SUPPLEMENT TO CHAPTER 6

An Economic Interpretation of the Dual Problem and the Simplex Method

Section 6.1 describes the essence of duality theory. The end of this section then mentions several important applications of duality theory. We now turn to still another application, namely, its use in providing an economic interpretation of the dual problem and the resulting insights for analyzing the primal problem. You already have seen one example when we discussed shadow prices in Sec. 4.9. We now will describe how this interpretation extends to the entire dual problem and then to the simplex method.

The economic interpretation of duality is based directly upon the typical interpretation for the primal problem (linear programming problem in our standard form) presented in Sec. 3.2. To refresh your memory, we have summarized this interpretation of the primal problem in Table 1.

Interpretation of the Dual Problem

To see how this interpretation of the primal problem leads to an economic interpretation for the dual problem,[1] note in Table 6.4 that W is the value of Z (total profit) at the current iteration. Because

$$W = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m,$$

each $b_i y_i$ can thereby be interpreted as the current *contribution to profit* by having b_i units of resource i available for the primal problem. Thus,

> The dual variable y_i is interpreted as the contribution to profit per unit of resource i ($i = 1, 2, \ldots, m$), when the current set of basic variables is used to obtain the primal solution.

TABLE 1 Economic interpretation of the primal problem

Quantity	Interpretation
x_j	Level of activity j ($j = 1, 2, \ldots, n$)
c_j	Unit profit from activity j
Z	Total profit from all activities
b_i	Amount of resource i available ($i = 1, 2, \ldots, m$)
a_{ij}	Amount of resource i consumed by each unit of activity j

[1]Actually, several slightly different interpretations have been proposed. The one presented here seems to us to be the most useful because it also directly interprets what the simplex method does in the primal problem.

In other words, the y_i values (or y_i^* values in the optimal solution) are just the **shadow prices** discussed in Sec. 4.9.

For example, when iteration 2 of the simplex method finds the optimal solution for the Wyndor problem, it also finds the optimal values of the dual variables (as shown in the bottom row of Table 6.5) to be $y_1^* = 0$, $y_2^* = \frac{3}{2}$, and $y_3^* = 1$. These are precisely the shadow prices found in Sec. 4.9 for this problem through graphical analysis. Recall that the resources for the Wyndor problem are the production capacities of the three plants being made available to the two new products under consideration, so that b_i is the number of hours of production time per week being made available in Plant i for these new products, where $i = 1, 2, 3$. As discussed in Sec. 4.9, the shadow prices indicate that individually increasing any b_i by 1 would increase the optimal value of the objective function (total weekly profit in units of thousands of dollars) by y_i^*. Thus, y_i^* can be interpreted as the contribution to profit per unit of resource i when using the optimal solution.

This interpretation of the dual variables leads to our interpretation of the overall dual problem. Specifically, since each unit of activity j in the primal problem consumes a_{ij} units of resource i,

> $\sum_{i=1}^{m} a_{ij} y_i$ is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity j were used ($j = 1, 2, \ldots, n$).

For the Wyndor problem, 1 unit of activity j corresponds to producing 1 batch of product j per week, where $j = 1, 2$. The mix of resources consumed by producing 1 batch of product 1 is 1 hour of production time in Plant 1 and 3 hours in Plant 3. The corresponding mix per batch of product 2 is 2 hours each in Plants 2 and 3. Thus, $y_1 + 3y_3$ and $2y_2 + 2y_3$ are interpreted as the current contributions to profit (in thousands of dollars per week) of these respective mixes of resources per batch produced per week of the respective products.

For each activity j, this same mix of resources (and more) probably can be used in other ways as well, but no alternative use should be considered if it is less profitable than 1 unit of activity j. Since c_j is interpreted as the unit profit from activity j, each functional constraint in the dual problem is interpreted as follows:

> $\sum_{i=1}^{m} a_{ij} y_i \geq c_j$ says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity j; otherwise, we would not be making the best possible use of these resources.

For the Wyndor problem, the unit profits (in thousands of dollars per week) are $c_1 = 3$ and $c_2 = 5$, so the dual functional constraints with this interpretation are $y_1 + 3y_3 \geq 3$ and $2y_2 + 2y_3 \geq 5$. Similarly, the interpretation of the nonnegativity constraints is the following:

> $y_i \geq 0$ says that the contribution to profit of resource i ($i = 1, 2, \ldots, m$) must be nonnegative: otherwise, it would be better not to use this resource at all.

The objective

$$\text{Minimize} \qquad W = \sum_{i=1}^{m} b_i y_i$$

can be viewed as minimizing the total implicit value of the resources consumed by the activities. For the Wyndor problem, the total implicit value (in thousands of dollars per week) of the resources consumed by the two products is $W = 4y_1 + 12y_2 + 18y_3$.

This interpretation can be sharpened somewhat by differentiating between basic and nonbasic variables in the primal problem for any given BF solution $(x_1, x_2, \ldots, x_{n+m})$. Recall that the *basic* variables (the only variables whose values can be nonzero) *always* have a coefficient of *zero* in row 0. Therefore, referring again to Table 6.4 and the accompanying equation for z_j, we see that

$$\sum_{i=1}^{m} a_{ij} y_i = c_j, \qquad \text{if } x_j > 0 \qquad (j = 1, 2, \ldots, n),$$

$$y_i = 0, \qquad \text{if } x_{n+i} > 0 \qquad (i = 1, 2, \ldots, m).$$

(This is one version of the complementary slackness property discussed in Sec. 6.2.) The economic interpretation of the first statement is that whenever an activity j operates at a strictly positive level ($x_j > 0$), the marginal value of the resources it consumes *must equal* (as opposed to exceeding) the unit profit from this activity. The second statement implies that the marginal value of resource i is *zero* ($y_i = 0$) whenever the supply of this resource is not exhausted by the activities ($x_{n+i} > 0$). In economic terminology, such a resource is a "free good"; the price of goods that are oversupplied must drop to zero by the law of supply and demand. This fact is what justifies interpreting the objective for the dual problem as minimizing the total implicit value of the resources *consumed,* rather than the resources *allocated.*

To illustrate these two statements, consider the optimal BF solution (2, 6, 2, 0, 0) for the Wyndor problem. The basic variables are x_1, x_2, and x_3, so their coefficients in row 0 are zero, as shown in the bottom row of Table 6.5. This bottom row also gives the corresponding dual solution: $y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$, with surplus variables $(z_1^* - c_1) = 0$ and $(z_2^* - c_2) = 0$. Since $x_1 > 0$ and $x_2 > 0$, both these surplus variables and direct calculations indicate that $y_1^* + 3y_3^* = c_1 = 3$ and $2y_2^* + 2y_3^* = c_2 = 5$. Therefore, the implicit value of the resources consumed per batch of the respective products produced does indeed equal the respective unit profits. The slack variable for the constraint on the amount of Plant 1 capacity used is $x_3 > 0$, so the marginal value of adding any Plant 1 capacity would be zero ($y_1^* = 0$).

Interpretation of the Simplex Method

The interpretation of the dual problem also provides an economic interpretation of what the simplex method does in the primal problem. The *goal* of the simplex method is to find how to use the available resources in the most profitable feasible way. To attain this goal, we must reach a BF solution that satisfies all the *requirements* on profitable use of the resources (the constraints of the dual problem). These requirements comprise the *condition for optimality* for the algorithm. For any given BF solution, the requirements (dual constraints) associated with the basic variables are automatically satisfied (with equality). However, those associated with nonbasic variables may or may not be satisfied.

In particular, if an original variable x_j is nonbasic so that activity j is not used, then the current contribution to profit of the resources that would be required to undertake each unit of activity j

$$\sum_{i=1}^{m} a_{ij} y_i$$

may be smaller than, larger than, or equal to the unit profit c_j obtainable from the activity. If it is smaller, so that $z_j - c_j < 0$ in row 0 of the simplex tableau, then these resources can be used more profitably by initiating this activity. If it is larger ($z_j - c_j > 0$), then these resources already are being assigned elsewhere in a more profitable way, so they should not be diverted to activity j. If $z_j - c_j = 0$, there would be no change in profitability by initiating activity j.

Similarly, if a slack variable x_{n+i} is nonbasic so that the total allocation b_i of resource i is being used, then y_i is the current contribution to profit of this resource on a marginal basis. Hence, if $y_i < 0$, profit can be increased by cutting back on the use of this resource (i.e., increasing x_{n+i}). If $y_i > 0$, it is worthwhile to continue fully using this resource, whereas this decision does not affect profitability if $y_i = 0$.

Therefore, what the simplex method does is to examine all the nonbasic variables in the current BF solution to see which ones can provide a *more profitable use of the resources* by being increased. If *none* can, so that no feasible shifts or reductions in the current proposed use of the resources can increase profit, then the current solution must be optimal. If one or more can, the simplex method selects the variable that, if increased by 1, would *improve the profitability* of the use of the resources the most. It then actually increases this variable (the entering basic variable) as much as it can until the marginal values of the resources change. This increase results in a new BF solution with a new row 0 (dual solution), and the whole process is repeated.

The economic interpretation of the dual problem considerably expands our ability to analyze the primal problem. However, Sec. 6.1 describes how this interpretation is just one ramification of the relationships between the two problems. Section 6.2 delves into these relationships more deeply.

PROBLEM

1. Consider the simplex tableaux for the Wyndor Glass Co. problem given in Table 4.8. For each tableau, give the economic interpretation of the following items:

(a) Each of the coefficients of the slack variables (x_3, x_4, x_5) in row 0

(b) Each of the coefficients of the decision variables (x_1, x_2) in row 0

(c) The resulting choice for the entering basic variable (or the decision to stop after the final tableau)