SUPPLEMENT 2 TO CHAPTER 9

The Construction of Initial BF Solutions for Transportation Problems

Section 9.2 presents the transportation simplex method for the transportation problem. The initialization step for this algorithm involves finding an initial BF solution. Section 9.2 briefly outlines and illustrates one method called the *northwest corner rule* for doing this. However, there actually are several available methods for doing this that are based on the general procedure outlined below. We will present two of these other methods—Vogel's approximation method and Russell's approximation method—which are designed to seek a very good BF solution (an "approximation" of an optimal solution), which should tend to reduce the number of iterations of the transportation simplex method that will be needed to reach an optimal solution. (We also will mention one other intuitive method in Prob. 9S2-4.) For completeness, we will begin with the northwest corner rule again before later comparing all three methods.

The general procedure for constructing an initial BF solution outllined below selects the $m + n - 1$ basic variables one at a time. After each selection, a value that will satisfy one additional constraint (thereby eliminating that constraint's row or column from further consideration for providing allocations) is assigned to that variable. Thus, after $m + n - 1$ selections, an entire basic solution has been constructed in such a way as to satisfy all the constraints. A number of different criteria, including the three mentioned above, have been proposed for selecting the basic variables. We will present and illustrate these three criteria after outlining the general procedure.

General Procedure[1] for Constructing an Initial BF Solution. To begin, all source rows and destination columns of the transportation simplex tableau are initially under consideration for providing a basic variable (allocation).

1. From the rows and columns still under consideration, select the next basic variable (allocation) according to some criterion.
2. Make that allocation large enough to exactly use up the remaining supply in its row or the remaining demand in its column (whichever is smaller).

[1]In Sec. 4.1 we pointed out that the simplex method is an example of the algorithms (systematic solution procedures) so prevalent in OR work. Note that this procedure also is an algorithm, where each successive execution of the (four) steps constitutes an iteration.

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3. Eliminate that row or column (whichever had the smaller remaining supply or demand) from further consideration. (If the row and column have the same remaining supply and demand, then arbitrarily select the *row* as the one to be eliminated. The column will be used later to provide a *degenerate* basic variable, i.e., a circled allocation of zero.)
4. If only one row or only one column remains under consideration, then the procedure is completed by selecting every *remaining* variable (i.e., those variables that were neither previously selected to be basic nor eliminated from consideration by eliminating their row or column) associated with that row or column to be basic with the only feasible allocation. Otherwise, return to step 1.

Alternative Criteria for Step 1

1. *Northwest corner rule:* Begin by selecting x_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ (that is, move one column to the *right*) if source i has any supply remaining. Otherwise, next select $x_{i+1,j}$ (that is, move one row *down*).

Example. To make this description more concrete, we now illustrate the general procedure on the Metro Water District problem (see Table 9.12) with the northwest corner rule being used in step 1. Because $m = 4$ and $n = 5$ in this case, the procedure would find an initial BF solution having $m + n - 1 = 8$ basic variables.

As shown in Table 1, the first allocation is $x_{11} = 30$, which exactly uses up the demand in column 1 (and eliminates this column from further consideration). This first iteration leaves a supply of 20 remaining in row 1, so next select $x_{1,1+1} = x_{12}$ to be a basic variable. Because this supply is no larger than the demand of 20 in column 2, all of it is allocated, $x_{12} = 20$, and this row is eliminated from further consideration. (Row 1 is chosen for elimination rather than column 2 because of the parenthetical instruction in step 3 of the general procedure.) Therefore, select $x_{1+1,2} = x_{22}$ next. Because the remaining demand of 0 in column 2 is less than the supply of 60 in row 2, allocate $x_{22} = 0$ and eliminate column 2.

Continuing in this manner, we eventually obtain the entire *initial BF solution* shown in Table 1, where the circled numbers are the values of the basic variables ($x_{11} = 30, \ldots, x_{45} = 50$) and all the other variables (x_{13}, etc.) are nonbasic variables equal to zero. Arrows have been added to show the order in which the basic variables (allocations) were selected. The value of Z for this solution is

$$Z = 16(30) + 16(20) + \cdots + 0(50) = 2{,}470 + 10M.$$

2. *Vogel's approximation method:* For each row and column remaining under consideration, calculate its **difference,** which is defined as *the arithmetic difference between the smallest and next-to-the-smallest unit cost c_{ij} still remaining in that row or column.* (If two unit costs tie for being the smallest remaining in a row or column, then the *difference* is 0.) In that row or column having the *largest difference,* select the variable having the *smallest remaining unit cost.* (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

Example. Now let us apply the general procedure to the Metro Water District problem by using the criterion for Vogel's approximation method to select the next basic variable in step 1. With this criterion, it is more convenient to work with parameter tables (rather than with complete transportation simplex tableaux), beginning with the one shown in Table 9.12. At each iteration, after the difference for every row and column remaining under consideration is calculated and displayed, the largest difference is circled and the

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TABLE 1 Initial BF solution from the Northwest Corner Rule

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 (30) →	16 (20) ↓	13	22	17	50	
	2	14	14 (0) →	13 (60) ↓	19	15	60	
	3	19	19	20 (10) →	23 (30) →	*M* (10) ↓	50	
	4(*D*)	*M*	0	*M*	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

smallest unit cost in its row or column is enclosed in a box. The resulting selection (and value) of the variable having this unit cost as the next basic variable is indicated in the lower right-hand corner of the current table, along with the row or column thereby being eliminated from further consideration (see steps 2 and 3 of the general procedure). The table for the next iteration is exactly the same except for deleting this row or column and subtracting the last allocation from its supply or demand (whichever remains).

Applying this procedure to the Metro Water District problem yields the sequence of parameter tables shown in Table 2, where the resulting initial BF solution consists of the eight basic variables (allocations) given in the lower right-hand corner of the respective parameter tables.

This example illustrates two relatively subtle features of the general procedure that warrant special attention. First, note that the final iteration selects *three* variables (x_{31}, x_{32}, and x_{33}) to become basic instead of the single selection made at the other iterations. The reason is that only *one* row (row 3) remains under consideration at this point. Therefore, step 4 of the general procedure says to select *every* remaining variable associated with row 3 to be basic.

Second, note that the allocation of $x_{23} = 20$ at the next-to-last iteration exhausts *both* the remaining supply in its row *and* the remaining demand in its column. However, rather than eliminate both the row and column from further consideration, step 3 says to eliminate *only the row,* saving the column to provide a *degenerate* basic variable later. Column 3 is, in fact, used for just this purpose at the final iteration when $x_{33} = 0$ is selected as one of the basic variables. For another illustration of this same phenomenon, see Table 1 where the allocation of $x_{12} = 20$ results in eliminating only row 1, so that column 2 is saved to provide a degenerate basic variable, $x_{22} = 0$, at the next iteration.

Although a zero allocation might seem irrelevant, it actually plays an important role. You will see soon that the transportation simplex method must know *all* $m + n - 1$ basic variables, including those with value zero, in the current BF solution.

3. *Russell's approximation method:* For each source row i remaining under consideration, determine its $\bar{u}_i$, which is the largest unit cost c_{ij} still remaining in that row. For each destination column j remaining under consideration, determine its $\bar{v}_j$, which is the largest unit cost c_{ij} still remaining in that column. For each variable x_{ij} not previously selected in these rows and columns, calculate $\Delta_{ij} = c_{ij} - \bar{u}_i - \bar{v}_j$. Select the variable having the *largest* (in absolute terms) *negative* value of Δ_{ij}. (Ties may be broken arbitrarily.)

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TABLE 2 Initial BF solution from Vogel's approximation method

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	*M*	50	0
	4(*D*)	*M*	0	*M*	[0]	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	(19)	15	Eliminate column 4	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
	4(*D*)	*M*	0	*M*	[0]	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	(15)	Eliminate row 4(*D*)	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	[13]	17	50	(3)
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	2	14	14	13	[15]	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	($M - 15$)	Eliminate column 5	

		Destination				
		1	**2**	**3**	**Supply**	**Row Difference**
Source	2	14	14	[13]	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	(7)	Eliminate row 2	

		Destination				
		1	**2**	**3**	**Supply**	
Source	3	19	19	20	50	
Demand		30	20	0	Select $x_{31} = 30$ $x_{32} = 20$ $x_{33} = 0$	$Z = 2,460$

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TABLE 3 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

Example. Using the criterion for Russell's approximation method in step 1, we again apply the general procedure to the Metro Water District problem (see Table 9.12). The results, including the sequence of basic variables (allocations), are shown in Table 3.

At iteration 1, the largest unit cost in row 1 is $\bar{u}_1 = 22$, the largest in column 1 is $\bar{v}_1 = M$, and so forth. Thus,

$$\Delta_{11} = c_{11} - \bar{u}_1 - \bar{v}_1 = 16 - 22 - M = -6 - M.$$

Calculating all the Δ_{ij} values for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$ shows that $\Delta_{45} = 0 - 2M$ has the largest negative value, so $x_{45} = 50$ is selected as the first basic variable (allocation). This allocation exactly uses up the supply in row 4, so this row is eliminated from further consideration.

Note that eliminating this row changes $\bar{v}_1$ and $\bar{v}_3$ for the next iteration. Therefore, the second iteration requires recalculating the Δ_{ij} with $j = 1, 3$ as well as eliminating $i = 4$. The largest negative value now is

$$\Delta_{15} = 17 - 22 - M = -5 - M,$$

so $x_{15} = 10$ becomes the second basic variable (allocation), eliminating column 5 from further consideration.

The subsequent iterations proceed similarly, but you may want to test your understanding by verifying the remaining allocations given in Table 3. As with the other procedures in this book, you should find your IOR Tutorial useful for doing the calculations involved and illuminating the approach. (See the interactive procedure for finding an initial BF solution.)

Comparison of Alternative Criteria for Step 1. Now let us compare these three criteria for selecting the next basic variable. The main virtue of the northwest corner rule is that it is quick and easy. However, because it pays no attention to unit costs c_{ij}, the solution obtained often will be far from optimal. (Note in Table 1 that $x_{35} = 10$ even though $c_{35} = M$, where M symbolically represents a *huge* positive number that was meant to prevent any allocation in this spot.) Expending a little more effort to find a good initial BF solution might greatly reduce the number of iterations then required by the transportation simplex method to reach an optimal solution (see Probs. 9S2-5 and 9S2-6). Finding such a solution is the objective of the other two criteria.

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Vogel's approximation method has been a popular criterion for several decades,[2] partially because it is relatively easy to implement by hand. Since the *difference* represents the minimum extra unit cost incurred by failing to make an allocation to the cell having the smallest unit cost in that row or column, this criterion does take costs into account in an effective way.

Russell's approximation method provides another excellent criterion[3] that is still quick to implement on a computer (but not manually). Although it is unclear as to which is more effective *on average,* this criterion *frequently* does obtain a better solution than Vogel's. (For the example, Vogel's approximation method happened to find the optimal solution with $Z = 2{,}460$, whereas Russell's misses slightly with $Z = 2{,}570$.) For a large problem, it may be worthwhile to apply both criteria and then use the better solution to start the iterations of the transportation simplex method.

One distinct advantage of Russell's approximation method is that it is patterned directly after step 1 for the transportation simplex method, which somewhat simplifies the overall computer code. In particular, the $\bar{u}_i$ and $\bar{v}_j$ values have been defined in such a way that the relative values of the $c_{ij} - \bar{u}_i - \bar{v}_j$ *estimate* the relative values of $c_{ij} - u_i - v_j$ that will be obtained when the transportation simplex method reaches an optimal solution.

PROBLEMS

9S2-1. Consider the transportation problem having the following parameter table:

		Destination			
		1	**2**	**3**	**Supply**
	1	6	3	5	4
Source	2	4	*M*	7	3
	3	3	4	3	2
Demand		4	2	3	

(a) Use Vogel's approximation method manually (don't use the interactive procedure in IOR Tutorial) to select the first basic variable for an initial BF solution.
(b) Use Russell's approximation method manually to select the first basic variable for an initial BF solution.
(c) Use the northwest corner rule manually to construct a complete initial BF solution.

D,I **9S2-2.** Consider the transportation problem having the following parameter table:

		Destination					
		1	**2**	**3**	**4**	**5**	**Supply**
	1	2	4	6	5	7	4
	2	7	6	3	*M*	4	6
Source	3	8	7	5	2	5	6
	4	0	0	0	0	0	4
Demand		4	4	2	5	5	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.
(a) Northwest corner rule.
(b) Vogel's approximation method.
(c) Russell's approximation method.

[2]N. V. Reinfeld and W. R. Vogel: *Mathematical Programming,* Prentice-Hall, Englewood Cliffs, NJ, 1958.

[3]E. J. Russell: "Extension of Dantzig's Algorithm to Finding an Initial Near-Optimal Basis for the Transportation Problem," *Operations Research,* **17:** 187–191, 1969.

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D,I **9S2-3.** Consider the transportation problem having the following parameter table:

		Destination						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
	1	13	10	22	29	18	0	5
	2	14	13	16	21	*M*	0	6
Source	3	3	0	*M*	11	6	0	7
	4	18	9	19	23	11	0	4
	5	30	24	34	36	28	0	3
Demand		3	5	4	5	6	2	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.
(a) Northwest corner rule.
(b) Vogel's approximation method.
(c) Russell's approximation method.

I **9S2-4.** Consider the transportation problem having the following parameter table (as first shown in Prob. 9.2-1).

		Destination				
		1	**2**	**3**	**4**	**Supply**
	1	7	4	1	4	1
Source	2	4	6	7	2	1
	3	8	5	4	6	1
	4	6	7	6	3	1
Demand		1	1	1	1	

Construct an initial BF solution by applying the general procedure for the initialization step of the transportation simplex method. However, rather than using one of the three criteria for step 1 presented in this supplement, use the minimum cost criterion given next for selecting the next basic variable. (With the corresponding interactive routine in your OR Courseware, choose the *Northwest Corner Rule,* since this choice actually allows the use of any criterion.)

> **Minimum cost criterion:** From among the rows and columns still under consideration, select the variable x_{ij} having the smallest unit cost c_{ij} to be the next basic variable. (Ties may be broken arbitrarily.)

D,I **9S2-5.** Consider the transportation problem having the following parameter table:

		Destination				
		1	**2**	**3**	**4**	**Supply**
	1	3	7	6	4	5
Source	2	2	4	3	2	2
	3	4	3	8	5	3
Demand		3	3	2	2	

Use each of the following criteria to obtain an initial BF solution. In each case, interactively apply the transportation simplex method, starting with this initial solution, to obtain an optimal solution. Compare the resulting number of iterations for the transportation simplex method.
(a) Northwest corner rule.
(b) Vogel's approximation method.
(c) Russell's approximation method.

9S2-6. The Energetic Company needs to make plans for the energy systems for a new building.

The energy needs in the building fall into three categories: (1) electricity, (2) heating water, and (3) heating space in the building. The daily requirements for these three categories (all measured in the same units) are

Electricity	20 units
Water heating	10 units
Space heating	30 units

The three possible sources of energy to meet these needs are electricity, natural gas, and a solar heating unit that can be installed on the roof. The size of the roof limits the largest possible solar heater to 30 units, but there is no limit to the electricity and natural gas available. Electricity needs can be met only by purchasing electricity (at a cost of $50 per unit). Both other energy needs can be met by any source or combination of sources. The unit costs are shown in the following table.

	Electricity	**Natural Gas**	**Solar Heater**
Water heating	$90	$60	$30
Space heating	80	50	40

The objective is to minimize the total cost of meeting the energy needs.
(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
D,I **(b)** Use the northwest corner rule to obtain an initial BF solution for this problem.

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D,I **(c)** Starting with the initial BF solution from part (*b*), interactively apply the transportation simplex method to obtain an optimal solution.

D,I **(d)** Use Vogel's approximation method to obtain an initial BF solution for this problem.

D,I **(e)** Starting with the initial BF solution from part (*d*), interactively apply the transportation simplex method to obtain an optimal solution.

I **(f)** Use Russell's approximation method to obtain an initial BF solution for this problem.

D,I **(g)** Starting with the initial BF solution obtained from part (*f*), interactively apply the transportation simplex method to obtain an optimal solution. Compare the number of iterations required by the transportation simplex method here and in parts (*c*) and (*e*).

9S2-7. Reconsider the transportation problem formulated in Prob. 9.1-6*a*.

D,I **(a)** Use each of the three criteria presented in this supplement to obtain an initial BF solution, and time how long you spend for each one. Compare both these times and the values of the objective function for these solutions.

C **(b)** Obtain an optimal solution for this problem. For each of the three initial BF solutions obtained in part (*a*), calculate the percentage by which its objective function value exceeds the optimal one.

D,I **(c)** For each of the three initial BF solutions obtained in part (*a*), interactively apply the transportation simplex method to obtain (and verify) an optimal solution. Time how long you spend in each of the three cases. Compare both these times and the number of iterations needed to reach an optimal solution.

9S2-8. Follow the instructions of Prob. 9S2-7 for the transportation problem formulated in Prob. 9.1-7*a*.

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