CHAPTER 26

The Application of Queueing Theory

As described in Chap. 17, queueing theory has enjoyed a prominent place among the modern analytical techniques of OR. However, the emphasis has been on developing a descriptive mathematical theory. Thus, queueing theory is not directly concerned with achieving the goal of OR: optimal decision making. Rather, it develops information on the behavior of queueing systems. This theory provides part of the information needed to conduct an OR study attempting to find the best design for a queueing system.

Section 17.10 discusses the *application* of queueing theory in the broader context of an overall OR study. This chapter expands considerably further on this same topic. It begins by introducing three examples that will be used for illustration throughout the chapter. Section 26.2 discusses the basic considerations for decision making in this context. The following two sections then develop decision models for the *optimal* design of queueing systems. The last model requires the incorporation of *travel-time models,* which are presented in Sec. 26.5.

26.1 EXAMPLES

Example 1—How Many Repairers?

SIMULATION, INC., a small company that makes gidgets for analog computers, has 10 gidget-making machines. However, because these machines break down and require repair frequently, the company has only enough operators to operate eight machines at a time, so two machines are available on a standby basis for use while other machines are down. Thus, eight machines are always operating whenever no more than two machines are waiting to be repaired, but the number of operating machines is reduced by 1 for each additional machine waiting to be repaired.

The time until any given operating machine breaks down has an exponential distribution, with a mean of 20 days. (A machine that is idle on a standby basis cannot break down.) The time required to repair a machine also has an exponential distribution, with a mean of 2 days. Until now the company has had just one repairer to repair these machines, which has frequently resulted in reduced productivity because fewer than eight machines are operating. Therefore, the company is considering hiring a second repairer, so that two machines can be repaired simultaneously.

Thus, the queueing system to be studied has the repairers as its servers and the machines requiring repair as its customers, where the problem is to choose between having

one or two servers. (Notice the analogy between this problem and the County Hospital emergency room problem described in Sec. 17.1.) With one slight exception, this system fits the *finite calling population variation* of the *M*/*M*/*s* model presented in Sec. 17.6, where $N = 10$ machines, $\lambda = \frac{1}{20}$ customer per day (for each operating machine), and $\mu = \frac{1}{2}$ customer per day. The exception is that the λ_0 and λ_1 parameters of the birth-and-death process are changed from $\lambda_0 = 10\lambda$ and $\lambda_1 = 9\lambda$ to $\lambda_0 = 8\lambda$ and $\lambda_1 = 8\lambda$. (All the other parameters are the same as those given in Sec. 17.6.) Therefore, the C_n factors for calculating the P_n probabilities change accordingly (see Sec. 17.5).

Each repairer costs the company approximately $280 per day. However, the estimated *lost profit* from having fewer than eight machines operating to produce gidgets is $400 per day for each machine down. (The company can sell the full output from eight operating machines, but not much more.)

The analysis of this problem will be pursued in Secs. 26.3 and 26.4.

Example 2—Which Computer?

EMERALD UNIVERSITY is making plans to lease a supercomputer to be used for scientific research by the faculty and students. Two models are being considered: one from the MBI Corporation and the other from the CRAB Company. The MBI computer costs more but is somewhat faster than the CRAB computer. In particular, if a sequence of typical jobs were run continuously for one 24-hour day, the number completed would have a Poisson distribution with a mean of 30 and 25 for the MBI and the CRAB computers, respectively. It is estimated that an average of 20 jobs will be submitted per day and that the time from one submission to the next will have an exponential distribution with a mean of 0.05 day. The leasing cost per day would be $5,000 for the MBI computer and $3,750 for the CRAB computer.

Thus, the queueing system of concern has the computer as its (single) server and the jobs to be run as its customers. Furthermore, this system fits the *M*/*M*/1 model presented at the beginning of Sec. 17.6. With 1 day as the unit of time, $\lambda = 20$ customers per day, and $\mu = 30$ and 25 customers per day with the MBI and the CRAB computers, respectively. You will see in Secs. 26.3 and 26.4 how the decision was made between the two computers.

Example 3—How Many Tool Cribs?

The MECHANICAL COMPANY is designing a new plant. This plant will need to include one or more tool cribs in the factory area to store tools required by the shop mechanics. The tools will be handed out by clerks as the mechanics arrive and request them and will be returned to the clerks when they are no longer needed. In existing plants, there have been frequent complaints from supervisors that their mechanics have had to waste too much time traveling to tool cribs and waiting to be served, so it appears that there should be *more* tool cribs and *more* clerks in the new plant. On the other hand, management is exerting pressure to reduce overhead in the new plant, and this reduction would lead to *fewer* tool cribs and *fewer* clerks. To resolve these conflicting pressures, an OR study is to be conducted to determine just how many tool cribs and clerks the new plant should have.

Each tool crib constitutes a queueing system, with the clerks as its servers and the mechanics as its customers. Based on previous experience, it is estimated that the time required by a tool crib clerk to service a mechanic has an exponential distribution, with a mean of $\frac{1}{2}$ minute. Judging from the anticipated number of mechanics in the entire factory area for the new plant, it is also predicted that they would require this service

randomly but at a mean rate of 2 mechanics per minute. Therefore, it was decided to use the *M*/*M*/*s* model of Sec. 17.6 to represent each queueing system. With 1 hour as the unit of time, $\mu = 120$. If only one tool crib were to be provided, λ also would be 120. With more than one tool crib, this mean arrival rate would be divided among the different queueing systems.

The total cost to the company of each tool crib clerk is about $20 per hour. The capital recovery costs, upkeep costs, and so forth associated with each tool crib provided are estimated to be $16 per working hour. While a mechanic is busy, the value to the company of his or her output averages about $48 per hour.

Sections 26.3 and 26.4 include discussions of how this (and additional) information was used to make the required decisions.

26.2 DECISION MAKING

Queueing-type situations that require decision making arise in a wide variety of contexts. For this reason, it is not possible to present a meaningful decision-making procedure that is applicable to all these situations. Instead, this section attempts to give a broad conceptual picture of a typical approach.

Designing a queueing system often involves making one or a combination of the following decisions:

1. Number of servers at a service facility.
2. Efficiency of the servers.
3. Number of service facilities.

When such problems are formulated in terms of a queueing model, the corresponding decision variables usually are s (number of servers at each facility), μ (mean service rate per busy server), and λ (mean arrival rate at each facility). The *number of service facilities* is directly related to λ because, assuming a uniform workload among the facilities, λ equals the total mean arrival rate to all facilities divided by the number of facilities. (Section 17.10 also mentions two other possible decisions when designing a queueing system, namely, the amount of waiting space in the queue and any priorities for different categories of customers, but we will focus in this chapter on the three types of decisions listed above.)

Refer to Sec. 26.1 and note how the three examples there respectively illustrate situations involving these three decisions. In particular, the decision facing Simulation, Inc., in Example 1 is *how many repairers* (servers) to provide. The problem for Emerald University in Example 2 is *how fast a computer* (server) is needed. The problem facing Mechanical Company in Example 3 is *how many tool cribs* (service facilities) to install as well as *how many clerks* (servers) to provide at each facility.

The first kind of decision is particularly common in practice. However, the other two also arise frequently, particularly for the internal service systems described in Sec. 17.3. One example illustrating a decision on the efficiency of the servers is the selection of the type of materials-handling equipment (the servers) to purchase to transport certain kinds of loads (the customers). Another such example is the determination of the size of a maintenance crew (where the entire crew is one server). Other decisions concern the number of service facilities, such as copy centers, computer facilities, tool cribs, storage areas, and so on, to distribute throughout an area.

All the specific decisions discussed here involve the general question of the *appropriate level of service* to provide in a queueing system. As mentioned at the beginning of Chap. 17 and in Sec. 17.10, decisions regarding the amount of service capacity to provide usually are based primarily on two considerations: (1) the cost incurred by providing the service, as

shown in Fig. 26.1, and (2) the amount of waiting time for that service, as suggested in Fig. 26.2. Figure 26.2 can be obtained by using the appropriate waiting-time equation from queueing theory. (For better conceptualization, we have drawn these figures and the subsequent two figures as smooth curves even though the level of service may be a discrete variable.)

These two considerations create conflicting pressures on the decision maker. The objective of reducing service costs recommends a minimal level of service. On the other hand, long waiting times are undesirable, which recommends a high level of service. Therefore, it is necessary to strive for some type of compromise. To assist in finding this compromise, Figs. 26.1 and 26.2 may be combined, as shown in Fig. 26.3. The problem is thereby reduced to selecting the point on the curve of Fig. 26.3 that gives the best balance between the average delay in being serviced and the cost of providing that service. Reference to Figs. 26.1 and 26.2 indicates the corresponding level of service.

FIGURE 26.1
Service cost as a function of service level.

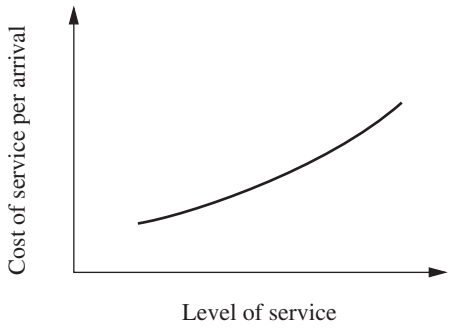


FIGURE 26.2
Expected waiting time as a function of service level.

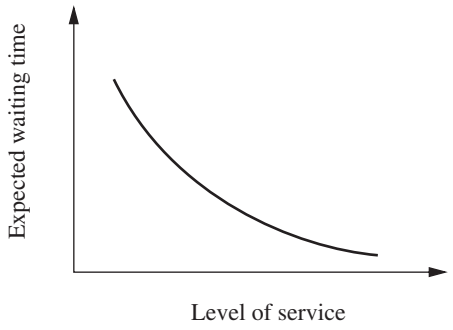
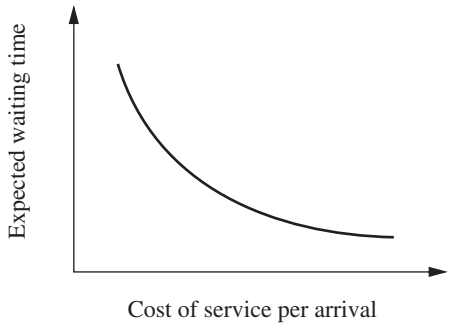


FIGURE 26.3
Relationship between average delay and service cost.



Obtaining the proper balance between delays and service costs requires answers to such questions as, How much expenditure on service is equivalent (in its detrimental impact) to a customer's being delayed 1 unit of time? Thus, to compare service costs and waiting times, it is necessary to adopt (explicitly or implicitly) a common measure of their impact. The natural choice for this common measure is cost, which then requires estimation of the cost of waiting.

Because of the diversity of waiting-line situations, no single process for estimating the cost of waiting is generally applicable. However, we shall discuss the basic considerations involved for several types of situations.

One broad category is where the customers are *external* to the organization providing the service; i.e., they are *outsiders* bringing their business to the organization. Consider first the case of *profit-making* organizations (typified by the commercial service systems described in Sec. 17.3). From the viewpoint of the decision maker, the cost of waiting probably consists primarily of the *lost profit* from *lost business.* This loss of business may occur immediately (because the customer grows impatient and leaves) or in the future (because the customer is sufficiently irritated that he or she does not come again). This kind of cost is quite difficult to estimate, and it may be necessary to revert to other criteria, such as a tolerable probability distribution of waiting times. When the customer is not a human being, but a job being performed on order, there may be more readily identifiable costs incurred, such as those caused by idle in-process inventories or increased expediting and administrative effort.

Now consider the type of situation where service is provided on a *nonprofit* basis to customers *external* to the organization (typical of social service systems and some transportation service systems described in Sec. 17.3). In this case, the cost of waiting usually is a *social cost* of some kind. Thus, it is necessary to evaluate the consequences of the waiting for the individuals involved and/or for society as a whole and to try to impute a monetary value to avoiding these consequences. Once again, this kind of cost is quite difficult to estimate, and it may be necessary to revert to other criteria.

A situation may be more amenable to estimating waiting costs if the customers are *internal* to the organization providing the service (as for the internal service systems discussed in Sec. 17.3). For example, the customers may be machines (as in Example 1 in Sec. 26.1) or employees (as in Example 3) of a firm. Therefore, it may be possible to identify directly some of or all the costs associated with the idleness of these customers. Typically, what is being wasted by this idleness is *productive output,* in which case the waiting cost becomes the *lost profit* from *all lost productivity.*

Given that the *cost of waiting* has been evaluated explicitly, the remainder of the analysis is conceptually straightforward. The objective is to determine the level of service that minimizes the total of the expected cost of service and the expected cost of waiting for that service. This concept is depicted in Fig. 26.4, where WC denotes *waiting cost,* SC denotes *service cost,* and TC denotes *total cost.* Thus, the mathematical statement of the objective is to

$$\text{Minimize} \qquad E(\text{TC}) = E(\text{SC}) + E(\text{WC}).$$

The next three sections are concerned with the application of this concept to various types of problems. Thus, Sec. 26.3 describes how $E(\text{WC})$ can be expressed mathematically. Section 26.4 then focuses on $E(\text{SC})$ to formulate the overall objective function $E(\text{TC})$ for several basic design problems (including some with multiple decision variables, so that the level-of-service axis in Fig. 26.4 then requires more than one dimension). Section 26.4 also introduces the fact that when a decision on the number of service facilities is required, time spent in traveling to and from a facility should be included in the analysis (as part of the total time waiting for service). Section 26.5 discusses how to determine the expected value of this travel time.

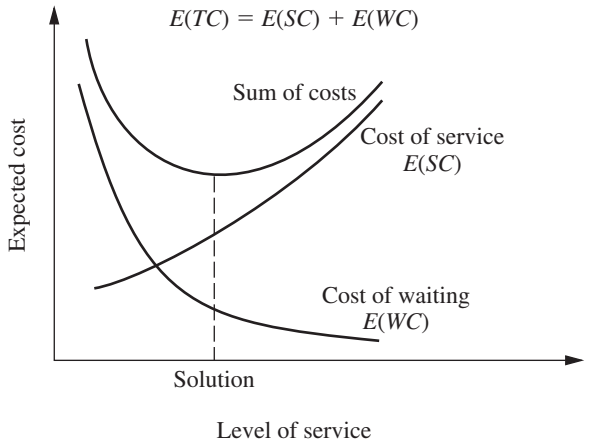


FIGURE 26.4
Conceptual solution procedure for many waiting-line problems.

26.3 FORMULATION OF WAITING-COST FUNCTIONS

To express $E(\text{WC})$ mathematically, we must first formulate a *waiting-cost function* that describes how the actual waiting cost being incurred varies with the current behavior of the queueing system. The form of this function depends on the context of the individual problem. However, most situations can be represented by one of the two basic forms described next.

The *g(N)* Form

Consider first the situation discussed in the preceding section where the queueing system *customers* are *internal* to the organization providing the service, and so the primary cost of waiting may be the *lost profit from lost productivity.* The *rate* at which productive output is lost sometimes is essentially *proportional* to the number of customers in the queueing system. However, in many cases there is not enough productive work available to keep all the members of the calling population continuously busy. Therefore, little productive output may be lost by having just a few members idle, waiting for service in the queueing system, whereas the loss may increase greatly if a few more members are made idle because they require service. Consequently, the primary property of the queueing system that determines the *current rate* at which waiting costs are being incurred is N, the number of customers in the system. Thus, the form of the waiting-cost function for this kind of situation is that illustrated in Fig. 26.5, namely, a function of N. We shall denote this form by $g(N)$.

The $g(N)$ function is constructed for a particular situation by estimating $g(n)$, the waiting-cost rate incurred when $N = n$, for $n = 1, 2, \ldots$, where $g(0) = 0$. After computing the P_n probabilities for a given design of the queueing system, we can calculate

$$E(\text{WC}) = E(g(N)).$$

Because N is a random variable, this calculation is made by using the expression for the expected value of a *function* of a *discrete* random variable

$$E(\text{WC}) = \sum_{n=0}^{\infty} g(n)P_n.$$

The Linear Case. For the special case where $g(N)$ is a *linear function* (i.e., when the waiting cost is proportional to N), then

$$g(N) = C_w N,$$

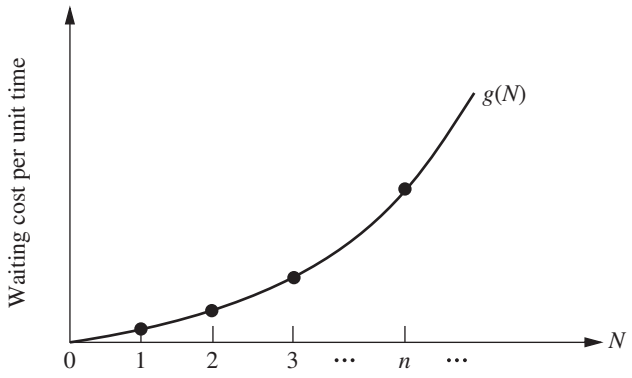


FIGURE 26.5
The waiting-cost function as a function of *N*.

TABLE 26.1 Calculation of *E*(WC) for Example 1

		s = 1		*s* = 2	
N* = *n	***g*(*n*)**	P_n	$g(n)P_n$	P_n	$g(n)P_n$
0	0	0.271	0	0.433	0
1	0	0.217	0	0.346	0
2	0	0.173	0	0.139	0
3	400	0.139	56	0.055	24
4	800	0.097	78	0.019	16
5	1,200	0.058	70	0.006	8
6	1,600	0.029	46	0.001	0
7	2,000	0.012	24	3×10^{-4}	0
8	2,400	0.003	7	4×10^{-5}	0
9	2,800	7×10^{-4}	0	4×10^{-6}	0
10	3,200	7×10^{-5}	0	2×10^{-7}	0
E(WC)			\$281 per day		\$48 per day

where C_w is the cost of waiting per unit time for each customer. In this case, *E*(WC) reduces to

$$W(\text{WC}) = C_w \sum_{n=0}^{\infty} nP_n = C_w L.$$

Example 1—How Many Repairers? For Example 1 of Sec. 26.1, Simulation, Inc., has two standby widget-making machines, so there is no lost productivity as long as the number of customers (machines requiring repair) in the system does not exceed 2. However, for each *additional* customer (up to the maximum of 10 total), the estimated lost profit is \$400 per day. Therefore,

$$g(n) = \begin{cases} 0 & \text{for } n = 0, 1, 2 \\ 400(n-2) & \text{for } n = 3, 4, \ldots, 10, \end{cases}$$

as shown in Table 26.1. Consequently, after calculating the P_n probabilities as described in Sec. 26.1, *E*(WC) is calculated by summing the rightmost column of Table 26.1 for each of the two cases of interest, namely, having one repairer ($s = 1$) or two repairers ($s = 2$).

The $h(\mathscr{W})$ Form

Now consider the cases discussed in Sec. 26.2 where the queueing system *customers* are *external* to the organization providing the service. Three major types of queueing systems described in Sec. 17.3—commercial service systems, transportation service systems, and social service systems—typically fall into this category. In the case of commercial service systems, the primary cost of waiting may be the lost profit from lost future business. For transportation service systems and social systems, the primary cost of waiting may be in the form of a social cost. However, for either type of cost, its magnitude tends to be affected greatly by the size of the waiting times experienced by the customers. Thus, the primary property of the queueing system that determines the waiting cost currently being incurred is $\mathscr{W}$, the waiting time in the system for the *individual* customers. Consequently, the form of the waiting-cost function for this kind of situation is that illustrated in Fig. 26.6, namely, a function of $\mathscr{W}$. We shall denote this form by $h(\mathscr{W})$.

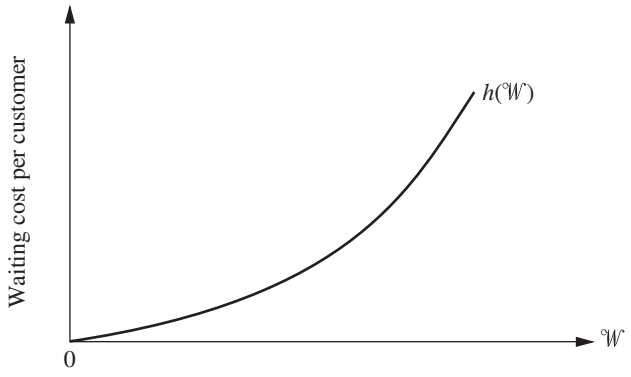
Note that the example of a $h(\mathscr{W})$ function shown in Fig. 26.6 is a nonlinear function where the slope keeps increasing as $\mathscr{W}$ increases. Although $h(\mathscr{W})$ sometimes is a simple linear function instead, it is fairly common to have this kind of nonlinear function. An increasing slope reflects a situation where the *marginal cost* of extending the waiting time keeps increasing. A customer may not mind a "normal" wait of reasonable length, in which case there may be virtually no negative consequences for the organization providing the service in terms of lost profit from lost future business, a social cost, etc. However, if the wait extends even further, the customer may become increasingly exasperated, perhaps even missing deadlines. In such a situation, the negative consequences to the organization may rapidly become relatively severe.

One way of constructing the $h(\mathscr{W})$ function is to estimate $h(w)$ (the waiting cost incurred when a customer's waiting time $\mathscr{W} = w$) for several values of w and then to fit a polynomial to these points. The expectation of this function of a continuous random variable is then defined as

$$E(h(\mathscr{W})) = \int_0^\infty h(w) f_{\mathscr{W}}(w)\, dw,$$

where $f_{\mathscr{W}}(w)$ is the probability density function of $\mathscr{W}$. However, because $E(h(\mathscr{W}))$ is the expected waiting cost *per customer* and $E(\text{WC})$ is the expected waiting cost *per unit time,* these two quantities are not equal in this case. To relate them, it is necessary to

FIGURE 26.6
The waiting-cost function as a function of $\mathscr{W}$.



multiply $E(h(\mathscr{W}))$ by the expected *number of customers per unit time* entering the queueing system. In particular, if the mean arrival rate is a constant λ, then

$$E(\text{WC}) = \lambda E(h(\mathscr{W})) = \lambda \int_0^\infty h(w) f_{\mathscr{W}}(w)\, dw.$$

Example 2—Which Computer? Because the faculty and students of Emerald University would experience different turnaround times with the two computers under consideration (see Sec. 26.1), the choice between the computers required an evaluation of the consequences of making them wait for their jobs to be run. Therefore, several leading scientists on the faculty were asked to evaluate these consequences.

The scientists agreed that one major consequence is a *delay in getting research done.* Little effective progress can be made while one is awaiting the results from a computer run. The scientists estimated that it would be worth \$500 to reduce this delay by a day. Therefore, this component of waiting cost was estimated to be \$500 per day, that is, $500\mathscr{W}$, where $\mathscr{W}$ is expressed in days.

The scientists also pointed out that a second major consequence of waiting is a *break in the continuity of the research.* Although a short delay (a fraction of a day) causes little problem in this regard, a longer delay causes significant wasted time in having to gear up to resume the research. The scientists estimated that this wasted time would be roughly proportional to the *square* of the delay time. Dollar figures of \$100 and \$400 were then imputed to the value of being able to avoid this consequence entirely rather than having a wait of $\frac{1}{2}$ day and 1 day, respectively. Therefore, this component of the waiting cost was estimated to be $400\mathscr{W}^2$.

This analysis yields

$$h(\mathscr{W}) = 500\mathscr{W} + 400\mathscr{W}^2.$$

Because

$$f_{\mathscr{W}}(w) = \mu(1 - \rho)e^{-\mu(1-\rho)w}$$

for the *M*/*M*/1 model (see Sec. 17.6) fitting this single-server queueing system,

$$E(h(\mathscr{W})) = \int_0^\infty (500w + 400w^2)\mu(1 - \rho)e^{-\mu(1-\rho)w}\, dw,$$

where $\rho = \lambda/\mu$ for a single-server system. Since $\mu(1 - \rho) = (\mu - \lambda)$, the values of μ and λ presented in Sec. 26.1 give

$$\mu(1 - \rho) = \begin{cases} 10 & \text{for MBI computer} \\ 5 & \text{for CRAB computer.} \end{cases}$$

Evaluating the integral for these two cases yields

$$E(h(\mathscr{W})) = \begin{cases} 58 & \text{for MBI computer} \\ 132 & \text{for CRAB computer.} \end{cases}$$

The result represents the expected waiting cost (in dollars) for each person arriving with a job to be run. Because $\lambda = 20$, the total expected waiting cost per day becomes

$$E(\text{WC}) = \begin{cases} \$1{,}160 \text{ per day} & \text{for MBI computer} \\ \$2{,}640 \text{ per day} & \text{for CRAB computer.} \end{cases}$$

The Linear Case. In preparation for considering the next example, consider now the special case where $h(\mathscr{W})$ is a linear function,

$$h(\mathscr{W}) = C_w\mathscr{W},$$

where C_w is the cost of waiting per unit time for each customer. In this case, $E(\text{WC})$ reduces to

$$E(\text{WC}) = \lambda E(C_w \mathscr{W}) = C_w(\lambda W) = C_w L.$$

Note that this result is identical to the result when $g(N)$ is a linear function. Consequently, when the total waiting cost incurred by the queueing system is simply *proportional* to the total waiting time, it does not matter whether the $g(N)$ or the $h(\mathscr{W})$ form is used for the waiting-cost function.

Example 3—How Many Tool Cribs? As indicated in Sec. 26.1, the value to the Mechanical Company of a busy mechanic's output averages about \$48 per hour. Thus, $C_w = 48$. Consequently, for each tool crib the expected waiting cost per hour is

$$E(\text{WC}) = 48L,$$

where L represents the expected number of mechanics waiting (or being served) at the tool crib.

26.4 DECISION MODELS

We mentioned in Sec. 26.2 that three common decision variables in designing queueing systems are s (number of servers), μ (mean service rate for each server), and λ (mean arrival rate at each service facility). We shall now formulate models for making some of these decisions.

Model 1—Unknown *s*

Model 1 is designed for the case where both μ and λ are fixed at a particular service facility, but where a decision must be made on the number of servers to have on duty at the facility.

Formulation of Model 1.

Definition:	C_s = marginal cost of a server per unit time.	
Given:	μ, λ, C_s.	
To find:	s.	
Objective:	Minimize	$E(\text{TC}) = C_s s + E(\text{WC})$.

Because only a few alternative values of s normally need to be considered, the usual way of solving this model is to calculate $E(\text{TC})$ for these values of s and select the minimizing one. Section 17.10 describes and illustrates this approach for the linear case where $E(\text{WC}) = C_w L$. The example presented there uses an Excel template that has been provided in your OR Courseware for performing these calculations when the queueing system fits the *M*/*M*/*s* queueing model. However, as long as the queueing model is tractable, it often is not very difficult to perform these calculations yourself for other cases, as illustrated by the following example.

Example 1—How Many Repairers? For Example 1 of Sec. 26.1, each repairer (server) costs SIMULATION, INC. approximately \$280 per day. Thus, with 1 day as the unit of time, $C_s = 280$. Using the values of $E(\text{WC})$ calculated in Table 26.1 then yields the results shown in Table 26.2, which indicate that the company should continue having just one repairer.

TABLE 26.2 Calculation of *E*(TC) in dollars per day for Example 1

s	$C_s s$	E(WC)	E(TC)
1	\$280	\$281	\$561 per day ← minimum
2	\$560	\$ 48	\$608 per day
≥3	≥\$840	≥\$ 0	≥\$840 per day

Model 2—Unknown μ and s

Model 2 is designed for the case where both the efficiency of service, measured by μ, and the number of servers s at a service facility need to be selected.

Alternative values of μ may be available because there is a choice on the *quality* of the servers. For example, when the servers will be materials-handling units, the quality of the units to be purchased affects their service rate for moving loads.

Another possibility is that the *speed* of the servers can be adjusted mechanically. For example, the speed of machines frequently can be adjusted by changing the amount of power consumed, which also changes the cost of operation.

Still another type of example is the selection of the number of crews (the servers) and the size of each crew (which determines μ) for jointly performing a certain task. The task might be maintenance work, or loading and unloading operations, or inspection work, or setup of machines, and so forth.

In many cases, only a few alternative values of μ are available, e.g., the efficiency of the alternative types of materials-handling equipment or the efficiency of the alternative crew sizes.

Formulation of Model 2.

Definitions: $f(\mu)$ = marginal cost of server per unit time when mean service rate is μ.
A = set of feasible values of μ.
Given: λ, $f(\mu)$, A.
To find: μ, s.
Objective: Minimize $E(\text{TC}) = f(\mu)s + E(\text{WC})$, subject to $\mu \in A$.

Example 2—Which Computer? For Example 2 in Sec. 26.1, EMERALD UNIVERSITY needs to make a decision about which supercomputer to lease. It is known that $\mu = 30$ for the MBI computer and $\mu = 25$ for the CRAB computer, where 1 day is the unit of time. These computers are the only two being considered by Emerald University, so

$$A = \{25, 30\}.$$

Because the leasing cost per day is \$3,750 for the CRAB computer ($\mu = 25$) and \$5,000 for the MBI computer ($\mu = 30$),

$$f(\mu) = \begin{cases} 3{,}750 & \text{for } \mu = 25 \\ 5{,}000 & \text{for } \mu = 30. \end{cases}$$

The supercomputer chosen will be the only one available to the faculty and students, so the number of servers (supercomputers) for this queueing system is restricted to $s = 1$. Hence,

$$E(\text{TC}) = f(\mu) + E(\text{WC}),$$

where $E(\text{WC})$ is given in Sec. 26.3 for the two alternatives. Thus,

$$E(\text{TC}) = \begin{cases} 3{,}750 + 2{,}640 = \$6{,}390 \text{ per day} & \text{for CRAB computer} \\ 5{,}000 + 1{,}160 = \$6{,}160 \text{ per day} & \text{for MBI computer.} \end{cases}$$

Consequently, the decision was made to lease the MBI supercomputer.

The Application of Model 2 to Other Situations. This example illustrates a case where the number of feasible values of μ is *finite* but the value of s is fixed. If s were not fixed, a two-stage approach could be used to solve such a problem. First, for each individual value of μ, set $C_s = f(\mu)$, and solve for the value of s that minimizes $E(\text{TC})$ for model 1. Second, compare these minimum $E(\text{TC})$ for the alternative values of μ, and select the one giving the overall minimum.

When the number of feasible values of μ is *infinite* (such as when the speed of a machine or piece of equipment is set mechanically within some feasible interval), another two-stage approach sometimes can be used to solve the problem. First, for each individual value of s, *analytically* solve for the value of μ that minimizes $E(\text{TC})$. [This approach requires setting to zero the derivative of $E(\text{TC})$ with respect to μ and then solving this equation for μ, which can be readily done only when analytical expressions are available for both $f(\mu)$ and $E(\text{WC})$.] Second, compare these minimum $E(\text{TC})$ for the alternative values of s, and select the one giving the overall minimum.

This analytical approach frequently is relatively straightforward for the case of $s = 1$ (see Prob. 26.4-11). However, because far fewer and less convenient analytical results are available for multiple-server versions of queueing models, this approach is either difficult (requiring computer calculations with numerical methods to solve the equation for μ) or completely impossible when $s > 1$. Therefore, a more practical approach is to consider only a relatively small number of representative values of μ and to use available tabulated results for the appropriate queueing model to obtain (or approximate) $E(\text{TC})$ for these μ values.

A Special Result with Model 2. Fortunately, under certain fairly common circumstances described next, $s = 1$ (and its minimizing value of μ) *must* yield the overall minimum $E(\text{TC})$ for model 2, so $s > 1$ cases need not be considered at all.

Optimality of a Single Server. Under certain conditions, $s = 1$ necessarily is *optimal* for model 2.

The primary conditions[1] are that

1. The value of μ minimizing $E(\text{TC})$ for $s = 1$ is feasible.
2. Function $f(\mu)$ is either *linear* or *concave* (as defined in Appendix 2).

In effect, this optimality result indicates that it is better to concentrate service capacity into one fast server rather than dispersing it among several slow servers. Condition 2 says that this concentrating of a given amount of service capacity can be done without increasing the cost of service. Condition 1 says that it must be possible to make μ sufficiently large that a single server can be used to full advantage.

To understand why this result holds, consider any other solution to model 2, $(s, \mu) = (s^*, \mu^*)$, where $s^* > 1$. The service capacity of this system (as measured by the mean rate of service completions when all servers are working) is $s^*\mu^*$. We shall now compare this solution with the corresponding single-server solution $(s, \mu) = (1, s^*\mu^*)$ having the *same* service capacity. In particular, Table 26.3 compares the mean rate at

[1]There also are minor restrictions on the queueing model and the waiting-cost function. However, any of the constant service-rate queueing models presented in Chap. 17 for $s \geq 1$ are allowed. If the $g(N)$ form is used for the waiting-cost function, it can be any *increasing* function. If the $h(\mathscr{W})$ form is used, it can be any linear function or any convex function (as defined in Appendix 2), which fits most cases of interest.

TABLE 26.3 Comparison of service efficiency for Model 2 solutions

	Mean Rate of Service Completions
$N = n$	$(s, \mu) = (s^*, \mu^*)$ versus $(s, \mu) = (1, s^*\mu^*)$
$n = 0$	$0 = 0$
$n = 1, 2, \ldots, s^* - 1$	$n\mu^* < s^*\mu^*$
$n \geq s^*$	$s^*\mu^* = s^*\mu^*$

which service completions occur for each given number of customers in the system $N = n$. This table shows that the service efficiency of the (s^*, μ^*) solution sometimes is worse but never is better than for the $(1, s^*\mu^*)$ solution because it can use the full service capacity only when there are at least s^* customers in the system, whereas the single-server solution uses the full capacity whenever there are *any* customers in the system. Because this lower service efficiency can only increase waiting in the system, $E(\text{WC})$ must be larger for (s^*, μ^*) than for $(1, s^*\mu^*)$. Furthermore, the expected service cost must be at least as large because condition 2 [and $f(0) = 0$] implies that

$$f(\mu^*)s \geq f(s^*\mu^*).$$

Therefore, $E(\text{TC})$ is larger for (s^*, μ^*) than $(1, s^*\mu^*)$. Finally, note that condition 1 implies that there is a feasible solution with $s = 1$ that is at least as good as $(1, s^*\mu^*)$. The conclusion is that *any* $s > 1$ solution *cannot* be optimal for model 2, so $s = 1$ must be optimal.[2]

This result is still of some use even when one or both conditions fail to hold. If μ cannot be made sufficiently large to permit a single server, it still suggests that a *few* fast servers should be preferred to many slow ones. If condition 2 does not hold, we still know that $E(\text{WC})$ is minimized by concentrating any given amount of service capacity into a single server, so the best $s = 1$ solution must be at least nearly optimal unless it causes a *substantial* increase in service cost.

Model 3—Unknown λ and s

Model 3 is designed especially for the case where it is necessary to select both the *number of service facilities* and the *number of servers* s at each facility. In the typical situation, a population (such as the employees in an industrial building) must be provided with a certain service, so a decision must be made as to what proportion of the population (and therefore what value of λ) should be assigned to each service facility. Examples of such facilities include employee facilities (drinking fountains, vending machines, and restrooms), storage facilities, and reproduction equipment facilities. It may sometimes be clear that only a single server should be provided at each facility (e.g., one drinking fountain or one copy machine), but s often is also a decision variable.

[2]For a rigorous proof of this result, see S. Stidham, Jr., "On the Optimality of Single-Server Queueing Systems," *Operations Research,* **18:** 708–732, 1970. This result focuses on minimizing $E(\text{TC})$ when $E(\text{WC})$ is based on waiting time in the system. However, if waiting costs are incurred only while waiting in the queue, markedly different results occur. For example, see X. Chao and C. Scott, "Several Results on the Design of Queueing Systems," *Operations Research,* **48:** 965–970, 2000. Furthermore, even when waiting time in the system is the relevant quantity, if the concern is to avoid extremely long waiting times as much as possible rather than minimizing $E(\text{TC})$, then several slow servers become superior to one fast server when the service-time distribution is so highly variable that it possesses some infinite higher moments. For an analysis of this alternative viewpoint, see A. Scheller-Wolf, "Necessary and Sufficient Conditions for Delay Moments in FIFO Multiserver Queues with an Application Comparing s Slow Servers with One Fast One," *Operations Research,* **51:** 748–758, 2003.

To simplify our presentation, we shall require in model 3 that λ and s be the same for all service facilities. However, it should be recognized that a slight improvement in the indicated solution might be achieved by permitting minor deviations in these parameters at individual facilities. This should be investigated as part of the detailed analysis that generally follows the application of the mathematical model.

Formulation of Model 3.

Definitions:	C_s = marginal cost of server per unit time.
	C_f = fixed cost of service per service facility per unit time.
	λ_p = mean arrival rate for entire calling population.
	n = number of service facilities = λ_p/λ.
Given:	μ, C_s, C_f, λ_p.
To find:	λ, s.
Objective:	Minimize E(TC), subject to $\lambda = \lambda_p/n$, where $n = 1, 2, \ldots$.

Finding *E*(TC). It might appear at first glance that the appropriate expression for the expected total cost per unit time of all the facilities should be

$$E(\text{TC}) \stackrel{?}{=} n[(C_f + C_s s) + E(\text{WC})],$$

where E(WC) here represents the expected waiting cost per unit time for *each* facility. However, if this expression actually were valid, it would imply that $n = 1$ *necessarily* is optimal for model 3. The reasoning is completely analogous to that for the optimality of a single-server result for model 2; namely, any solution $(n, s) = (n^*, s^*)$ with $n^* > 1$ has higher service costs than the $(n, s) = (1, n^*s^*)$ solution, and it *also* has a higher expected waiting cost because it sometimes makes less effective use of the available service capacity. In particular, it sometimes has idle servers at one facility while customers are waiting at another facility, so the mean rate of service completions would be less than if the customers had access to *all* the servers at one common facility.

Because there are many situations where it obviously would *not* be optimal to have just one service facility (e.g., the number of restrooms in a 50-story building), something must be wrong with this expression. Its deficiency is that it considers only the cost of service and the cost of waiting *at the service facilities* while totally ignoring the cost of the time wasted in *traveling* to and from the facilities. Because travel time would be prohibitive with only one service facility for a large population, enough separate facilities must be distributed throughout the calling population to hold travel time down to a reasonable level.

Thus, letting the random variable T be the round-trip travel time for a customer coming to and going back from one of the service facilities, we see that the total time lost by the customer actually is $\mathscr{W} + T$. (Recall from Chap. 17 that $\mathscr{W}$ is the waiting time in the queueing system *after* the customer arrives.) Therefore, a customer's *total* cost for time lost should be based on $\mathscr{W} + T$ rather than just $\mathscr{W}$. To simplify the analysis, let us separate this total cost into the sum of the waiting-time cost based on $\mathscr{W}$ (or N) and the travel-time cost based on T. We shall also assume that the travel-time cost is proportional to T, where C_t is the cost of each unit of travel time for each customer. For ease of presentation, suppose that the probability distribution of T is the same for each service facility, so that $C_tE(T)$ is the *expected travel cost* for each arrival at any of the service facilities. The resulting expression for E(TC) is

$$E(\text{TC}) = n[(C_f + C_s s) + E(\text{WC}) + \lambda C_t E(T)]$$

because λ is the expected number of arrivals *per unit time* at each facility. Consequently, by evaluating (or estimating) $E(T)$ for each case of interest, model 3 can be solved by calculating E(TC) for various values of s for each n and then selecting the solution giving

the overall minimum. The next section discusses how to evaluate $E(T)$ and also solves an example (Example 3 of Sec. 26.1) fitting model 3.

26.5 THE EVALUATION OF TRAVEL TIME

As discussed in Sec. 26.4, one of the important considerations for deciding how many service facilities to provide is the amount of time that customers must spend traveling to and from a facility. Therefore, *the expected round-trip travel time* $E(T)$ for a customer is one of the components of the objective function for model 3, the decision model that is concerned with deciding on the number of service facilities. We now shall elaborate on how to determine $E(T)$.

$E(T)$ can be interpreted as the *average travel time* spent by customers in coming both to and from a given service facility. Therefore, the value of $E(T)$ depends very much upon the characteristics of the individual situation. However, we shall illustrate a rather general approach to evaluating $E(T)$ by developing a basic travel-time model and then calculating $E(T)$ for the more complicated situation involved in Example 3. In both cases it is assumed that the portion of the population assigned to the service facility under consideration is *distributed uniformly* throughout the assigned area, that each arrival returns to its *original location* after receiving service, and that the average speed of travel does *not* depend upon the distance traveled. Another basic assumption is that all travel is *rectilinear,* i.e., it progresses along a system of *orthogonal* paths (aisles, streets, highways, and so on) that are *parallel* to the main sides of the area under consideration.

A Basic Travel-Time Model

Description: Rectangular area and rectilinear travel, as shown in Fig. 26.7.

Definitions: T = travel time (round trip) for an arrival.
v = average velocity (speed) of customers in traveling to and from a facility.
a, b, c, d = respective distances from a facility to a boundary of the area assigned to the facility, as shown in Fig. 26.7.

Given: v, a, b, c, d.

To find: Expected value of T, $E(T)$.

Using an orthogonal (x, y) coordinate system, Fig. 26.7 shows the coordinates (x, y) of the location of a *particular* customer. The x and y coordinates of the location from which a *random* arrival comes actually are *random variables* X and Y, where X ranges from $-a$ to c and Y ranges from $-b$ to d. Because the total round-trip distance traveled by the random arrival is

$$D = 2(|X| + |Y|)$$

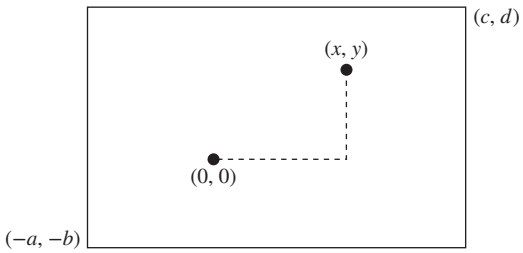


FIGURE 26.7
Graphical representation of a basic travel-time model, where the service facility is at (0, 0) and a random arrival comes from (and returns to) some location (x, y).

and

$$T = \frac{D}{v},$$

it follows that

$$E(T) = \frac{2}{v}\,(E\{|X|\} + E\{|Y|\}),$$

Thus, the problem is reduced to identifying the probability distributions of $|X|$ and $|Y|$ and then calculating their means.

First consider $|X|$. Its probability distribution can be obtained directly from the distribution of X. Because the customers are assumed to be distributed uniformly throughout the assigned area, and because the *height* of the rectangular area is the *same* for all possible values of $X = x$, X must have a *uniform distribution* between $-a$ and c, as shown in Fig. 26.8*a*. Because $|x| = |-x|$, adding the probability density function values at x and $-x$ then yields the probability distribution of $|X|$ shown in Fig. 26.8*b*.

Therefore, noting that $|x| = x$ for $x \geq 0$,

$$\begin{aligned} E\{|X|\} &= \int_0^{\max\{a,\,c\}} x f_{|x|}(x)\,dx \\ &= \int_0^{\min\{a,\,c\}} \frac{2x}{a+c}\,dx + \int_{\min\{a,\,c\}}^{\max\{a,\,c\}} \frac{x}{a+c}\,dx \\ &= \frac{1}{2}\,\frac{1}{a+c}\,[(\min\{a,\,c\})^2 + (\max\{a,\,c\})^2] \\ &= \frac{a^2+c^2}{2(a+c)}. \end{aligned}$$

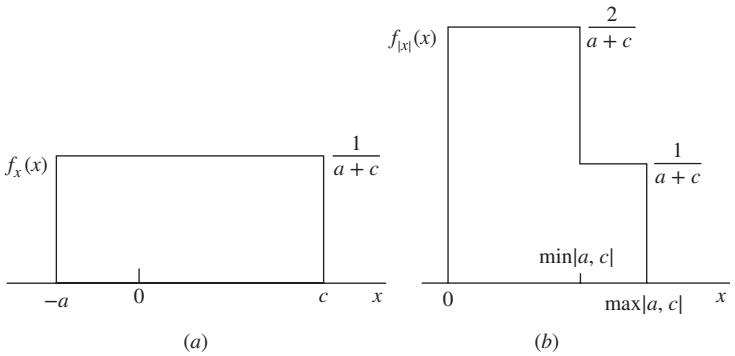
The analysis for $|Y|$ is completely analogous, where the *width* of the rectangular area for possible values of $Y = y$ now determines the probability distribution of Y.

The result is that

$$E\{|Y|\} = \frac{b^2+d^2}{2(b+d)}.$$

Consequently,

$$E(T) = \frac{1}{v}\left(\frac{a^2+c^2}{a+c} + \frac{b^2+d^2}{b+d}\right).$$



■ **FIGURE 26.8**
Probability density functions of (*a*) X; (*b*) $|X|$.

Example 3—How Many Tool Cribs? For the new plant being designed for the MECHANICAL COMPANY (see Sec. 26.1), the layout of the portion of the factory area where the mechanics will work is shown in Fig. 26.9. The three *possible* locations for tool cribs are identified as Locations 1, 2, and 3, where access to these locations will be provided by a system of orthogonal aisles parallel to the sides of the indicated area. The coordinates are given in units of *feet*. The mechanics will be distributed quite uniformly throughout the area shown, and each mechanic will be assigned to the *nearest* tool crib. It is estimated that the mechanics will walk to and from a tool crib at an average speed of slightly less than 3 miles/hour, so v is set at $v = 15{,}000$ feet/hour.

The three basic alternatives being considered are

Alternative 1: Have *three* tool cribs—use Locations 1, 2, and 3;
Alternative 2: Have *one* tool crib—use Location 2;
Alternative 3: Have *two* tool cribs—use Locations 1 and 3.

The calculation of $E(T)$ for each alternative is given next, followed by the use of model 3 to make the choice among them.

Alternative 1 ($n = 3$): If all three locations were used, *each* tool crib would service a 300×300 foot *square* area. Therefore, this case is just a special case of the basic travel-time model just presented, where $a = c = 150$ and $b = d = 150$. Consequently,

$$E(T) = \frac{1}{15{,}000 \text{ ft/hr}} \left(\frac{150^2 + 150^2}{150 + 150} + \frac{150^2 + 150^2}{150 + 150} \right) \text{ ft}$$

$$= \frac{1}{15{,}000 \text{ ft/hr}} (300 \text{ ft})$$

$$= 0.02 \text{ hr}.$$

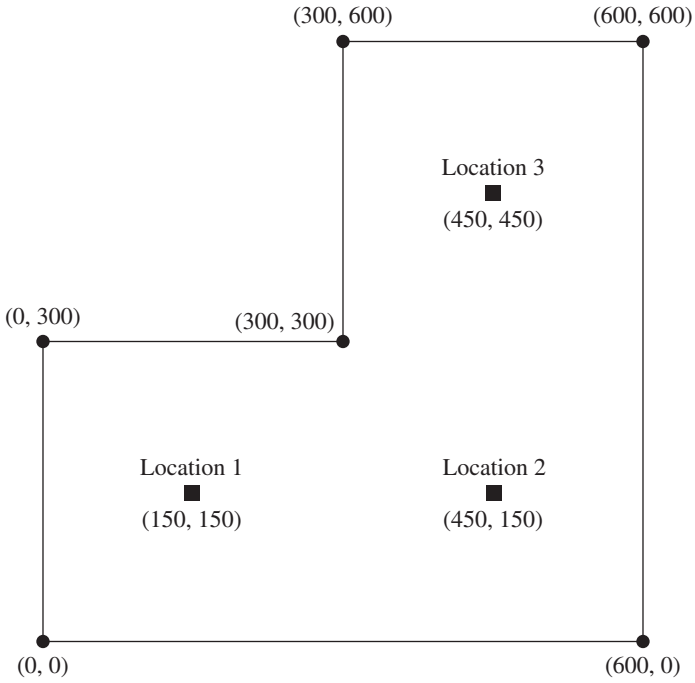


FIGURE 26.9
Layout for Example 3.

Alternative 2 ($n = 1$)*:* With just *one* tool crib (in Location 2) to service the entire area shown in Fig. 26.9, the derivation of $E(T)$ is a little more complicated than it is for the basic traveltime model. The first step is to relabel Location 2 as the original (0, 0) for an (x, y) coordinate system, so that 450 would be subtracted from the first coordinates shown and 150 would be subtracted from the second coordinates. The probability density function for X is then obtained by dividing the *height* for each possible value of $X = x$ by the total area (so that the area under the probability density function curve equals 1), as given in Fig. 26.10*a*. Combining the values for x and $-x$ then yields the probability distribution of $|X|$ shown in Fig. 26.10*b*.

Hence,

$$E\{|X|\} = \int_0^{450} x\, f_{|X|}(x)\, dx$$

$$= \int_0^{150} x \left(\frac{1}{225}\right) dx + \int_{150}^{450} x \left(\frac{1}{900}\right) dx$$

$$= \frac{150^2}{450} + \frac{450^2 - 150^2}{1{,}800} = 150.$$

We suggest that you now try the same approach (using the *width* of the area rather than the height) to derive $E\{|Y|\}$. You will find that the probability distribution of $|Y|$ is *identical* to that for $|X|$, so $E\{|Y|\} = 150$. As a result,

$$E(T) = \frac{2}{15{,}000}\,(150 + 150)$$

$$= 0.04 \text{ hr}.$$

Alternative 3 ($n = 2$)*:* With tool cribs in just Locations 1 and 3, the areas assigned to them would be divided by a line segment between (300, 300) and (600, 0) in Fig. 26.9. Notice that the two areas and their tool cribs are located symmetrically with respect to this line segment. Therefore, $E(T)$ is the same for both, so we shall derive it just for the tool crib in Location 1. (You might try it for the other tool crib for practice—see Prob. 26.5-3.)

Proceeding just as for Alternative 2, relabel Location 1 as the origin (0, 0) for an (x, y) coordinate system, so that 150 would be subtracted from all coordinates shown in Fig. 26.9. This relabeling leads directly to the probability density function of X, and then of $|X|$, shown in Fig. 26.11. As a result,

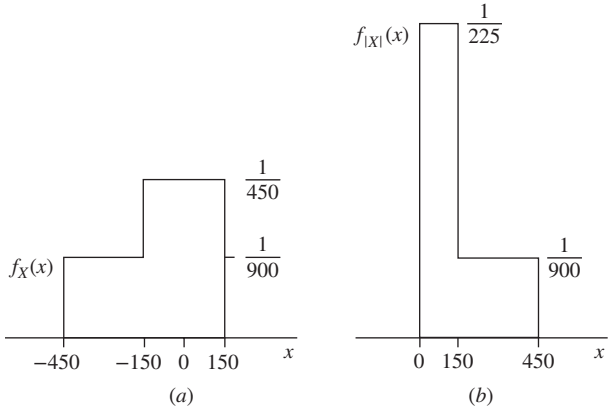
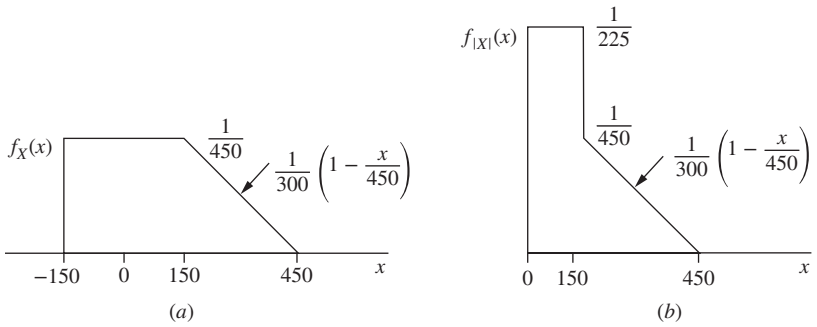
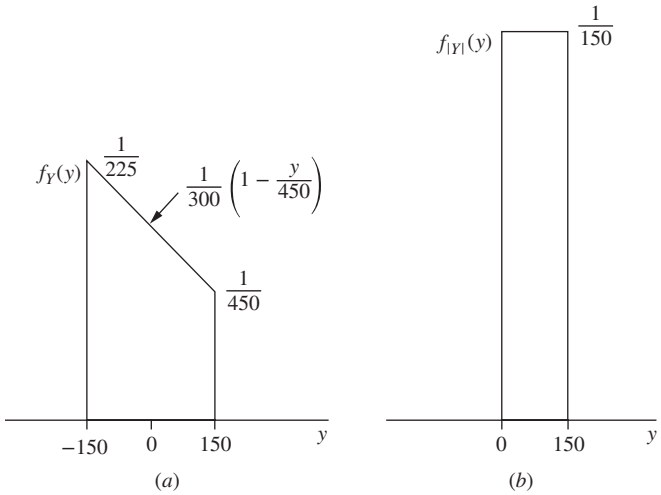


FIGURE 26.10
Probability density functions of (*a*) X and (*b*) $|X|$ for a tool crib at Location 2 of Fig. 26.9 under Alternative 2 (no other tool cribs).



■ **FIGURE 26.11**
Probability density functions of (*a*) X and (*b*) $|X|$ for a tool crib at Location 1 of Fig. 26.9 under Alternative 3 (the only other tool crib is at Location 3).



■ **FIGURE 26.12**
Probability density functions of (*a*) Y and (*b*) $|Y|$ for a tool crib at Location 1 of Fig. 26.9 under Alternative 3 (the only other tool crib is at Location 3).

$$E\{|\mathrm{X}|\} = \frac{1}{225}\int_0^{150} x\,dx + \frac{1}{300}\int_{150}^{450}\left(1-\frac{x}{450}\right)x\,dx$$

$$= \frac{1}{225}\left[\frac{x^2}{2}\right]_0^{150} + \frac{1}{300}\left[\frac{x^2}{2} - \frac{x^3}{1{,}350}\right]_{150}^{450}$$

$$= \frac{1}{225}\,\frac{150^2}{2} + \frac{1}{300}\left(\frac{450^2}{2} - \frac{450^3}{1{,}350}\right) - \frac{1}{300}\left(\frac{150^2}{2} - \frac{150^3}{1{,}350}\right)$$

$$= 133\tfrac{1}{3}.$$

Next, the probability density function of Y is obtained by using the *width* of the area assigned to the tool crib at Location 1 for each possible value of $Y = y$ and then dividing by the size of the area, as given in Fig. 26.12*a*. This result then yields the *uniform* distribution of $|Y|$ shown in Fig. 26.12*b*. Thus,

$$E\{|\mathrm{Y}|\} = \frac{1}{150}\int_0^{150} y\,dy$$

$$= 75.$$

TABLE 26.4 Calculation of E(TC), in dollars per hour for Example 3

n	λ	s	L	$E(T)$	$C_f + C_s s$	E(WC)	$\lambda C_t E(T)$	E(TC)
1	120	1	∞	0.04	\$36	∞	\$230.40	∞
1	120	2	1.333	0.04	\$56	\$64.00	\$230.40	\$350.40
1	120	3	1.044	0.04	\$76	\$50.11	\$230.40	\$356.51
2	60	1	1.000	0.0278	\$36	\$48.00	\$ 80.00	\$328.00
2	60	2	0.534	0.0278	\$56	\$25.63	\$ 80.00	\$323.26
3	40	1	0.500	0.02	\$36	\$24.00	\$ 38.40	\$295.20
3	40	2	0.344	0.02	\$56	\$16.51	\$ 38.40	\$332.73

Consequently,

$$E(T) = \frac{2}{15{,}000}(133\tfrac{1}{3} + 75)$$

$$= 0.0278 \text{ hr.}$$

Applying Model 3: Because $E(T)$ now has been evaluated for the three alternatives under consideration, the stage is set for using model 3 from Sec. 26.4 to choose among these alternatives. Most of the data required for this model are given in Sec. 26.1, namely,

$\mu = 120$ per hour, $C_f = \$16$ per hour,
$C_s = \$20$ per hour,
$\lambda_p = 120$ per hour, $C_t = \$48$ per hour,

where the $M/M/s$ model given in Sec. 17.6 is used to calculate L and so on. In addition, the end of Sec. 26.3 gives $E(\text{WC}) = 48L$ in dollars per hour. Therefore,

$$E(\text{TC}) = n\left[(16 + 20s) + 48L + \frac{120}{n}\,48E(T)\right].$$

The resulting calculation of E(TC) for various s values for each n is given in Table 26.4, which indicates that the *overall minimum* E(TC) of \$295.20 per hour is obtained by having three tool cribs (so $\lambda = 40$ for each), with one clerk at each tool crib.

26.6 CONCLUSIONS

This chapter has discussed the application of queueing theory for *designing* queueing systems. Every individual problem has its own special characteristics, so no standard procedure can be prescribed to fit every situation. Therefore, the emphasis has been on introducing fundamental considerations and approaches that can be adapted to most cases. We have focused on three particularly common decision variables (s, μ, and λ) as a vehicle for introducing and illustrating these concepts. However, there are many other possible decision variables (e.g., the size of a waiting room for a queueing system) and many more complicated situations (e.g., designing a *priority* queueing system) that can also be analyzed in a similar way.

The time required to *travel* to and from a service facility sometimes is an important consideration. A rather general approach to evaluating expected travel time has been introduced by applying it to some relatively simple cases. However, once again, many more complicated situations can also be analyzed quite similarly. We have discussed the incorporation of travel-time information into the overall analysis only in the context of

determining the *number* of service facilities to provide when *customers* must travel to the nearest facility. But travel-time models also can be very useful when the *servers* must travel to the customer from the service facility (e.g., fire trucks and ambulances), as well as in other contexts.

Another useful area for the application of queueing theory is the development of policies for *controlling* queueing systems, e.g., for *dynamically* adjusting the number of servers or the service rate to compensate for changes in the number of customers in the system. Considerable research has been conducted in this area.

Queueing theory has proved to be a very useful tool, and its use is continuing to grow as recognition of the many guises of queueing systems grows.

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Same templates as provided for Chap. 17

"Ch. 26—Application of QT" LINGO File for Selected Examples

See Appendix 1 for documentation of the software.

PROBLEMS

To the left of each of the following problems (or their parts), we have inserted a T whenever one of the templates for this chapter (and Chap. 17) can be useful.

26.2-1. For each kind of queueing system listed in Prob. 17.3-1, briefly describe the nature of the *cost of service* and the *cost of waiting* that would need to be considered in designing the system.

26.3-1. Suppose that a queueing system fits the $M/M/1$ model described in Sec. 17.6, with $\lambda = 2$ and $\mu = 4$. Evaluate the expected waiting cost per unit time $E(\text{WC})$ for this system when its waiting-cost function has the form

(a) $g(N) = 10N + 2N^2$.
(b) $h(\mathscr{W}) = 25\mathscr{W} + \mathscr{W}^3$.

26.3-2. Follow the instructions of Prob. 26.3-1 for the following waiting-cost functions.

(a) $$g(N) = \begin{cases} 10N & \text{for } N = 1, 2 \\ 6N^2 & \text{for } N = 3, 4, 5 \\ N^3 & \text{for } N > 5. \end{cases}$$

(b) $$h(\mathscr{W}) = \begin{cases} \mathscr{W} & \text{for } 0 \le \mathscr{W} \le 1 \\ \mathscr{W}^2 & \text{for } \mathscr{W} \ge 1. \end{cases}$$

26.4-1. A certain queueing system has a Poisson input, with a mean arrival rate of 4 customers per hour. The service-time distribution is exponential, with a mean of 0.2 hour. The marginal cost of providing each server is \$20 per hour, where it is estimated that the cost that is incurred by having each customer *idle* (i.e., in the queueing system) is \$120 per hour for the first customer and \$180 per hour for each additional customer. Determine the number of servers that should be assigned to the system to minimize the expected total cost per hour. [*Hint:* Express E(WC) in terms of L, P_0, and ρ, and then use the template for the M/M/s model in your OR Courseware.]

26.4-2. Reconsider Prob. 17.6-10. The total compensation for the new employee would be \$16 per hour, which is just half that for the cashier. It is estimated that the grocery store incurs lost profit due to lost future business of \$0.16 for each minute that each customer has to wait (including service time). The manager now wants to determine on an expected total cost basis whether it would be worthwhile to hire the new person.

(a) Which decision model presented in Sec. 26.4 applies to this problem? Why?

(b) Use this model to determine whether to continue the status quo or to adopt the proposal.

26.4-3. The Southern Railroad Company has been subcontracting for the painting of its railroad cars as needed. However, management has decided that the company can save money by doing this work itself. A decision now needs to be made to choose between two alternative ways of doing this.

Alternative 1 is to provide two paint shops, where painting is done by hand (one car at a time in each shop), for a total hourly cost of \$70. The painting time for a car would be 6 hours. Alternative 2 is to provide one spray shop involving an hourly cost of \$100. In this case, the painting time for a car (again done one at a time) would be 3 hours. For both alternatives, the cars arrive according to a Poisson process with a mean rate of 1 every 5 hours. The cost of idle time per car is \$100 per hour.

(a) Use Fig. 17.8 to estimate L, L_q, W, and W_q for Alternative 1.

(b) Find these same measures of performance for Alternative 2.

(c) Determine and compare the expected total cost per hour for these alternatives.

26.4-4. The production of tractors at the Jim Buck Company involves producing several subassemblies and then using an assembly line to assemble the subassemblies and other parts into finished tractors. Approximately three tractors per day are produced in this way. An in-process inspection station is used to inspect the subassemblies before they enter the assembly line. At present there are two inspectors at the station, and they work together to inspect each subassembly. The inspection time has an exponential distribution, with a mean of 15 minutes. The cost of providing this inspection system is \$40 per hour.

A proposal has been made to streamline the inspection procedure so that it can be handled by only one inspector. This inspector would begin by visually inspecting the exterior of the subassembly, and she would then use new efficient equipment to complete the inspection. Although this process with just one inspector would slightly increase the mean of the distribution of inspection times from 15 minutes to 16 minutes, it also would reduce the variance of this distribution to only 40 percent of its original value. Because of the expense involved with purchasing and operating the new inspection equipment, the capitalized cost of this proposed inspection system would be \$40 per hour, just as for the current inspection system.

The subassemblies arrive at the inspection station according to a Poisson process at a mean rate of 3 per hour. The cost of having the subassemblies wait to begin inspection at the inspection station (thereby increasing in-process inventory and possibly disrupting subsequent production) is estimated to be \$20 per hour for each subassembly.

Management now needs to make a decision about whether to continue the status quo or adopt the proposal.

T **(a)** Find the main measures of performance—L, L_q, W, W_q—for the current queueing system.

(b) Repeat part (*a*) for the proposed queueing system.

(c) What conclusions can you draw about what management should do from the results in parts (*a*) and (*b*)?

(d) Determine and compare the expected total cost per hour for the status quo and the proposal.

26.4-5. The car rental company, Try Harder, has been subcontracting for the maintenance of its cars in St. Louis. However, due to long delays in getting its cars back, the company has decided to open its own maintenance shop to do this work more quickly. This shop will operate 42 hours per week.

Alternative 1 is to hire two mechanics (at a cost of \$1,500 per week each), so that two cars can be worked on at a time. The time required by a mechanic to service a car has an Erlang distribution, with a mean of 5 hours and a shape parameter of $k = 8$.

Alternative 2 is to hire just one mechanic (for \$1,500 per week) but to provide some additional special equipment (at a capitalized cost of \$1,250 per week) to speed up the work. In this case, the maintenance work on each car is done in two stages, where the time required for each stage has an Erlang distribution with the shape parameter $k = 4$, where the mean is 2 hours for the first stage and 1 hour for the second stage.

For both alternatives, the cars arrive according to a Poisson process at a mean rate of 0.3 car per hour (during work hours). The company estimates that its net lost revenue due to having its cars unavailable for rental is \$150 per week per car.

(a) Use Fig. 17.10 to estimate L, L_q, W, and W_q for alternative 1.

(b) Find these same measures of performance for alternative 2.

(c) Determine and compare the expected total cost per week for these alternatives.

26.4-6. A certain small car-wash business is currently being analyzed to see if costs can be reduced. Customers arrive according to a Poisson process at a mean rate of 15 per hour, and only one car can be washed at a time. At present the time required to wash a car has an exponential distribution, with a mean of 4 minutes. It also has been noticed that if there are already 4 cars waiting (including the one being washed), then any additional arriving customers leave and take their business elsewhere. The lost incremental profit from each such lost customer is \$6.

Two proposals have been made. Proposal 1 is to add certain equipment, at a capitalized cost of \$6 per hour, which would reduce the expected washing time to 3 minutes. In addition, each arriving customer would be given a guarantee that if she had to wait longer than $\frac{1}{2}$ hour (according to a time slip she receives upon arrival) before her car is ready, then she receives a free car wash (at a marginal cost of \$4 for the company). This guarantee would be well posted and advertised, so it is believed that no arriving customers would be lost.

Proposal 2 is to obtain the most advanced equipment available, at an increased cost of \$20 per hour, and each car would be sent through two cycles of the process in succession. The time required for a cycle has an exponential distribution, with a mean of 1 minute, so total expected washing time would be 2 minutes. Because of the increased speed and effectiveness, it is believed that essentially no arriving customers would be lost.

The owner also feels that because of the loss of customer goodwill (and consequent lost future business) when customers have to wait, a cost of \$0.20 for each minute that a customer has to wait before her car wash begins should be included in the analysis of all alternatives.

Evaluate the expected total cost per hour $E(\text{TC})$ of the status quo, proposal 1, and proposal 2 to determine which one should be chosen.

26.4-7. The Seabuck and Roper Company has a large warehouse in southern California to store its inventory of goods until they are needed by the company's many furniture stores in that area. A single crew with four members is used to unload and/or load each truck that arrives at the loading dock of the warehouse. Management currently is downsizing to cut costs, so a decision needs to be made about the future size of this crew.

Trucks arrive at the loading dock according to a Poisson process at a mean rate of 1 per hour. The time required by a crew to unload and/or load a truck has an exponential distribution (regardless of crew size). The mean of this distribution with the four-member crew is 15 minutes. If the size of the crew were to be changed, it is estimated that the mean service rate of the crew (now $\mu = 4$ customers per hour) would be proportional to its size.

The cost of providing each member of the crew is \$20 per hour. The cost that is attributable to having a truck not in use (i.e., a truck standing at the loading dock) is estimated to be \$30 per hour.

(a) Identify the customers and servers for this queueing system. How many servers does it currently have?

T **(b)** Use the appropriate Excel template to find the various measures of performance for this queueing system with four members on the crew. (Set $t = 1$ hour in the Excel template for the waiting-time probabilities.)

T **(c)** Repeat (*b*) with three members.

T **(d)** Repeat part (*b*) with two members.

(e) Should a one-member crew also be considered? Explain.

(f) Given the previous results, which crew size do you think management should choose?

(g) Use the cost figures to determine which crew size would minimize the expected total cost per hour.

(h) Assume now that the mean service rate of the crew is proportional to the square root of its size. What should the size be to minimize expected total cost per hour?

26.4-8. Trucks arrive at a warehouse according to a Poisson process with a mean rate of 4 per hour. Only one truck can be loaded at a time. The time required to load a truck has an exponential distribution with a mean of $10/n$ minutes, where n is the number of loaders ($n = 1, 2, 3, \ldots$). The costs are (i) \$18 per hour for each loader and (ii) \$20 per hour for each truck being loaded or waiting in line to be loaded. Determine the number of loaders that minimizes the expected hourly cost.

26.4-9. A company's machines break down according to a Poisson process at a mean rate of 3 per hour. Nonproductive time on any machine costs the company \$60 per hour. The company employs a maintenance person who repairs machines at a mean rate of μ machines per hour (when continuously busy) if the company pays that person a wage of $\$5\mu$ per hour. The repair time has an exponential distribution.

Determine the hourly wage that minimizes the company's total expected cost.

26.4-10. Jake's Machine Shop contains a grinder for sharpening the machine cutting tools. A decision must now be made on the speed at which to set the grinder.

The grinding time required by a machine operator to sharpen the cutting tool has an exponential distribution, where the mean $1/\mu$. can be set at 0.5 minute, 1 minute, or 1.5 minutes, depending upon the speed of the grinder. The running and maintenance costs go up rapidly with the speed of the grinder, so the estimated cost per minute is \$1.60 for providing a mean of 0.5 minute, \$0.40 for a mean of 1.0 minute, and \$0.20 for a mean of 1.5 minutes.

The machine operators arrive randomly to sharpen their tools at a mean rate of 1 every 2 minutes. The estimated cost of an operator being away from his or her machine to the grinder is \$0.80 per minute.

T **(a)** Obtain the various measures of performance for this queueing system for each of the three alternative speeds for the grinder. (Set $t = 5$ minutes in the Excel template for the waiting time probabilities.)

(b) Use the cost figures to determine which grinder speed minimizes the expected total cost per minute.

26.4-11. Consider the special case of model 2 where (1) any $\mu > \lambda/s$ is feasible and (2) both $f(\mu.)$ and the waiting-cost function are linear functions, so that

$$E(\text{TC}) = C_r s\mu + C_w L,$$

where C_r is the marginal cost per unit time for each unit of a server's mean service rate and C_w is the cost of waiting per unit time for each customer. The optimal solution is $s = 1$ (by the optimality of a single-server result), and

$$\mu = \lambda + \sqrt{\frac{\lambda C_w}{C_r}}$$

for any queueing system fitting the $M/M/1$ model presented in Sec. 17.6.

Show that this μ is indeed optimal for the $M/M/1$ model.

26.4-12. Consider a harbor with a single dock for unloading ships. The ships arrive according to a Poisson process at a mean rate of λ ships per week, and the service-time distribution is exponential with a mean rate of μ unloadings per week. Assume that harbor facilities are owned by the shipping company, so that the objective is to balance the cost associated with idle ships with the cost of running the dock. The shipping company has no control over the arrival rate λ (that is, λ is fixed); however, by changing the size of the unloading crew, and so on, the shipping company can adjust the value of μ as desired.

Suppose that the expected cost per unit time of running the unloading dock is $D\mu$. The waiting cost for each idle ship is some constant (C) times the *square* of the total waiting time (including loading time). The shipping company wishes to adjust μ so that the expected total cost (including the waiting cost for idle ships) per unit time is minimized. Derive this optimal value of μ in terms of D and C.

26.4-13. Consider a queueing system with two types of customers. Type 1 customers arrive according to a Poisson process with a mean rate of 5 per hour. Type 2 customers also arrive according to a Poisson process with a mean rate of 5 per hour. The system has two servers, and both serve both types of customers. For types 1 and 2, service times have an exponential distribution with a mean of 10 minutes. Service is provided on a first-come-first-served basis.

Management now wants you to compare this system's design of having both servers serve both types of customers with the alternative design of having one server serve just type 1 customers and the other server serve just type 2 customers. Assume that this alternative design would not change the probability distribution of service times.

(a) Without doing any calculations, indicate which design would give a smaller expected total number of customers in the system. What result are you using to draw this conclusion?

T **(b)** Verify your conclusion in part (*a*) by finding the expected total number of customers in the system under the original design and then under the alternative design.

26.4-14. Reconsider Prob. 17.6-32.

(a) Formulate part (*a*) to fit as closely as possible a special case of one of the decision models presented in Sec. 26.4. (Do not solve.)

(b) Describe Alternatives 2 and 3 in queueing theory terms, including their relationship (if any) to the decision models presented in Sec. 26.4. Briefly indicate why, in comparison with Alternative 1, each of these other alternatives might decrease the total number of operators (thereby increasing their utilization) needed to achieve the required production rate. Also point out any dangers that might prevent this decrease.

26.4-15. Consider the formulation of the County Hospital emergency room problem as a preemptive priority queueing system, as presented in Sec. 17.8. Suppose that the following inputted costs are assigned to making patients wait (*excluding* treatment time): \$10 per hour for stable cases, \$1,000 per hour for serious cases, and \$100,000 per hour for critical cases. The cost associated with having an additional doctor on duty would be \$40 per hour. Referring to Table 17.3, determine on an expected-total-cost basis whether there should be one or two doctors on duty.

26.5-1. Consider a factory whose floor area is a square with 600 feet on each side. Suppose that one service facility of a certain kind is provided in the center of the factory. The employees are distributed uniformly throughout the factory, and they walk to and from the facility at an average speed of 3 miles per hour along a system of orthogonal aisles.

Find the expected travel time $E(T)$ per arrival.

26.5-2. A certain large shop doing light fabrication work uses a single central storage facility (dispatch station) for material in in-process storage. The typical procedure is that each employee personally delivers his finished work (by hand, tote box, or hand cart) and receives new work and materials at the facility. Although this procedure worked well in earlier years when the shop was smaller, it appears that it may now be advisable to divide the shop into two semi-independent parts, with a separate storage facility for each one. You have been assigned the job of comparing the use of two facilities and of one facility from a cost standpoint.

The factory has the shape of a rectangle 150 by 100 yards. Thus, by letting 1 yard be the unit of distance, the (x, y) coordinates of the corners are (0, 0), (150, 0), (150, 100), and (0, 100). With this coordinate system, the existing facility is located at (50, 50), and the location available for the second facility is (100, 50).

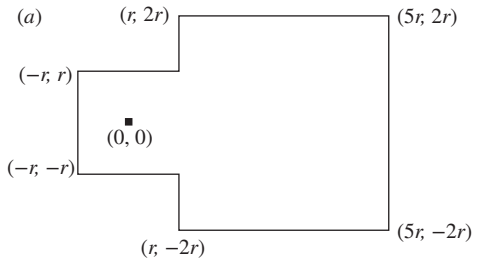
Each facility would be operated by a single clerk. The time required by a clerk to service a caller has an exponential distribution, with a mean of 2 minutes. Employees arrive at the present facility according to a Poisson input process at a mean rate of 24 per hour. The employees are rather uniformly distributed throughout the shop, and if the second facility were installed, each employee would normally use the nearer of the two facilities. Employees walk at an average speed of about 5,000 yards per hour. All aisles are parallel to the outer walls of the shop. The net cost of providing each facility is estimated to be about \$40 per hour, plus \$30 per hour for the clerk. The estimated total cost of an employee being idled by traveling or waiting at the facility is \$50 per hour.

Given the preceding cost factors, which alternative minimizes the expected total cost?

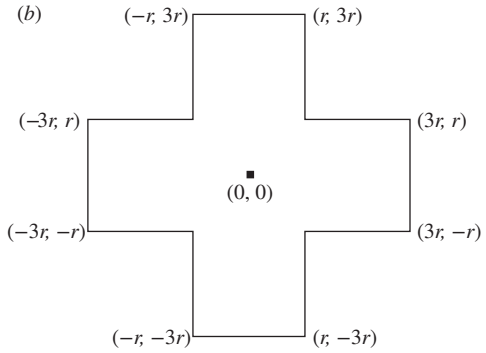
26.5-3. Consider *Alternative 3* (tool cribs in Locations 1 and 3) for the example illustrated in Fig. 26.9. Derive $E(T)$ for the tool crib in Location 3 by using the probability density functions of X and Y directly for this tool crib.

26.5-4. Suppose that the calling population for a particular service facility is uniformly distributed over *each* area shown, where the service facility is located at (0, 0). Making the same assumptions as in Sec. 26.5, derive the expected round-trip travel time per arrival $E(T)$ in terms of the average velocity v and the distance r.

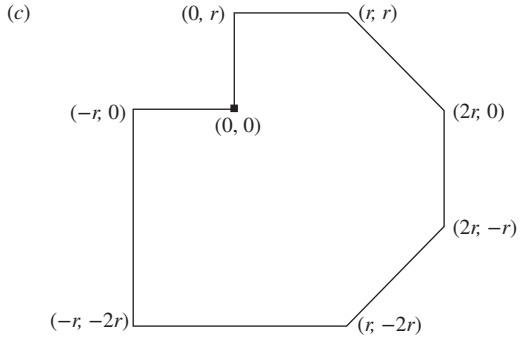
(*a*)



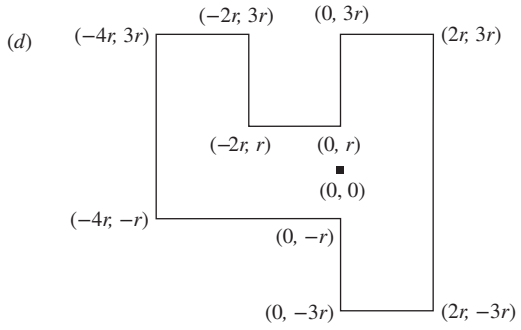
(*b*)



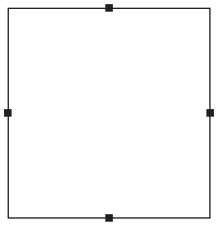
(*c*)



(*d*)



26.5-5. A job shop is being laid out in a square area with 600 feet on a side, and one of the decisions to be made is the *number* of facilities for the storage and shipping of final inventory. The capitalized cost associated with providing each facility would be \$20/hour. There are just four potential locations available for these facilities, one in the middle of each of the four sides of the square area as shown in the figure.



The loads to be moved to a storage and shipping facility would be distributed uniformly throughout the shop area and they become available according to a *Poisson* process at a mean rate of 90 per hour. Each time a load becomes available, an appropriate materials-handling vehicle would be sent from the *nearest* facility to pick it up (with an expected loading time of 3 minutes) and bring it there, where the cost would be \$80/hour for time spent in traveling, loading, and waiting to be unloaded. The vehicles would travel at a speed of 20,000 feet per hour along a system of orthogonal aisles parallel to the sides of the shop area.

Another decision to be made is the number of employees (m) to provide at each storage and shipping facility for unloading arriving vehicles. These m employees would work together on each vehicle, and the time required to unload it would have an *exponential* distribution, with a mean of $2/m$ minutes. The cost of providing each employee is \$30/hour.

Determine the number of facilities and the value of m at each that will minimize expected total cost per hour.