path or circuit.

Once `H` has been modified, if needed, the function follows Algorithm 1 from Section 10.5. There are two key ideas at the heart of this algorithm. The first is that, for a graph whose vertices all have even degree, if you pick any vertex to start at and follow edges at random but without repetition, you will definitely return to the original vertex and create a circuit. The second key idea is that (for a connected graph), if your circuit does not include all of the edges of the graph, then some vertex used in the existing circuit can be made the starting point for a new subcircuit. This subcircuit can then be spliced into the main circuit. This will eventually use all the edges and the result will be a Euler circuit.

The symbol `circuit` will hold the main circuit that the function finds in `H`. The circuit will be stored as a list of the edges through which the circuit passes and is initialized to the empty list. The main `While` loop consists of three parts: (1) determining the starting point for the subcircuit (named `subC`); (2) building the subcircuit; and (3) splicing the subcircuit into the main circuit.

The first step, finding the starting point for the subcircuit, depends on the state of the main circuit. If `circuit` is the empty list (i.e., this is the first pass through the main loop), then the starting point is the first edge in the graph. If the main circuit is not empty, then the else clause looks at the vertices in the main circuit to find one that has neighbors (since edges are deleted from `H` as they are added to the circuit, only vertices that are an endpoint of an unused edge have neighbors). The first vertex that has a neighbor is used as the starting point for the subcircuit. The `insertPoint` variable is used to keep track of the index, relative to `circuit`, of the starting vertex for the subcircuit. This is used when the subcircuit is spliced into the main circuit.

The second step is to build `subC`. The `buildingSub` symbol is used to control the `While` loop. It is initialized to true and is set to false once `subC` has returned to its starting vertex and is thus a circuit. The variable `v` is set to the last vertex currently included as part of the subcircuit and `w` represents a neighbor of `v`. To remove the edge between `v` and `w` from `H`, we use `EdgeDelete`.

After deleting the edge from `H`, the newest vertex is compared with the starting vertex to determine if the circuit has been closed. If the new vertex closes the circuit, then the `buildingSub` variable is set to false, which causes the loop to terminate. Otherwise, the `While` loop continues building the subcircuit.

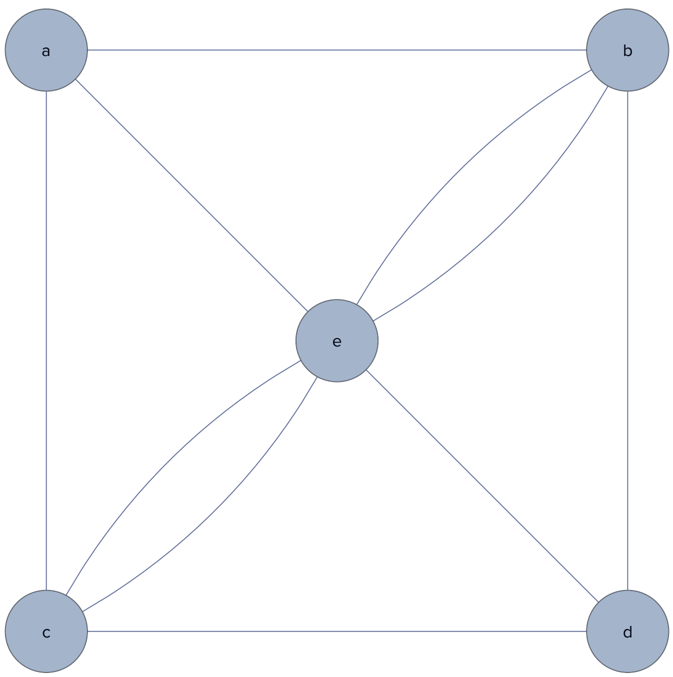
The third step, once the subcircuit has been built, is to splice it into the main circuit. If `circuit` is empty, it is merely set to `subC`. Otherwise, `Insert` is used to add the subcircuit. `Insert` takes a list as the first argument and adds the expression given as the second argument in the position specified by the third argument. The elements previously in that position and later and pushed down to make room. We give the third argument as `insertPoint+1` to insert the sublist between the edges at position `insertPoint` and position `insertPoint+1`.

The main `While` loop continues until all the edges of the graph have been included in the circuit, making circuit an Euler circuit for `H`. Then the final step is to either output the circuit, if the original graph has an Euler circuit, or to cut the temporary vertex and corresponding edges out of the circuit and produce the Euler path. For this, we use `Position`, with first argument a list and second argument a pattern to search for, to find the locations of the temporary vertex in the circuit. The vertex will appear twice and its location will be reported as a pair consisting of the position within `circuit` and a 1 or 2 corresponding to its location on the edge. The `Part` specification `[[All,1]]` returns the first element in all subexpressions and thus extracts the locations of the edges within the `circuit` list containing the temporary vertex. Once the locations of the temporary vertex have been found, the function uses `Join` to combine the part of the circuit after it has visited the temporary vertex with the part before. The result is the Euler path beginning at one of the vertices of odd degree and ending at the other.

As an example, consider Exercise 5 from Section 10.5.

In[199]:=
```
exercise3=Graph[{"a","b","c","d","e"},
  {"a"→"b","a"→"e","a"→"c","b"→"d","b"→"e",
   "b"→"e","c"→"d","c"→"e","c"→"e","d"→"e"},
  DirectedEdges→False,VertexSize→Medium,
  VertexLabels→Placed["Name",Center],
  VertexCoordinates→
    {{0,1},{1,1},{0,0},{1,0},{0.5,0.5}}]
```

Out[199]=



In[200]:=
```
exercise3EulerPath=findEulerianPath[exercise3]
```

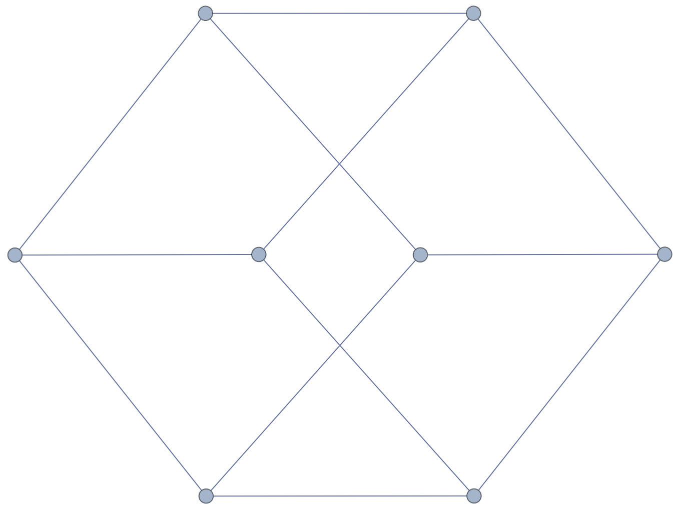
Out[200]=
```
{d⟷e,e⟷b,b⟷d,d⟷c,
  c⟷a,a⟷b,b→e,e→c,c⟷e,e⟷a}
```

Hamilton Circuits

Turning our attention to Hamilton circuits, the Wolfram Language provides the function `HamiltonianGraphQ` for determining whether or not the graph contains a Hamilton circuit. This function, like `EulerianGraphQ`, accepts a graph as the sole argument. It returns true or false depending on whether the graph has a Hamilton circuit.

In[201]:=
```
hcGraphExample=HypercubeGraph[3]
```

Out[201]=



In[202]:=
```
HamiltonianGraphQ[hcGraphExample]
```

Out[202]=
```
True
```

The `FindHamiltonianCycle` function, similar to `FindEulerianCycle`, accepts a graph as the argument and returns a list containing a Hamiltonian circuit. The optional second argument can be used to find more than one cycle.

In[203]:=
```
FindHamiltonianCycle[hcGraphExample]
```

Out[203]=
```
{{1⟷2,2⟷4,4⟷8,8⟷6,6⟷5,5⟷7,7⟷3,3⟷1}}
```

In[204]:=
```
FindHamiltonianCycle[hcGraphExample,All]
```

Out[204]=
```
{{1⟷3,3⟷7,7⟷8,8⟷4,4⟷2,2⟷6,6⟷5,5⟷1},
  {1⟷3,3⟷4,4⟷2,2⟷6,6⟷8,8⟷7,7⟷5,5⟷1},
  {1⟷2,2⟷6,6⟷8,8⟷4,4⟷3,3⟷7,7⟷5,5⟷1},
  {1⟷2,2⟷4,4⟷3,3⟷7,7⟷8,8⟷6,6⟷5,5⟷1},
  {1⟷2,2⟷6,6⟷5,5⟷7,7⟷8,8⟷4,4⟷3,3⟷1},
  {1⟷2,2⟷4,4⟷8,8⟷6,6⟷5,5⟷7,7⟷3,3⟷1}}
```

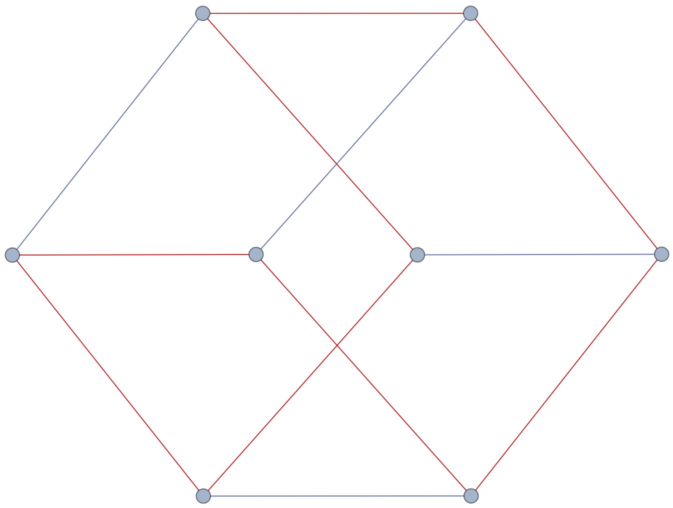
In[205]:=
```
hcGraphExampleCircuit=
  First[FindHamiltonianCycle[hcGraphExample]]
```

Out[205]=
```
{1⟷2,2⟷4,4⟷8,8⟷6,6⟷5,5⟷7,7⟷3,3⟷1}
```

We can use `HighlightGraph` to illustrate the path statically.

In[206]:=
```
HighlightGraph[hcGraphExample,hcGraphExampleCircuit]
```

Out[206]=

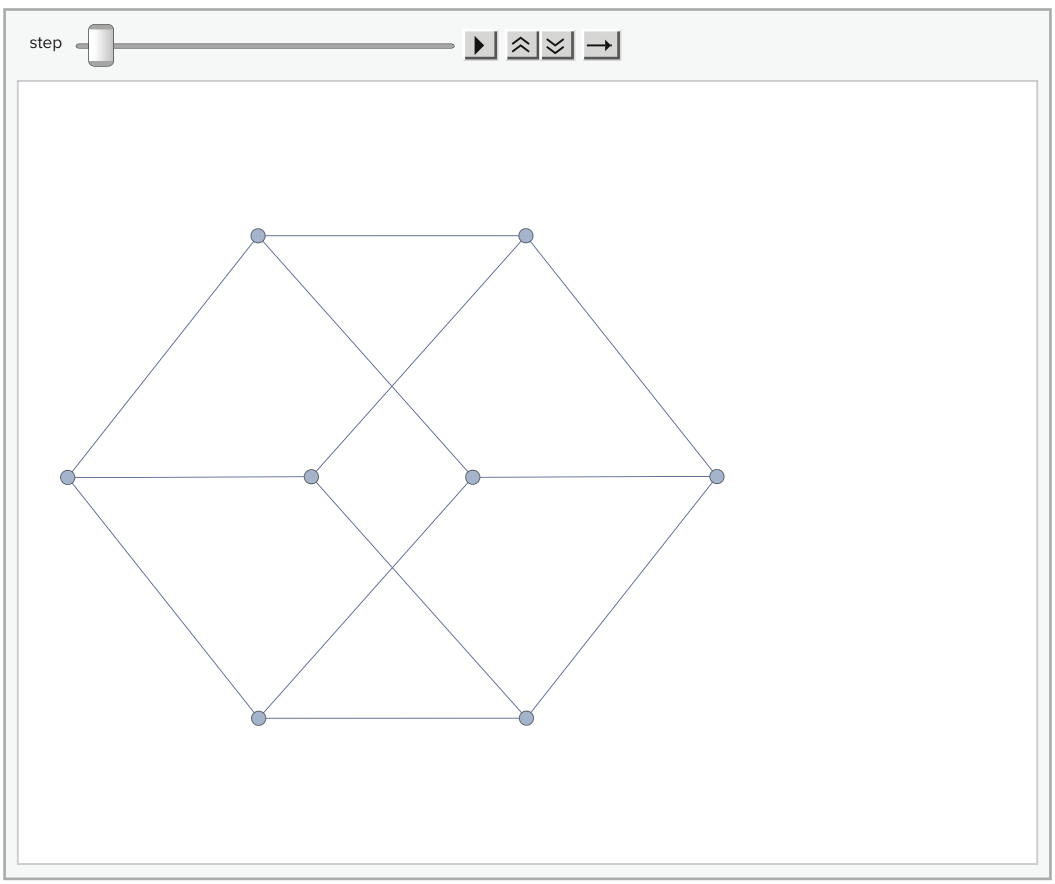


Moreover, our animation function `animatePath`, created above, works equally well here.

In[207]:=

```
animatePath[hcGraphExample,hcGraphExampleCircuit]
```

Out[207]=



10.6 Shortest-Path Problems

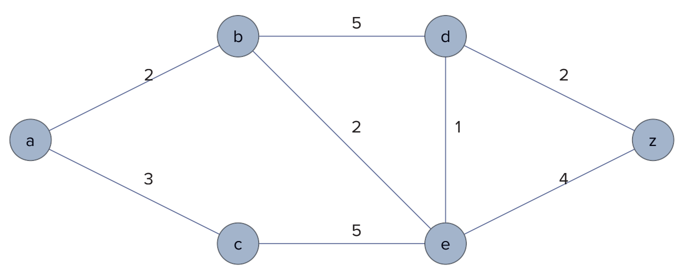
Among the most common problems in graph theory are shortest path problems. Generally, in shortest path problems, we wish to determine a path between two vertices of a weighted graph that is minimal in terms of the total weight of the edges in the path.

To define a `Graph` object with weighted edges, use the `EdgeWeight` option. The value associated with the option is the list of the weights of each edge. The weights must appear in the same order as they are displayed in the output by `EdgeList`. For graphs you define by listing the edges, this is identical to the order you give in the definition.

We reproduce Exercise 2 from Section 10.6 of the textbook to use as an example.

In[208]:=
```
exercise2=Graph[{"a","b","c","d","e","z"},
  {"a"→"b","a"→"c","b"→"d","b"→"e","c"→"e",
   "d"→"e","d"→"z","e"→"z"},DirectedEdges→False,
  EdgeWeight->{2,3,5,2,5,1,2,4},
  VertexCoordinates→
    {{0,.5},{1,1},{1,0},{2,1},{2,0},{3,.5}},
  VertexSize→Medium,VertexLabels→Placed["Name",Center],
  EdgeLabels→Placed["EdgeWeight",{.5,{-1,0}}]]
```

Out[208]=



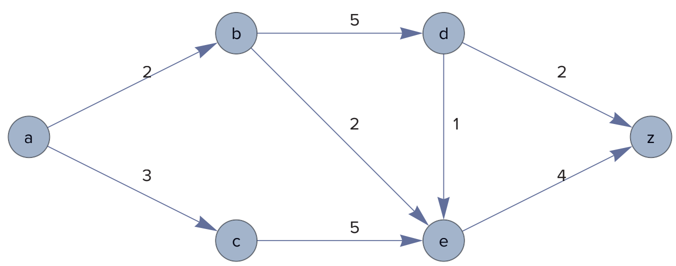
Note the use of the `EdgeLabels` option to display the edge weights. The second argument of `Placed` indicates that the label should appear halfway between the vertices and should be shifted to the right. The form of the second argument is $\{s, \{l_x, l_y\}\}$, where s indicates fraction along the edge that the label should appear and $\{l_x, l_y\}$, indicates what position within the label is to be located at that position on the line. For example, `{0,0}` indicates that the lower left corner of the label should appear on the edge, while `{-1,0}` means that a point to the left of the label should be placed at the edge. You can imagine the label surrounded by a box, and $\{l_x, l_y\}$ indicates, relative to the lower left corner of that box, where the label should be anchored to the edge. Also, the values are relative to the size of the label, so `{-1,0}` refers to a point to the left of the label at a distance equal to the width of the label.

A second approach to setting edge weights is by using the `EdgeWeight` property. Rather than listing all of the edges and then listing the weights separately using the option as shown above, we can wrap the edge definitions in the `Property` wrapper, setting the edge weights at the same time the edges are described.

In[209]:=
```
Graph[{"a","b","c","d","e","z"},
  {Property["a"→"b",EdgeWeight→2],
  Property["a"→"c",EdgeWeight→3],
  Property["b"→"d",EdgeWeight→5],
```

```
Property["b"→"e",EdgeWeight→2],
Property["c"→"e",EdgeWeight→5],
Property["d"→"e",EdgeWeight→1],
Property["d"→"z",EdgeWeight→2],
Property["e"→"z",EdgeWeight→4]},
VertexCoordinates→
  {{0,.5},{1,1},{1,0},{2,1},{2,0},{3,.5}},
VertexSize→Medium,VertexLabels→Placed["Name",Center],
EdgeLabels→Placed["EdgeWeight",{.5,{-1,0}}]]
```

Out[209]=



Note in the above that we omitted the `DirectedEdges` option to illustrate that there is no difference with a directed graph.

You can also define a weighted graph using the `WeightedAdjacencyGraph` function and an adjacency matrix. Unlike with `AdjacencyGraph`, the adjacency matrix must use `Infinity`, or ∞ (ESC inf ESC) to indicate that there is no edge between the corresponding vertices. In the example below, we give the list of vertices as the first argument, otherwise *Mathematica* will automatically use positive integers to name the vertices. Note that `WeightedAdjacencyMatrix` applied to a weighted graph returns the adjacency matrix with weights.

In[210]:=

```
exercise2matrix={{∞,2,3,∞,∞,∞},{2,∞,∞,5,2,∞},
  {3,∞,∞,∞,5,∞},{∞,5,∞,∞,1,2},{∞,2,5,1,∞,4},
  {∞,∞,∞,2,4,∞}};
exercise2matrix//MatrixForm
```

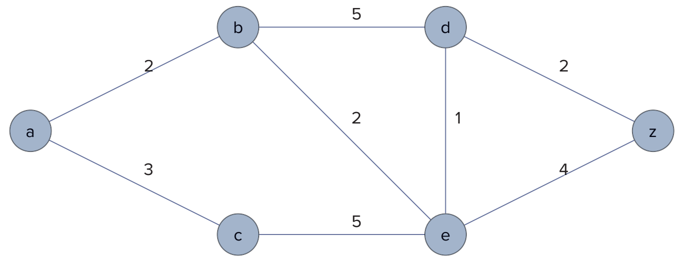
Out[211]//MatrixForm=

$$\begin{pmatrix} \infty & 2 & 3 & \infty & \infty & \infty \\ 2 & \infty & \infty & 5 & 2 & \infty \\ 3 & \infty & \infty & \infty & 5 & \infty \\ \infty & 5 & \infty & \infty & 1 & 2 \\ \infty & 2 & 5 & 1 & \infty & 4 \\ \infty & \infty & \infty & 2 & 4 & \infty \end{pmatrix}$$

In[212]:=

```
WeightedAdjacencyGraph[{"a","b","c","d","e","z"},
  exercise2matrix,
  VertexCoordinates→
    {{0,.5},{1,1},{1,0},{2,1},{2,0},{3,.5}},
  VertexSize→Medium,VertexLabels→Placed["Name",Center],
  EdgeLabels→Placed["EdgeWeight",{.5,{-1,0}}]]
```

Out[212]=



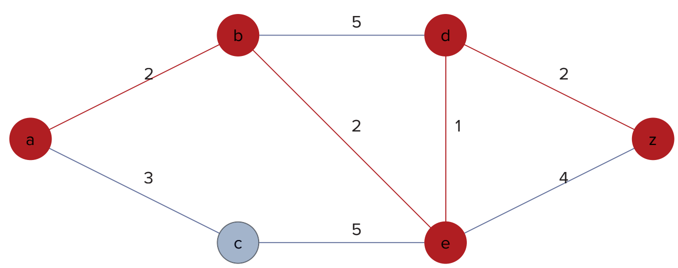
Now, we will make use of the Wolfram Language's implementation of path finding algorithms to compute the shortest path between a and z. To do this, we simply call the `FindShortestPath` function with three arguments: the graph and the names of the starting and ending vertices. The output will be a list of vertices beginning with the starting vertex and ending with the final vertex through which the path runs.

```
In[213]:= FindShortestPath[exercise2,"a","z"]
Out[213]= {a,b,e,d,z}
```

The `PathGraph` function can be used to help visualize this path. `PathGraph` accepts a list of vertices as input and produces the graph obtained by connecting successive vertices with an edge. By default, the graph produced is undirected. To obtain a directed path, assign the `DirectedEdges` option to `True`. Here, we will display the path in the graph by applying `HighlightGraph`, using `PathGraph` in the second argument.

```
In[214]:= HighlightGraph[exercise2,
            PathGraph@FindShortestPath[exercise2,"a","z"]]
```

Out[214]=



The length of the shortest path can be determined with the `GraphDistance` function and the same arguments.

```
In[215]:= GraphDistance[exercise2,"a","z"]
Out[215]= 7.
```

If the final argument, the destination vertex, is omitted from `GraphDistance`, the result will be a list of the lengths of the shortest paths to each vertex of the graph.

```
In[216]:= GraphDistance[exercise2,"a"]
Out[216]= {0,2.,3.,5.,4.,7.}
```

To determine the shortest path from every vertex to every other vertex, use the `GraphDistanceMatrix` function. Algorithm 2 in the Exercises of Section 10.6 describes the Floyd–Warshall algorithm (also known as simply the Floyd algorithm), which is one method for computing this matrix.

In[217]:=
```
GraphDistanceMatrix[exercise2]//MatrixForm
```

Out[217]//MatrixForm=

$$\begin{pmatrix} 0. & 2. & 3. & 5. & 4. & 7. \\ 2. & 0. & 5. & 3. & 2. & 5. \\ 3. & 5. & 0. & 6. & 5. & 8. \\ 5. & 3. & 6. & 0. & 1. & 2. \\ 4. & 2. & 5. & 1. & 0. & 3. \\ 7. & 5. & 8. & 2. & 3. & 0. \end{pmatrix}$$

10.7 Planar Graphs

This section explains how *Mathematica* can be used to explore the question of whether a graph is planar by manipulating graphs in order to produce homeomorphic graphs and applying Kuratowski's Theorem. We consider only undirected simple graphs in this section. The question of planarity for a directed graph can be answered by considering the underlying undirected graph, which can be obtained by applying the function `UndirectedGraph`.

Before looking at how *Mathematica* can help you manipulate graphs to apply Kuratowski's theorem manually, note that the function `PlanarGraphQ`, applied to a `Graph` object, will determine whether or not the graph is planar.

In[218]:=
```
PlanarGraphQ[HypercubeGraph[3]]
```

Out[218]=
```
True
```

In[219]:=
```
PlanarGraphQ[CompleteGraph[5]]
```

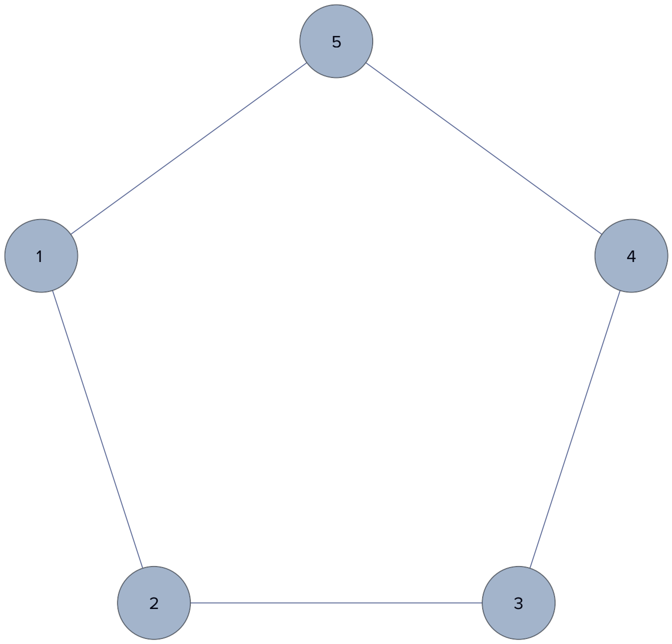
Out[219]=
```
False
```

Elementary Subdivisions, Smoothing, and Homeomorphic Graphs

Recall that an elementary subdivision refers to the process of modifying a graph by removing an edge $\{u, v\}$ and replacing it with a vertex w and new edges $\{u, w\}$ and $\{w, v\}$. Effectively, this splits the original edge into two by inserting a vertex in the middle of it. It is not difficult to perform an elementary subdivision. We will use a cycle graph as an example.

In[220]:=
```
subdivideExample=CycleGraph[5,VertexSize→Medium,
  VertexLabels→Placed["Name",Center]]
```

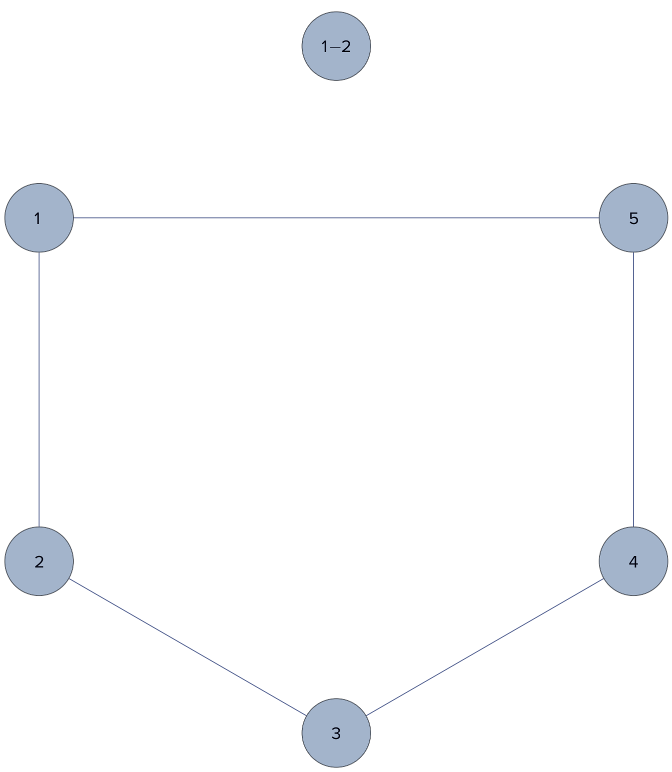
Out[220]=



We will subdivide the edge $\{1, 2\}$. To do this, we first need to introduce a vertex. We will give this vertex the name “1-2”. We apply `ToString` to the integers to create strings and combine them and the hyphen with the `StringJoin` (<>) operator. We use the `VertexAdd` function to add the vertex to the graph.

In[221]:=
```
subdivideExampleB=
  VertexAdd[subdivideExample,
    ToString[1]<>"-"<>ToString[2]]
```

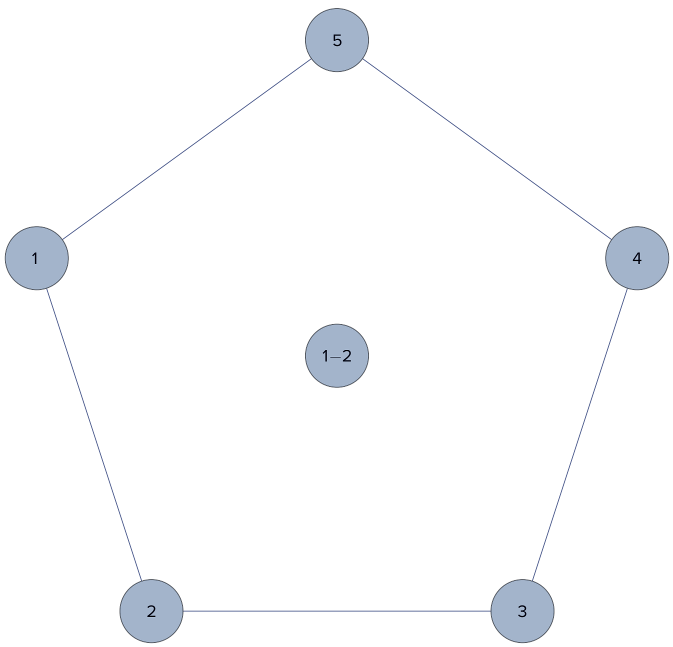
Out[221]=



Note that *Mathematica* has moved the original vertices. We can put them back in place by assigning the `VertexCoordinates` option in the new graph to the positions of the original vertices obtained from `GraphEmbedding`. We just need to append a location for the new vertex, which will be at the end of the list of vertices. Note that we need to apply `Graph` to the result of the `VertexAdd`, since that function will not accept the options normally available to graphs.

In[222]:=
```
subdivideExampleB=
  Graph[VertexAdd[subdivideExample,
      ToString[1]<>"-"<>ToString[2]],
    VertexCoordinates→
      Append[GraphEmbedding[subdivideExample],{0,0}]]
```

Out[222]=



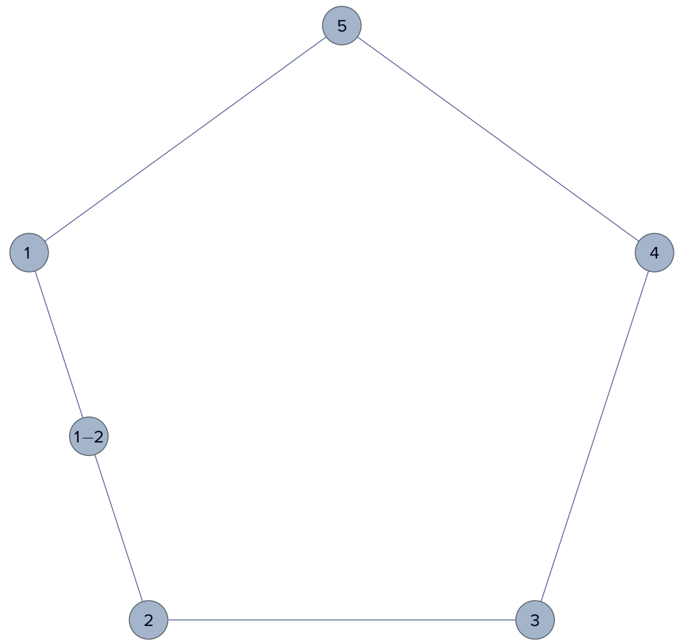
Now, we move the new vertex along the edge which is to be deleted. This is done by using `PropertyValue` and averaging the positions of the existing vertices.

In[223]:=
```
PropertyValue[{subdivideExampleB,"1-2"},
    VertexCoordinates]=
  (PropertyValue[{subdivideExample,1},VertexCoordinates]
    +PropertyValue[{subdivideExample,2},
      VertexCoordinates])/2
```

Out[223]=
```
{-0.769421,-0.25}
```

In[224]:=
```
subdivideExampleB
```

Out[224]=

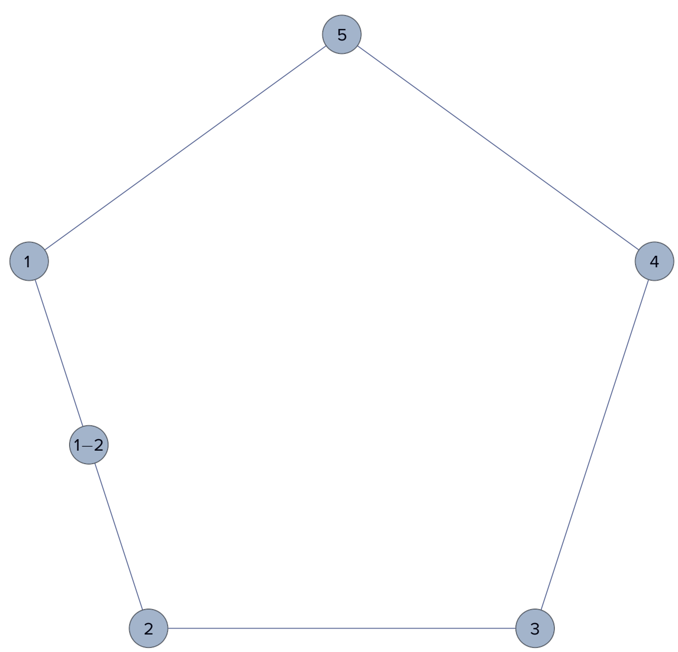


Finally, we simply remove the original edge with EdgeDelete and add the new ones with EdgeAdd. Recall that the second argument of EdgeDelete must be given using the special symbols ⊷ (ESC ue ESC) or ⟶(ESC de ESC), not as a Rule (->).

In[225]:=

```
subdivideExampleB=EdgeDelete[subdivideExampleB,1→2];
subdivideExampleB=
  EdgeAdd[subdivideExampleB,{1→"1-2",2→"1-2"}]
```

Out[225]=



Based on this example, we create the following function.

In[227]:=
```
subdivideGraph::nonedge=
  "Second argument must be an edge in the graph.";
subdivideGraph[G_Graph,E_Rule|E_UndirectedEdge]/;
    UndirectedGraphQ[G]:=Module[{H,e,newV,v},
  If[Head[E]===Rule,
    e=UndirectedEdge@@E,
    e=E
  ];
  If[!EdgeQ[G,e],Message[subdivideGraph::nonedge];
    Return[$Failed]];
  newV=ToString[e[[1]]]<>"-"<>ToString[e[[2]]];
  H=Graph[VertexAdd[G,newV],
    VertexCoordinates→Append[GraphEmbedding[G],
      {0,0}]];
  PropertyValue[{H,newV},VertexCoordinates]=
    (PropertyValue[{H,e[[1]]},VertexCoordinates]+
      PropertyValue[{H,e[[2]]},VertexCoordinates])/2;
  H=EdgeDelete[H,e];
  H=EdgeAdd[H,{e[[1]]→newV,newV→e[[2]]}];
  H
]
```

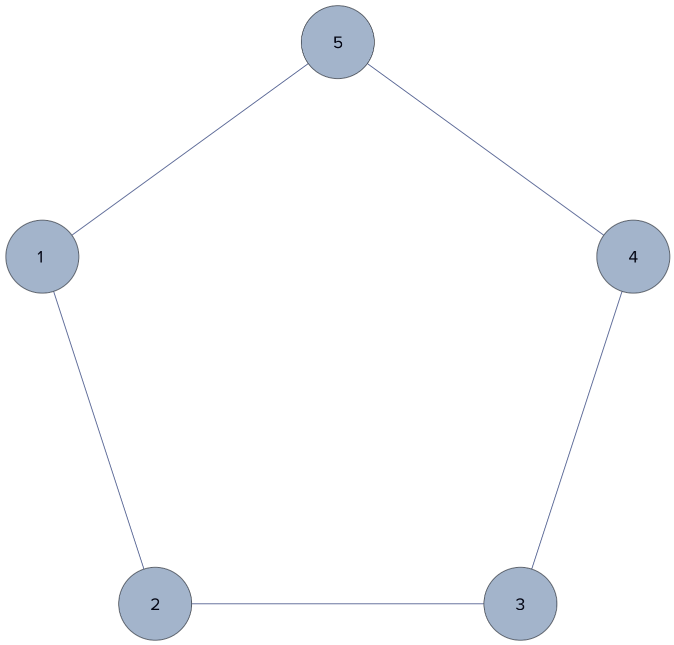
The inverse operation of elementary subdivision is referred to as smoothing. To be precise, let v be a vertex of degree 2 with neighbors u and w and such that u and w are not adjacent. We smooth the vertex v by deleting v and the edges incident to it and adding the edge $\{u, w\}$. Below, we have created a function to implement smoothing.

In[229]:=
```
smoothGraph::vertx="Cannot smooth this vertex.";
smoothGraph[G_Graph,v_]/;UndirectedGraphQ[G]:=
  Module[{N,H,e},
    N=AdjacencyList[G,v];
    If[Length[N]≠2||
       EdgeQ[G,UndirectedEdge[N[[1]],N[[2]]]],
      Message[smoothGraph::vertx];
      Return[$Failed]];
    H=VertexDelete[G,v];
    e=UndirectedEdge[N[[1]],N[[2]]];
    H=EdgeAdd[H,e];
    H
  ]
```

As an example, we can smooth the vertex “1-2” added above.

In[231]:=
```
smoothGraph[subdivideExampleB,"1-2"]
```

Out[231]=



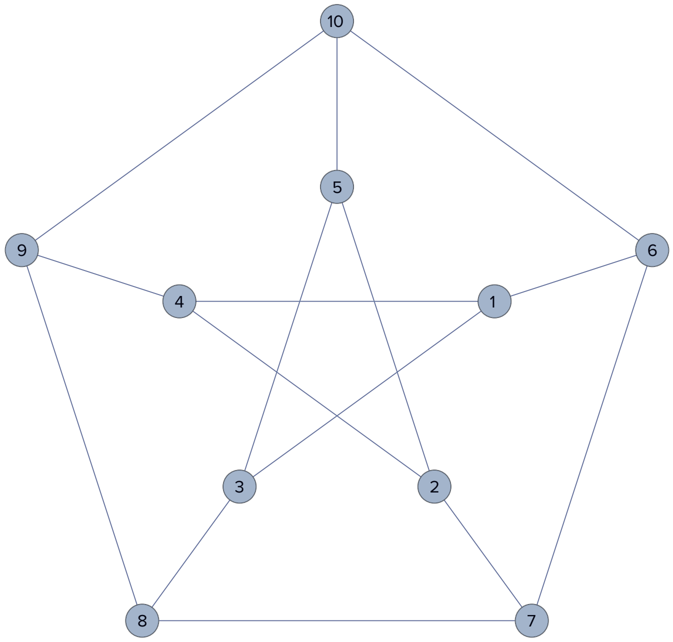
The textbook defines graphs to be homeomorphic if they can be obtained from the same graph from a sequence of elementary subdivisions. It is clear that if $G_1, G_2, G_3, \ldots, G_n$ is a sequence of graphs, each of which can be obtained from the previous by an elementary subdivision, then $G_n, \ldots, G_3, G_2, G_1$ is a sequence of graphs, each of which can be obtained from the previous by a smoothing. Hence, we can say that two graphs are homeomorphic if one can be transformed into the other by a sequence of elementary subdivisions and smoothings.

Applying Kuratowski’s Theorem

Recall that Kuratowski’s theorem asserts that a graph is nonplanar if and only if it contains a subgraph homeomorphic to either $K_{3,3}$ or K_5. Using the functions above and those for creating subgraphs, we can use *Mathematica* to manipulate a graph and confirm that it is nonplanar using Kuratowski’s theorem. We will illustrate this with the Petersen graph.

In[232]:=
```
petersen=PetersenGraph[5,2,VertexSize→Medium,
  VertexLabels→Placed["Name",Center]]
```

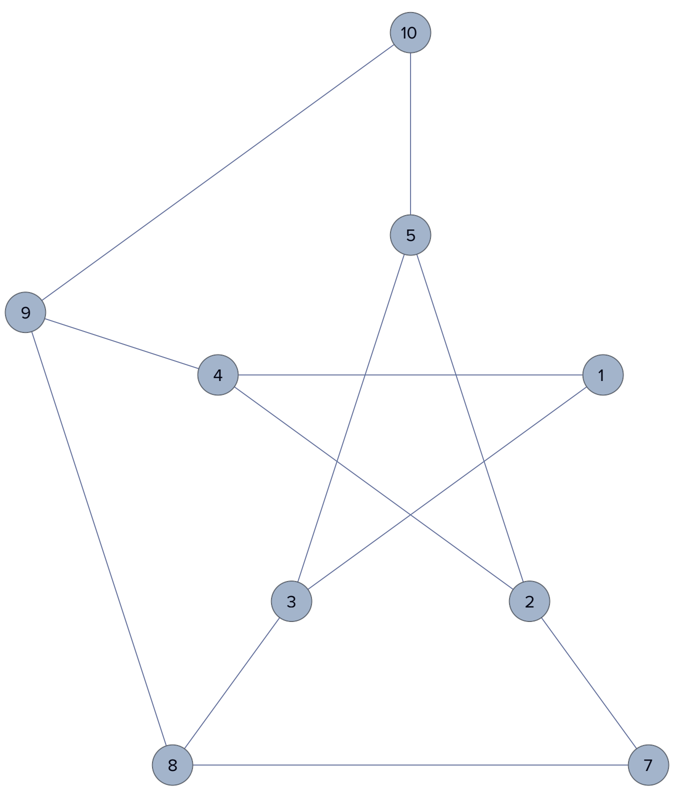
Out[232]=



First, we form the subgraph of the Petersen graph obtained by removing vertex 2 and the three edges incident to it.

In[233]:=
```
petersen1=VertexDelete[petersen,6]
```

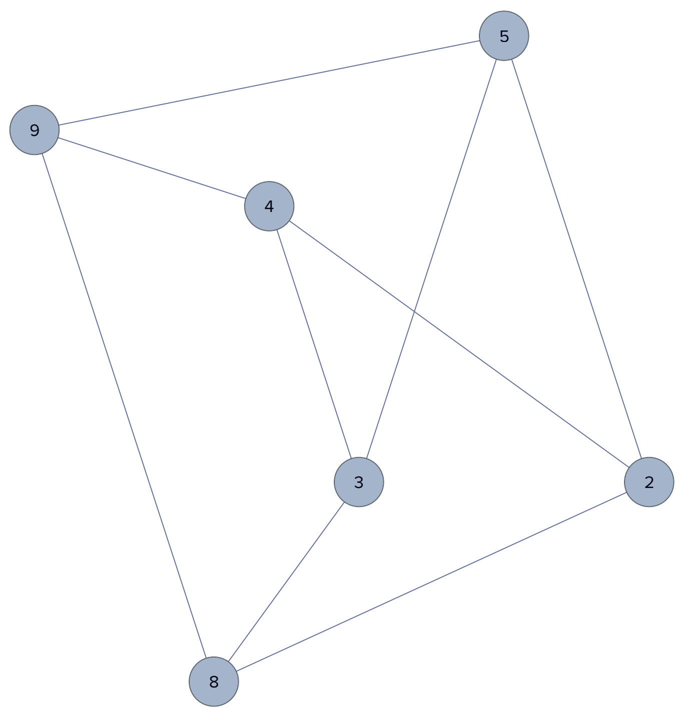
Out[233]=



Now, we notice that there are three vertices that are smoothable: 1, 7, and 10. That is to say, those three vertices have degree 2 and their neighbors are not adjacent.

In[234]:=
```
petersen2=smoothGraph[petersen1,1];
petersen3=smoothGraph[petersen2,7];
petersen4=smoothGraph[petersen3,10]
```

Out[234]=



We now observe that this graph has 6 vertices, each of which has degree 3, just like $K_{3,3}$. Therefore, there is a definite possibility that this graph is $K_{3,3}$. We confirm that is $K_{3,3}$ with `IsomorphicGraphQ`.

In[236]:=
```
IsomorphicGraphQ[petersen4,CompleteGraph[{3,3}]]
```

Out[236]=
```
True
```

This demonstrates that the Petersen graph has a subgraph that is homeomorphic to $K_{3,3}$ and hence is nonplanar.

10.8 Graph Coloring

In this section, we consider the problem of how to properly color a graph; that is, how to assign to each vertex of a graph a color such that no vertex has the same color as any of its neighbors.

It is worth noting that, in terms of computational complexity, Hamilton circuits and graph coloring are equivalently difficult problems.

We will create a function based on the algorithm described in the preface to Exercise 29 in Section 10.8 of the text. It can be shown that this algorithm will color a graph using at most one more color than the maximal degree of the graph. It is considered a greedy algorithm because it makes optimal choices at each step but never reconsiders its choices. That is to say, it does the best it can at every step but never backtracks to make improvements. Greedy algorithms often lead to good, but non-optimal, solutions.

The algorithm proceeds as follows. First, the vertices are sorted in order of descending degree. The first color is assigned to the first vertex in the list. Also assign color 1 to the first vertex in the list not adjacent to vertex 1, to the next vertex not adjacent to those already colored, etc. Then, move on to the second color. The first uncolored vertex in the list is assigned color 2, as are vertices further down the list not adjacent to ones previously assigned the second color. This continues until all of the vertices have been given a color.

Our first step in implementing this function will be to sort the list of vertices in decreasing order of degree. For this, we will make use of the Wolfram Language's very flexible `Sort` function. With no additional instructions, this function will sort a list of numbers in increasing numerical order and a list of strings in lexicographical order. The `Sort` function can accept an optional argument that allows us to specify the way in which the list is sorted. Specifically, `Sort` takes as an argument a Boolean-valued function on two arguments and returns true if the first argument precedes the second.

For our graph coloring procedure, we will create a helper function that takes a `Graph` object and returns the sorted list of vertices.

```
In[237]:= sortVertices[G_Graph]:=
            Sort[VertexList[G],
              VertexDegree[G,#1]≥VertexDegree[G,#2]&]
```

In order for our algorithm to color the vertices of a graph, we need to decide on what colors to use. Here, we will use the first of the indexed color schemes.

```
In[238]:= ColorData[1]

          ColorDataFunction[ Index: 1   Colors: ∞
                             Palette:                ]
```

The assertion that this `ColorDataFunction` has infinitely many colors is true in the sense that it will produce a color for any positive integer, although there is, of course, a limit on how many of them a person will be able to distinguish.

We now implement the greedy coloring algorithm.

```
In[239]:= greedyColorer[G_Graph]:=
            Module[{H=G,V,currentColor=0,
                colorFunction=ColorData[1],excludeSet,i},
              V=sortVertices[H];
              While[V≠{},currentColor++;
                PropertyValue[{H,V[[1]]},VertexStyle]=
                  colorFunction[currentColor];
                excludeSet=VertexList[NeighborhoodGraph[H,V[[1]]]];
```

```
        V=Delete[V,1];
        i=1;
        While[i≤Length[V],
          If[!MemberQ[excludeSet,V[[i]]],
            PropertyValue[{H,V[[i]]},VertexStyle]=
              colorFunction[currentColor];
            excludeSet=Union[excludeSet,
              VertexList[NeighborhoodGraph[H,V[[i]]]]];
            V=Delete[V,i],
            i++
          ]
        ];
      ];
      H
    ]
```

Note that the list V, which is initialized to the list of vertices, sorted in decreasing order of degree, is used to track which vertices still need to be assigned a color. When a vertex has been assigned a color, it is deleted from the list V using Delete. The Delete function removes from the list in the first argument the element at the position specified by the second argument.

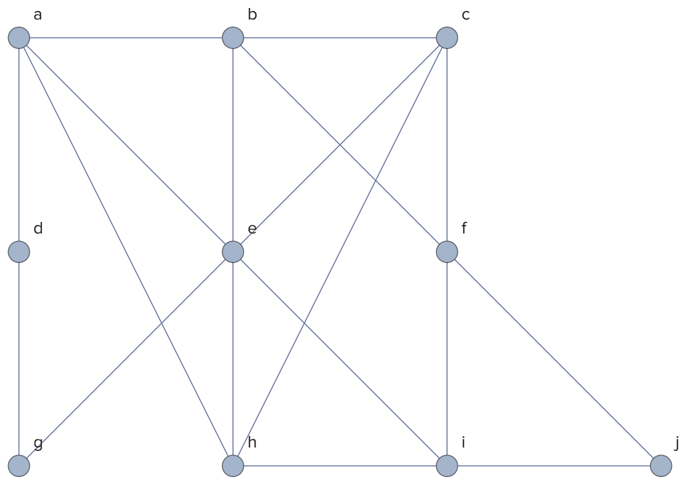
The excludeSet variable is used to store all vertices which cannot be assigned the current color. Each time a vertex is assigned a color, it and all of its neighbors are added to the excludeSet. As the function looks down the list of vertices that still need to have a color assigned, it checks to see if they are in this set.

The index i, which controls the inner While loop, is incremented in the else clause of the If statement that tests to see if a vertex can be assigned the color. If the vertex at index i is assigned the color, then it is removed from the list V, and thus the index i refers to a different vertex (the vertex previously in position $i + 1$).

As an example, we solve Exercise 29 of Section 10.8.

```
In[240]:= exercise29=
            Graph[{"a","b","c","d","e","f","g","h","i","j"},
              {"a"→"b","a"→"d","a"→"e","a"→"h","b"→"c",
               "b"→"e","b"→"f","c"→"e","c"→"f","c"→"h",
               "d"→"g","e"→"g","e"→"h","e"→"i","f"→"i",
               "f"→"j","h"→"i","i"→"j"},DirectedEdges→False,
              VertexCoordinates→
                {{0,2},{1,2},{2,2},{0,1},{1,1},{2,1},{0,0},
                {1,0},{2,0},{3,0}},VertexLabels→"Name",
              VertexSize→Small]
```

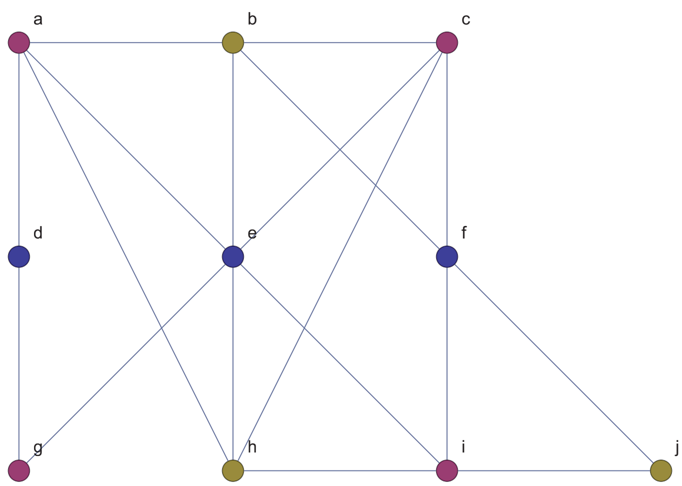
Out[240]=



In[241]:=

```
greedyColorer[exercise29]
```

Out[241]=



Solutions to Computer Projects and Computations and Explorations

Computations and Explorations 1

Display all simple graphs with four vertices.

Solution: To solve this problem, we will generate all possible edge sets and then construct the graphs based on these edge sets. The possible edge sets are all of the subsets of the set of all possible edges, which we obtain from the complete graph on the vertices. We generalize the question and have our function create all the simple graphs on *n* vertices.

In[242]:=

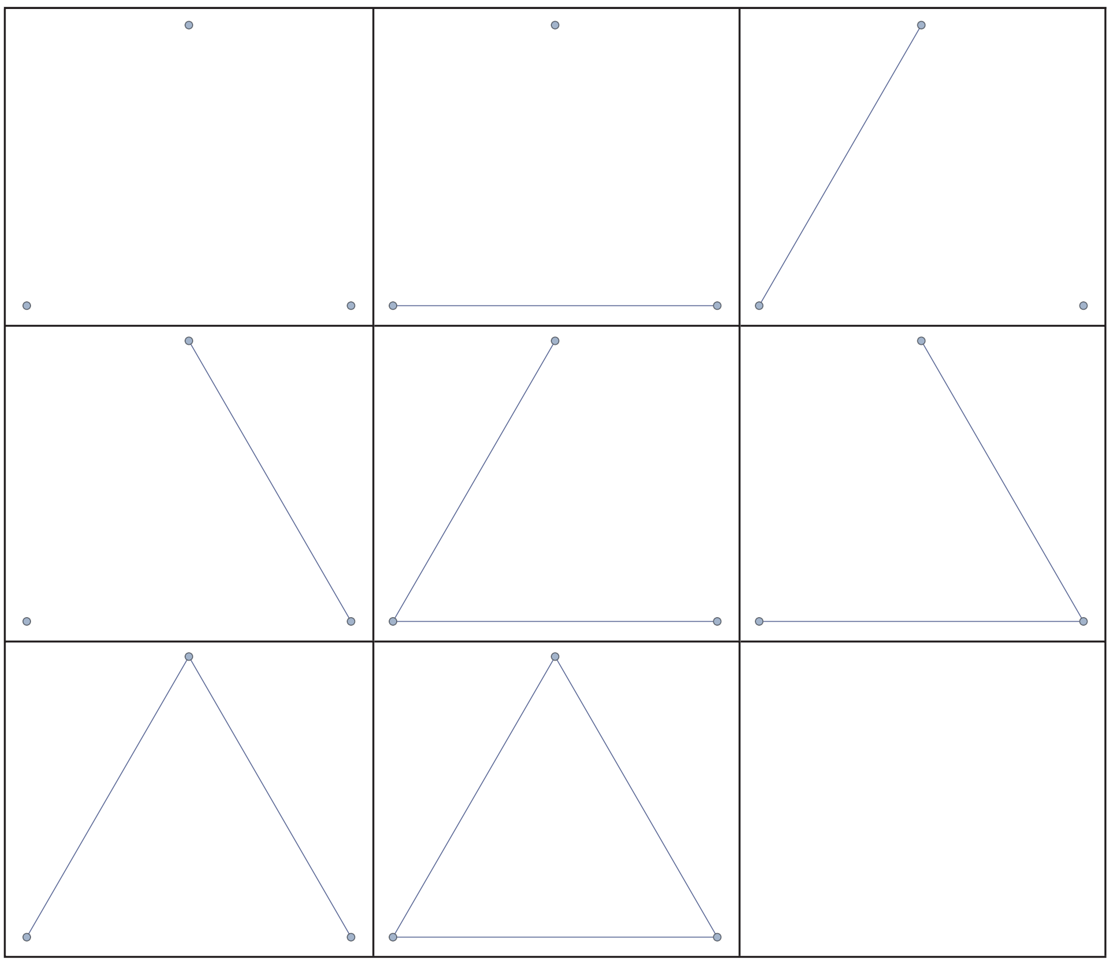
```
allGraphs[n_Integer]/;n>0:=
  Module[{cg,v,A={},V=Range[n],powerE,vCoords,E},
    cg=CompleteGraph[n];
    powerE=Subsets[EdgeList[cg]];
    vCoords=GraphEmbedding[cg];
    Do[AppendTo[A,Graph[V,E,VertexCoordinates→vCoords]],
```

```
            {E,powerE}];
          A
        ]
```

Recall that the complete graph on n vertices has $C(n,2)$ edges, so there are $2^{C(n,2)}$ graphs on n vertices. Thus, on 4 vertices, there are 64 graphs. For $n=3$, there are only 8 graphs, which is more manageable.

We use the `Partition` function to break the list of all graphs into triples and then apply `Grid` with the `Frame` option in order to display the results in a useful way. The second argument to `Partition` is the size of the sublists (the rows) and the last specifies padding, which is needed to create sublists of equal size. The third argument is the offset, which equals the second in order to not repeat elements. The fourth argument allows for the last sublist to be padded.

```
In[243]:=  Grid[Partition[allGraphs[3],3,3,{1,1},Null],Frame→All]
```



Computations and Explorations 2

Display a full set of nonisomorphic simple graphs with six vertices.

Solution: The solution to this exercise is very similar to the previous question. The only difference is that, once the list of graphs is generated, we remove those that are isomorphic to others by applying `DeleteDuplicates` with second argument `IsomorphicGraphQ`. This uses `IsomorphicGraphQ` to determine whether two elements are to be considered duplicates or not.

In[244]:=
```
nonIsoGraphs[n_Integer]/;n>0:=
  Module[{cg,v,A={},V=Range[n],powerE,vCoords,
      E,i,G,j},
    cg=CompleteGraph[n];
    powerE=Subsets[EdgeList[cg]];
    vCoords=GraphEmbedding[cg];
    Do[AppendTo[A,Graph[V,E,VertexCoordinates→vCoords]]
      ,{E,powerE}];
    DeleteDuplicates[A,IsomorphicGraphQ]
  ]
```

We apply this to five vertices, since six takes a bit more time to compute.

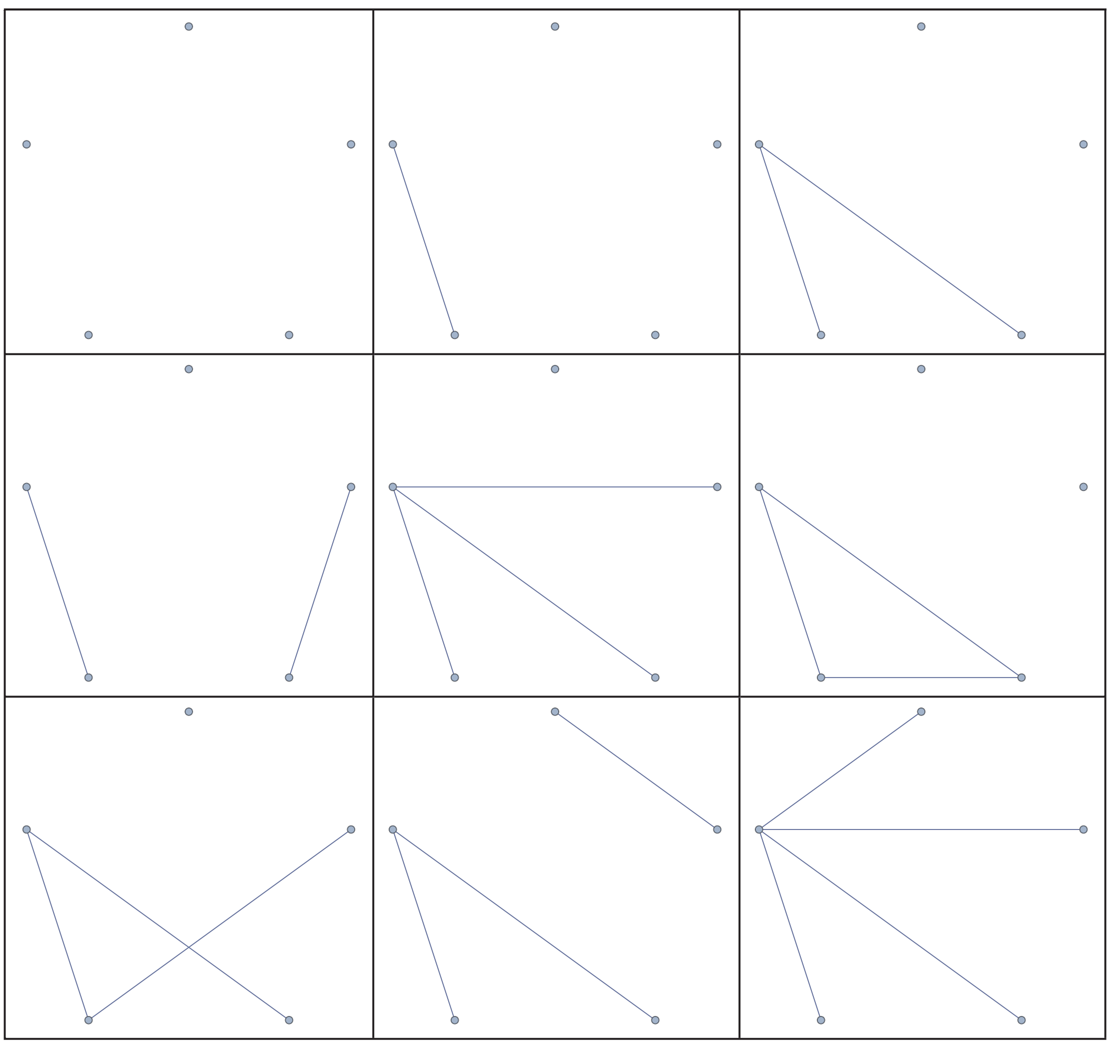
In[245]:=
```
nonIso5=nonIsoGraphs[5];
```

In[246]:=
```
Length[nonIso5]
```

Out[246]=
```
34
```

We see that there are 34 nonisomorphic graphs on 5 vertices. Here are the first nine.

In[247]:=
```
Grid[Partition[nonIso5[[1;;9]],3],Frame→All]
```



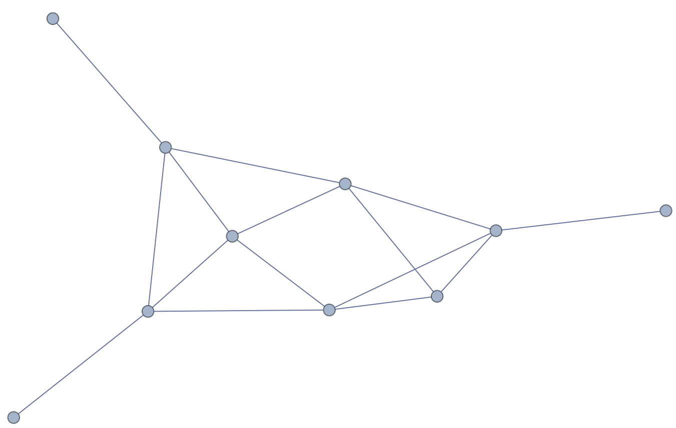
Computations and Explorations 9

Generate at random simple graphs with 10 vertices. Stop when you have constructed one with an Euler circuit. Display an Euler circuit in this graph.

Solution: To generate the random graphs, we will use the `RandomGraph` function. By passing this function a list containing a number of vertices and a number of edges, it produces a graph with that number of vertices and edges. To display a random graph on 10 vertices and 15 edges, we enter the following:

In[248]:=
```
RandomGraph[{10,15}]
```

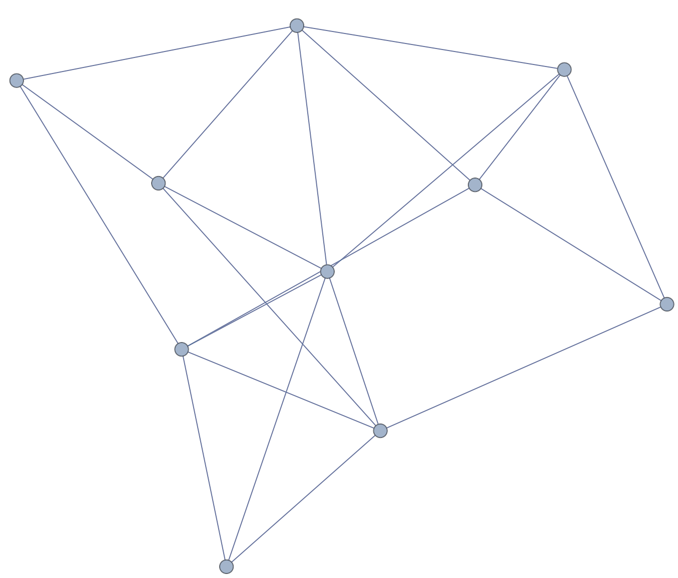
Out[248]=



Alternatively, the argument to `RandomGraph` can be a graph distribution, which is a probability distribution on the sample space consisting of graphs. There are six built-in graph distributions. We will use the `BernoulliGraphDistribution`, which allows you to specify a number of vertices and a probability. The probability indicates the independent probability that an edge will appear between any two vertices. It requires two parameters, the number of vertices and the probability. The following constructs a random graph with 10 vertices and with each edge as likely to appear as not.

In[249]:=
```
RandomGraph[BernoulliGraphDistribution[10,.5]]
```

Out[249]=



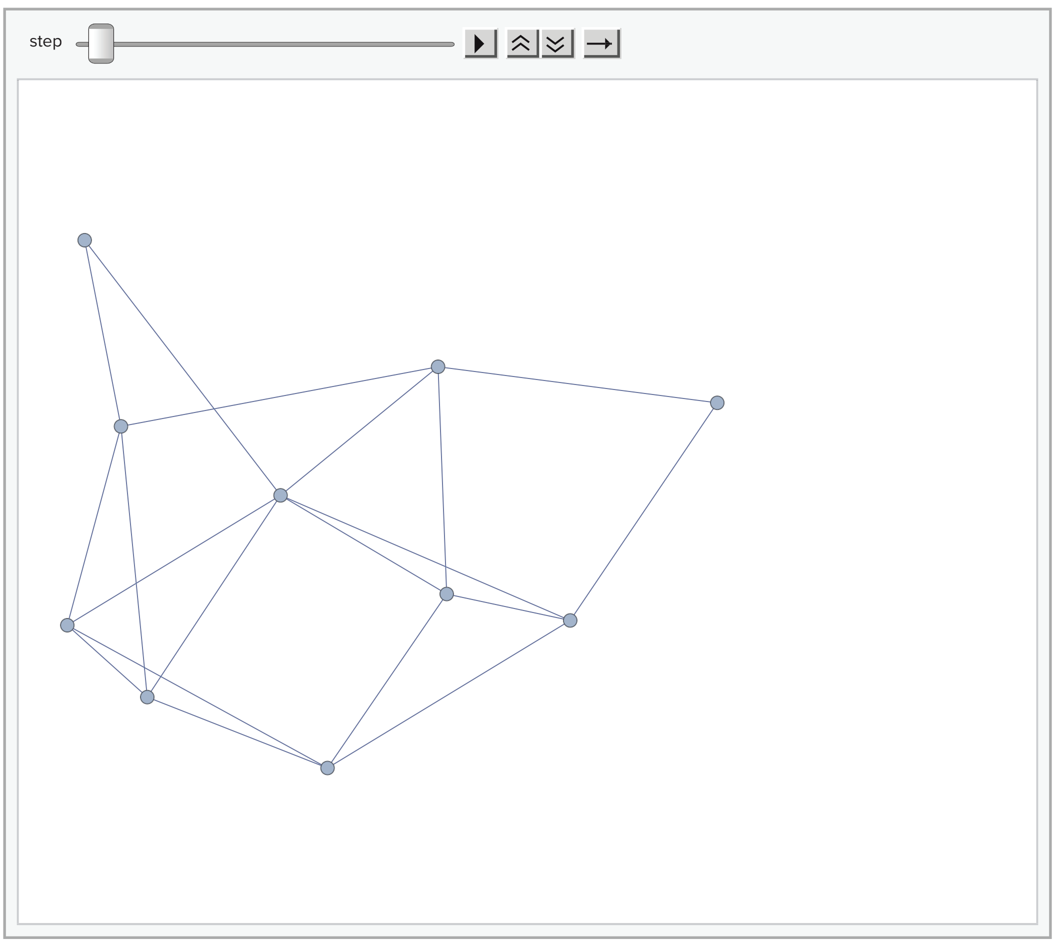
RandomGraph also accepts an optional second argument, a positive integer, causing it to produce a list of that many random graphs.

Recall the description of the EulerianGraphQ function. When this function is given a graph, it returns true or false depending on the existence of an Euler circuit.

To satisfy the requirements of this problem, we use RandomGraph to generate a random graph *G*. Then, we test it for an Euler circuit using EulerianGraphQ. As long as the randomly generated graph does not have an Euler circuit, we continue generating new random graphs. We display the path using FindEulerianCycle and the animatePath function we created in Section 10.5.

In[250]:=
```
generateEulerian[n_Integer]/;n>0:=Module[{G,path},
  While[!EulerianGraphQ[G],
    G=RandomGraph[BernoulliGraphDistribution[n,.5]]];
  animatePath[G,First[FindEulerianCycle[G]]]
]
```

In[251]:=
```
generateEulerian[10]
```



Computations and Explorations 13

Estimate the probability that a randomly generated simple graph with n vertices is connected for each possible integer n not exceeding ten by generating a set of random simple graphs and determining whether each is connected.

Solution: To solve this problem we will create a function that generates a number of random graphs of the specified size and counts the number that are connected. We use the `RandomGraph` function to create the random graphs and the `ConnectedGraphQ` function to test them for connectivity.

```
In[252]:= connectedProbability[n_Integer,max_Integer]/;
            n>0&&max>0:=Module[{G,i,count=0},
            For[i=1,i≤max,i++,
              G=RandomGraph[BernoulliGraphDistribution[n,.5]];
              If[ConnectedGraphQ[G],count++]
            ];
            count/max
          ]
```

```
In[253]:= Table[connectedProbability[i,100],{i,10}]
```

Out[253]= $\left\{1,\frac{9}{20},\frac{53}{100},\frac{13}{25},\frac{18}{25},\frac{81}{100},\frac{87}{100},\frac{24}{25},\frac{97}{100},\frac{97}{100}\right\}$

Exercises

1. Write a function in the Wolfram Language to find *all* maximal matchings for a bipartite graph.
2. Write functions in the Wolfram Language for calculating the adjacency and incidence matrices for a pseudograph.
3. Write a function in the Wolfram Language for creating a pseudograph from an incidence matrix.
4. Write a function in the Wolfram Language to automate the creation of vertex-colored graphs to illustrate graph isomorphisms, as was done with the graphs from Example 11 at the end of Section 10.3 of this manual.
5. Write a function in the Wolfram Language to find all of the minimal edge cuts of a given graph.
6. Use *Mathematica* to construct all regular graphs of degree n, given a positive integer n. (Regular graph is defined in the Exercises for Section 10.2.)
7. For vertices u and v in a simple, undirected and connected graph G, the local vertex connectivity $\kappa(u, v)$ is defined to be the minimum number of vertices that must be removed so that there is no path between vertex u and vertex v. Write a function in the Wolfram Language that calculates the local vertex connectivity of a graph and a pair of its vertices.
8. For vertices u and v in a simple, undirected and connected graph G, the local edge connectivity $\lambda(u, v)$ is defined to be the minimum number of edges that must be removed so that there is no path between vertex u and vertex v. Write a function in the Wolfram Language that calculates the local edge connectivity of a graph and a pair of its vertices.
9. Write a function in the Wolfram Language that computes the thickness of a nonplanar simple graph (see the Exercises in Section 10.7 for a definition of thickness).

10. Write a function in the Wolfram Language for finding an orientation of a simple graph. (An orientation of a graph is defined in the Supplementary Exercises of Chapter 10.)

11. Write a function in the Wolfram Language for finding the bandwidth of a simple graph. (The bandwidth of a graph is defined in the Supplementary Exercises of Chapter 10.)

12. Write a function in the Wolfram Language for finding the radius and diameter of a simple graph. (The radius and diameter of a graph are defined in the Supplementary Exercises of Chapter 10.)

13. Use *Mathematica* to find the minimum number of queens controlling an $n \times n$ chessboard for as many values of n as you can. Make use of the concept of a dominating set, described in the Supplementary Exercises of Chapter 10.

14. Write a function in the Wolfram Language for finding all self-complementary graphs on n vertices. (A self-complementary graph is a graph which is isomorphic to its own complement.) Use your function to display the self-complementary graphs for as large a n as possible.

15. Write a function in the Wolfram Language that finds a total coloring for a graph. A total coloring of a graph is an assignment of a color to each vertex and each edge such that: (a) no pair of adjacent vertices have the same color; (b) no two edges with a common endpoint have the same color; and (c) no edge has the same color as either of its endpoints.

16. A sequence of positive integers is called *graphic* if there is a simple graph that has this sequence as its degree sequence. In this context, the degree sequence of a graph is the nondecreasing sequence made up of the degrees of the vertices of the graph. Develop a function in the Wolfram Language for determining whether a sequence of positive integers is graphic and, if it is, to construct a graph with this degree sequence.