SUPPLEMENT TO CHAPTER 12

Some Innovative Uses of Binary Variables in Model Formulation

Chapter 12 has presented a number of examples where the *basic decisions* of the problem are of the *yes-or-no type,* so that *binary variables* are introduced to represent these decisions. We now will look at some other ways in which binary variables can be very useful. In particular, we will see that these variables sometimes enable us to take a problem whose natural formulation is intractable and *reformulate* it as a pure or mixed IP problem.

This kind of situation arises when the original formulation of the problem fits either an IP or a linear programming format *except* for minor disparities involving combinatorial relationships in the model. By expressing these combinatorial relationships in terms of questions that must be answered yes or no, **auxiliary binary variables** can be introduced to the model to represent these yes-or-no decisions. (Rather than being a decision variable for the original problem under consideration, an *auxiliary* binary variable is a binary variable that is introduced into the model of the problem simply to help formulate the model as a pure or mixed BIP model.) Introducing these variables reduces the problem to an MIP problem (or a *pure* IP problem if all the original variables also are required to have integer values).

Some cases that can be handled by this approach are discussed next, where the x_j denote the *original* variables of the problem (they may be either continuous or integer variables) and the y_i denote the *auxiliary* binary variables that are introduced for the reformulation.

Either-Or Constraints

Consider the important case where a choice can be made between two constraints, so that *only one* (either one) must hold (whereas the other one can hold but is not required to do so). For example, there may be a choice as to which of two resources to use for a certain purpose, so that it is necessary for only one of the two resource availability constraints to hold mathematically. To illustrate the approach to such situations, suppose that one of the requirements in the overall problem is that

$$\begin{aligned} &\text{Either} \quad 3x_1 + 2x_2 \le 18 \\ &\text{or} \quad\quad\;\; x_1 + 4x_2 \le 16, \end{aligned}$$

i.e., at least one of these two inequalities must hold but not necessarily both. This requirement must be reformulated to fit it into the linear programming format where *all* specified constraints must hold. Let M symbolize a huge positive number. Then this requirement can be rewritten as

$$\text{Either} \quad \begin{aligned} 3x_1 + 2x_2 &\le 18 \\ x_1 + 4x_2 &\le 16 + M \end{aligned}$$

$$\text{or} \quad \begin{aligned} 3x_1 + 2x_2 &\le 18 + M \\ x_1 + 4x_2 &\le 16. \end{aligned}$$

The key is that adding M to the right-hand side of such constraints has the effect of eliminating them, because they would be satisfied automatically by any solutions that satisfy the other constraints of the problem. (This formulation assumes that the set of feasible solutions for the overall problem is a bounded set and that M is large enough that it will not eliminate any feasible solutions.) This formulation is equivalent to the set of constraints

$$\begin{aligned} 3x_1 + 2x_2 &\le 18 + My \\ x_1 + 4x_2 &\le 16 + M(1 - y). \end{aligned}$$

Because the *auxiliary variable* y must be either 0 or 1, this formulation guarantees that one of the original constraints must hold while the other is, in effect, eliminated. This new set of constraints would then be appended to the other constraints in the overall model to give a pure or mixed IP problem (depending upon whether the x_j are integer or continuous variables).

This approach is related directly to the discussion at the beginning of this supplement about expressing combinatorial relationships in terms of questions that must be answered yes or no. The combinatorial relationship involved in the current example concerns the combination of the *other* constraints of the model with the *first* of the two *alternative* constraints and then with the *second.* Which of these two combinations of constraints is *better* (in terms of the value of the objective function that then can be achieved)? To rephrase this question in yes-or-no terms, we ask two complementary questions:

1. Should $x_1 + 4x_2 \le 16$ be selected as the constraint that must hold?
2. Should $3x_1 + 2x_2 \le 18$ be selected as the constraint that must hold?

Because exactly one of these questions is to be answered affirmatively, we let the binary terms y and $1 - y$, respectively, represent these yes-or-no decisions. Thus, $y = 1$ if the answer is yes to the first question (and no to the second), whereas $1 - y = 1$ (that is, $y = 0$) if the answer is yes to the second question (and no to the first). Since $y + 1 - y = 1$ (one yes) automatically, there is no need to add another constraint to force these two decisions to be mutually exclusive. (If separate binary variables y_1 and y_2 had been used instead to represent these yes-or-no decisions, then an additional constraint $y_1 + y_2 = 1$ would have been needed to make them mutually exclusive.)

A formal presentation of this approach is given next for a more general case.

K out of *N* Constraints Must Hold

Consider the case where the overall model includes a set of N possible constraints such that only some K of these constraints *must* hold. (Assume that $K < N$.) Part of the optimization process is to choose the *combination* of K constraints that permits the objective function to reach its best possible value. The $N - K$ constraints *not* chosen are, in effect, eliminated from the problem, although feasible solutions might coincidentally still satisfy some of them.

This case is a direct generalization of the preceding case, which had $K = 1$ and $N = 2$. Denote the N possible constraints by

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &\le d_1 \\ f_2(x_1, x_2, \ldots, x_n) &\le d_2 \\ &\vdots \\ f_N(x_1, x_2, \ldots, x_n) &\le d_N. \end{aligned}$$

Then, applying the same logic as for the preceding case, we find that an equivalent formulation of the requirement that some K of these constraints *must* hold is

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &\le d_1 + My_1 \\ f_2(x_1, x_2, \ldots, x_n) &\le d_2 + My_2 \\ &\vdots \\ f_N(x_1, x_2, \ldots, x_n) &\le d_N + My_N \\ \sum_{i=1}^{N} y_i &= N - K, \end{aligned}$$

and

$$y_i \text{ is binary,} \qquad \text{for } i = 1, 2, \ldots, N,$$

where M symbolizes a huge positive number. For each binary variable y_i ($i = 1, 2, \ldots, N$), note that $y_i = 0$ makes $My_i = 0$, which reduces the new constraint i to the original constraint i. On the other hand, $y_i = 1$ makes $(d_i + My_i)$ so large that (again assuming a bounded feasible region) the new constraint i is automatically satisfied by any solution that satisfies the other new constraints, which has the effect of eliminating the original constraint i. Therefore, because the constraints on the y_i guarantee that K of these variables will equal 0 and those remaining will equal 1, K of the original constraints will be unchanged and the other $(N - K)$ original constraints will, in effect, be eliminated. The choice of *which* K constraints should be retained is made by applying the appropriate algorithm to the overall problem so it finds an optimal solution for *all* the variables simultaneously.

Functions with *N* Possible Values

Consider the situation where a given function is required to take on any one of N given values. Denote this requirement by

$$f(x_1, x_2, \ldots, x_n) = d_1 \qquad \text{or} \qquad d_2, \ldots, \qquad \text{or} \qquad d_N.$$

One special case is where this function is

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_j x_j,$$

as on the left-hand side of a linear programming constraint. Another special case is where $f(x_1, x_2, \ldots, x_n) = x_j$ for a given value of j, so the requirement becomes that x_j must take on any one of N given values.

The equivalent IP formulation of this requirement is the following:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{N} d_i y_i \\ \sum_{i=1}^{N} y_i &= 1 \end{aligned}$$

and

y_i is binary, for $i = 1, 2, \ldots, N$.

so this new set of constraints would replace this requirement in the statement of the overall problem. This set of constraints provides an *equivalent* formulation because exactly one y_i must equal 1 and the others must equal 0, so exactly one d_i is being chosen as the value of the function. In this case, there are N yes-or-no questions being asked, namely, should d_i be the value chosen ($i = 1, 2, \ldots, N$)? Because the y_i respectively represent these *yes-or-no decisions,* the second constraint makes them *mutually exclusive alternatives.*

To illustrate how this case can arise, reconsider the Wyndor Glass Co. problem presented in Sec. 3.1. Eighteen hours of production time per week in Plant 3 currently is unused and available for the two new products *or* for certain future products that will be ready for production soon. In order to leave any remaining capacity in usable blocks for these future products, management now wants to impose the restriction that the production time used by the two current new products be 6 *or* 12 *or* 18 hours per week. Thus, the third constraint of the original model ($3x_1 + 2x_2 \leq 18$) now becomes

$$3x_1 + 2x_2 = 6 \quad \text{or} \quad 12 \quad \text{or} \quad 18.$$

In the preceding notation, $N = 3$ with $d_1 = 6$, $d_2 = 12$, and $d_3 = 18$. Consequently, management's new requirement should be formulated as follows:

$$3x_1 + 2x_2 = 6y_1 + 12y_2 + 18y_3$$
$$y_1 + y_2 + y_3 = 1$$

and

y_1, y_2, y_3 are binary.

The overall model for this new version of the problem then consists of the original model (see Sec. 3.1) plus this new set of constraints that replaces the original third constraint. This replacement yields a very tractable MIP formulation.

In general terms, for *all* the formulation possibilities with auxiliary binary variables discussed so far, we need to strike the same note of caution. This approach sometimes requires adding a relatively large number of such variables, which can make the model *computationally infeasible.* (Section 12.5 provides some perspective on the sizes of IP problems that can be solved.)

We now present two examples that illustrate a variety of formulation techniques with binary variables. For the sake of clarity, these examples have been kept very small. (**A somewhat larger formulation example**, with dozens of binary variables and constraints, is included in the Solved Examples section of the book's website for Chapter 12.) In actual applications, these formulations typically would be just a small part of a vastly larger model.

EXAMPLE 1 Making Choices When the Decision Variables Are Continuous

The Research and Development Division of the GOOD PRODUCTS COMPANY has developed three possible new products. However, to avoid undue diversification of the company's product line, management has imposed the following restriction:

Restriction 1: From the three possible new products, *at most two* should be chosen to be produced.

Each of these products can be produced in either of two plants. For administrative reasons, management has imposed a second restriction in this regard.

Restriction 2: Just one of the two plants should be chosen to be the sole producer of the new products.

The production cost per unit of each product would be essentially the same in the two plants. However, because of differences in their production facilities, the number of hours of production time needed per unit of each product might differ between the two plants. These data are given in Table 1, along with other relevant information, including marketing estimates of the number of units of each product that could be sold per week if it is produced. The objective is to choose the products, the plant, and the production rates of the chosen products so as to maximize total profit.

In some ways, this problem resembles a standard *product mix problem* such as the Wyndor Glass Co. example described in Sec. 3.1. In fact, if we changed the problem by dropping the two restrictions *and* by requiring each unit of a product to use the production hours given in Table 1 in *both plants* (so the two plants now perform different operations needed by the products), it would become just such a problem. In particular, if we let x_1, x_2, x_3 be the production rates of the respective products, the model then becomes

$$\text{Maximize} \qquad Z = 5x_1 + 7x_2 + 3x_3,$$

subject to

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 &\leq 30 \\ 4x_1 + 6x_2 + 2x_3 &\leq 40 \\ x_1 \qquad\qquad\quad &\leq \ 7 \\ x_2 \qquad &\leq \ 5 \\ x_3 &\leq \ 9 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

For the real problem, however, restriction 1 necessitates adding to the model the constraint

The number of strictly positive decision variables (x_1, x_2, x_3) must be ≤ 2.

TABLE 1 Data for Example 1 (the Good Products Co. problem)

	Production Time Used for Each Unit Produced			Production Time Available per Week
	Product 1	Product 2	Product 3	
Plant 1	3 hours	4 hours	2 hours	30 hours
Plant 2	4 hours	6 hours	2 hours	40 hours
Unit profit	5	7	3	(thousands of dollars)
Sales potential	7	5	9	(units per week)

This constraint does not fit into a linear or an integer programming format, so the key question is how to convert it to such a format so that a corresponding algorithm can be used to solve the overall model. If the decision variables were binary variables, then the constraint would be expressed in this format as $x_1 + x_2 + x_3 \leq 2$. However, with *continuous* decision variables, a more complicated approach involving the introduction of auxiliary binary variables is needed.

Requirement 2 necessitates replacing the first two functional constraints ($3x_1 + 4x_2 + 2x_3 \leq 30$ and $4x_1 + 6x_2 + 2x_3 \leq 40$) by the restriction

$$\begin{array}{ll} \text{Either} & 3x_1 + 4x_2 + 2x_3 \leq 30 \\ \text{or} & 4x_1 + 6x_2 + 2x_3 \leq 40 \end{array}$$

must hold, where the choice of which constraint must hold corresponds to the choice of which plant will be used to produce the new products. We discussed earlier how such an either-or constraint can be converted to a linear or an integer programming format, again with the help of an auxiliary binary variable.

Formulation with Auxiliary Binary Variables. To deal with requirement 1, we introduce three auxiliary binary variables (y_1, y_2, y_3) with the interpretation

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ can hold (can produce product } j) \\ 0 & \text{if } x_j = 0 \text{ must hold (cannot produce product } j), \end{cases}$$

for $j = 1, 2, 3$. To enforce this interpretation in the model with the help of M (a symbol for a huge positive number), we add the constraints

$$\begin{aligned} &x_1 \leq My_1 \\ &x_2 \leq My_2 \\ &x_3 \leq My_3 \\ &y_1 + y_2 + y_3 \leq 2 \\ &y_j \text{ is binary,} \qquad \text{for } j = 1, 2, 3. \end{aligned}$$

The either-or constraint and nonnegativity constraints give a *bounded* feasible region for the decision variables (so each $x_j \leq M$ throughout this region). Therefore, in each $x_j \leq My_j$ constraint, $y_j = 1$ allows any value of x_j in the feasible region, whereas $y_j = 0$ forces $x_j = 0$. (Conversely, $x_j > 0$ forces $y_j = 1$, whereas $x_j = 0$ allows either value of y_j.) Consequently, when the fourth constraint forces choosing at most two of the y_j to equal 1, this amounts to choosing at most two of the new products as the ones that can be produced.

To deal with requirement 2, we introduce another auxiliary binary variable y_4 with the interpretation

$$y_4 = \begin{cases} 1 & \text{if } 4x_1 + 6x_2 + 2x_3 \leq 40 \text{ must hold (choose Plant 2)} \\ 0 & \text{if } 3x_1 + 4x_2 + 2x_3 \leq 30 \text{ must hold (choose Plant 1).} \end{cases}$$

As discussed earlier, this interpretation is enforced by adding the constraints,

$$\begin{aligned} &3x_1 + 4x_2 + 2x_3 \leq 30 + My_4 \\ &4x_1 + 6x_2 + 2x_3 \leq 40 + M(1 - y_4) \\ &y_4 \text{ is binary.} \end{aligned}$$

Consequently, after we move all variables to the left-hand side of the constraints, the complete model is

$$\text{Maximize} \qquad Z = 5x_1 + 7x_2 + 3x_3,$$

subject to

$$
\begin{aligned}
x_1 &\le 7\\
x_2 &\le 5\\
x_3 &\le 9\\
x_1 - My_1 &\le 0\\
x_2 - My_2 &\le 0\\
x_3 - My_3 &\le 0\\
y_1 + y_2 + y_3 &\le 2\\
3x_1 + 4x_2 + 2x_3 - My_4 &\le 30\\
4x_1 + 6x_2 + 2x_3 + My_4 &\le 40 + M
\end{aligned}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0$$
$$y_j \text{ is binary}, \qquad \text{for } j = 1, 2, 3, 4.$$

This now is an MIP model, with three variables (the x_j) not required to be integer and four binary variables, so an MIP algorithm can be used to solve the model. When this is done (after substituting a large numerical value for M),[1] the optimal solution is $y_1 = 1$, $y_2 = 0$, $y_3 = 1$, $y_4 = 1$, $x_1 = 5\frac{1}{2}$, $x_2 = 0$, and $x_3 = 9$; that is, choose products 1 and 3 to produce, choose Plant 2 for the production, and choose the production rates of $5\frac{1}{2}$ units per week for product 1 and 9 units per week for product 3. The resulting total profit is \$54,500 per week.

EXAMPLE 2 Violating Proportionality

The SUPERSUDS CORPORATION is developing its marketing plans for next year's new products. For three of these products, the decision has been made to purchase a total of five TV spots for commercials on national television networks. The problem we will focus on is how to allocate the five spots to these three products, with a maximum of three spots (and a minimum of zero) for each product.

Table 2 shows the estimated impact of allocating zero, one, two, or three spots to each product. This impact is measured in terms of the *profit* (in units of millions of dollars) from the *additional sales* that would result from the spots, considering also the cost of producing the commercial and purchasing the spots. The objective is to allocate five spots to the products so as to maximize the total profit.

TABLE 2 Data for Example 2 (the Supersuds Corp. problem)

Number of TV Spots	Profit		
	Product		
	1	2	3
0	0	0	0
1	1	0	–1
2	3	2	2
3	3	3	4

[1]In practice, some care is taken to choose a value for M that definitely is large enough to avoid eliminating any feasible solutions, but as small as possible otherwise in order to avoid unduly enlarging the feasible region for the LP relaxation (described in Sec. 12.5) and to avoid numerical instability. For this example, a careful examination of the constraints reveals that the minimum feasible value of M is $M = 9$.

This small problem can be solved easily by dynamic programming (Chap. 11) or even by inspection. (The optimal solution is to allocate two spots to product 1, no spots to product 2, and three spots to product 3.) However, we will show two different BIP formulations for illustrative purposes. Such a formulation would become necessary if this small problem needed to be incorporated into a larger IP model involving the allocation of resources to marketing activities for all the corporation's new products.

One Formulation with Auxiliary Binary Variables. A natural formulation would be to let x_1, x_2, x_3 be the number of TV spots allocated to the respective products. The contribution of each x_j to the objective function then would be given by the corresponding column in Table 2. However, each of these columns violates the assumption of proportionality described in Sec. 3.3. Therefore, we cannot write a *linear* objective function in terms of these integer decision variables.

Now see what happens when we introduce an *auxiliary binary variable* y_{ij} for each positive integer value of $x_i = j$ ($j = 1, 2, 3$), where y_{ij} has the interpretation

$$y_{ij} = \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $y_{21} = 0$, $y_{22} = 0$, and $y_{23} = 1$ mean that $x_2 = 3$.) The resulting *linear* BIP model is

$$\text{Maximize} \quad Z = y_{11} + 3y_{12} + 3y_{13} + 2y_{22} + 3y_{23} - y_{31} + 2y_{32} + 4y_{33},$$

subject to

$$\begin{aligned} y_{11} + y_{12} + y_{13} &\leq 1 \\ y_{21} + y_{22} + y_{23} &\leq 1 \\ y_{31} + y_{32} + y_{33} &\leq 1 \\ y_{11} + 2y_{12} + 3y_{13} + y_{21} + 2y_{22} + 3y_{23} + y_{31} + 2y_{32} + 3y_{33} &= 5 \end{aligned}$$

and

each y_{ij} is binary.

Note that the first three functional constraints ensure that each x_i will be assigned just one of its possible values. (Here $y_{i1} + y_{i2} + y_{i3} = 0$ corresponds to $x_i = 0$, which contributes nothing to the objective function.) The last functional constraint ensures that $x_1 + x_2 + x_3 = 5$. The *linear* objective function then gives the total profit according to Table 2.

Solving this BIP model gives an optimal solution of

$$\begin{array}{lllll} y_{11} = 0, & y_{12} = 1, & y_{13} = 0, & \text{so} & x_1 = 2 \\ y_{21} = 0, & y_{22} = 0, & y_{23} = 0, & \text{so} & x_2 = 0 \\ y_{31} = 0, & y_{32} = 0, & y_{33} = 1, & \text{so} & x_3 = 3. \end{array}$$

Another Formulation with Auxiliary Binary Variables. We now redefine the above auxiliary binary variables y_{ij} as follows:

$$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the difference is that $y_{ij} = 1$ now if $x_i \geq j$ instead of $x_i = j$. Therefore,

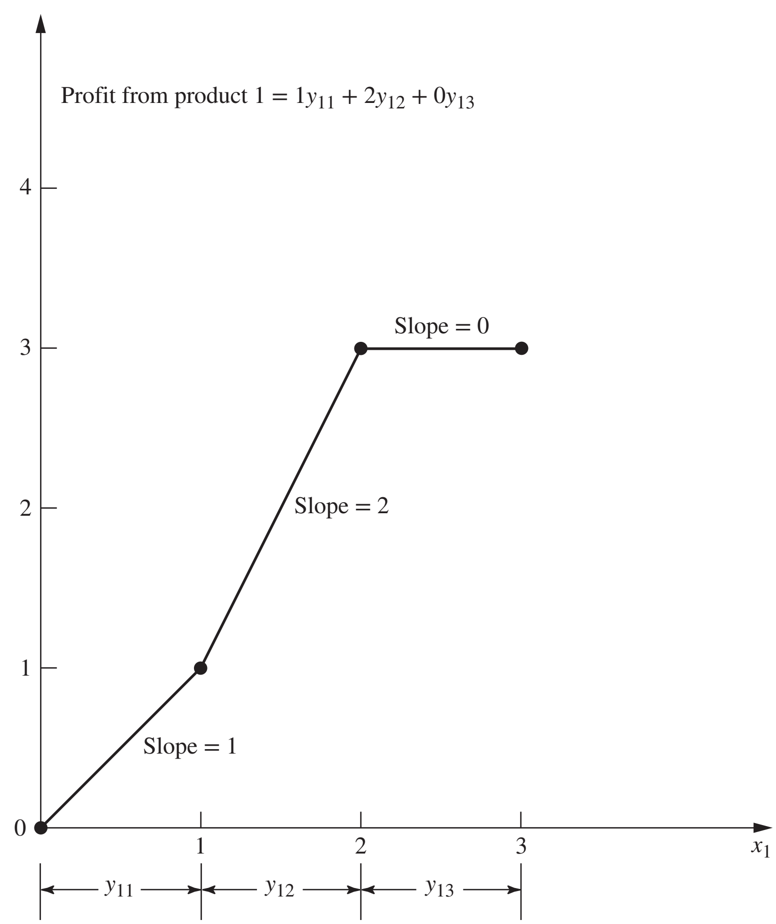
$$\begin{aligned}
x_i = 0 \quad &\Rightarrow \quad y_{i1} = 0, \quad y_{i2} = 0, \quad y_{i3} = 0,\\
x_i = 1 \quad &\Rightarrow \quad y_{i1} = 1, \quad y_{i2} = 0, \quad y_{i3} = 0,\\
x_i = 2 \quad &\Rightarrow \quad y_{i1} = 1, \quad y_{i2} = 1, \quad y_{i3} = 0,\\
x_i = 3 \quad &\Rightarrow \quad y_{i1} = 1, \quad y_{i2} = 1, \quad y_{i3} = 1,\\
\text{so } x_i &= y_{i1} + y_{i2} + y_{i3}
\end{aligned}$$

for $i = 1, 2, 3$. Because allowing $y_{i2} = 1$ is contingent upon $y_{i1} = 1$ and allowing $y_{i3} = 1$ is contingent upon $y_{i2} = 1$, these definitions are enforced by adding the constraints

$$y_{i2} \leq y_{i1} \quad \text{and} \quad y_{i3} \leq y_{i2}, \quad \text{for } i = 1, 2, 3.$$

The new definition of the y_{ij} also changes the objective function, as illustrated in Fig. 1 for the product 1 portion of the objective function. Since y_{11}, y_{12}, y_{13} provide the successive increments (if any) in the value of x_1 (starting from a value of 0), the coefficients of y_{11}, y_{12}, y_{13} are given by the respective *increments* in the product 1 column of Table 2 ($1 - 0 = 1$, $3 - 1 = 2$, $3 - 3 = 0$). These *increments* are the *slopes* in Fig. 1, yielding $1y_{11} + 2y_{12} + 0y_{13}$ for the product 1 portion of the objective function. Note that applying this approach to all three products still must lead to a *linear* objective function.

FIGURE 1
The profit from the additional sales of product 1 that would result from x_1 TV spots, where the slopes give the corresponding coefficients in the objective function for the second BIP formulation for Example 2 (the Supersuds Corp. problem).



After we bring all variables to the left-hand side of the constraints, the resulting complete BIP model is

$$\text{Maximize} \quad Z = y_{11} + 2y_{12} + 2y_{22} + y_{23} - y_{31} + 3y_{32} + 2y_{33},$$

subject to

$$y_{12} - y_{11} \leq 0$$
$$y_{13} - y_{12} \leq 0$$
$$y_{22} - y_{21} \leq 0$$
$$y_{23} - y_{22} \leq 0$$
$$y_{32} - y_{31} \leq 0$$
$$y_{33} - y_{32} \leq 0$$
$$y_{11} + y_{12} + y_{13} + y_{21} + y_{22} + y_{23} + y_{31} + y_{32} + y_{33} = 5$$

and

each y_{ij} is binary.

Solving this BIP model gives an optimal solution of

$$y_{11} = 1, \quad y_{12} = 1, \quad y_{13} = 0, \quad \text{so} \quad x_1 = 2$$
$$y_{21} = 0, \quad y_{22} = 0, \quad y_{23} = 0, \quad \text{so} \quad x_2 = 0$$
$$y_{31} = 1, \quad y_{32} = 1, \quad y_{33} = 1, \quad \text{so} \quad x_3 = 3.$$

There is little to choose between this BIP model and the preceding one other than personal taste. They have the same number of binary variables (the prime consideration in determining computational effort for BIP problems). They also both have some *special structure* (constraints for *mutually exclusive alternatives* in the first model and constraints for *contingent decisions* in the second) that can lead to speedup. The second model does have more functional constraints than the first.

Another example of a challenging IP formulation is given in the Solved Examples section for Chapter 12 on the book's website.

PROBLEMS

12S-1. The Research and Development Division of the Progressive Company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, an operations research study has been requested to find the most profitable product mix.

A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. Management's objective is to find the product mix that maximizes the total profit (total net revenue minus start-up costs).

	Product			
	1	2	3	4
Start-up cost	\$50,000	\$40,000	\$70,000	\$60,000
Marginal revenue	\$70	\$60	\$90	\$80

Let the continuous decision variables x_1, x_2, x_3, and x_4 be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:

1. No more than two of the products can be produced.
2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.
3. Either $5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6{,}000$
   or $4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6{,}000$.

(a) Introduce auxiliary binary variables to formulate a mixed BIP model for this problem.

C (b) Use the computer to solve this model.

12S-2. Suppose that a mathematical model fits linear programming except for the restriction that $|x_1 - x_2| = 0$, or 3, or 6. Show how to reformulate this restriction to fit an MIP model.

12S-3. Suppose that a mathematical model fits linear programming except for the restrictions that

1. At least one of the following two inequalities holds:

$$3x_1 - x_2 - x_3 + x_4 \leq 12$$
$$x_1 + x_2 + x_3 + x_4 \leq 15.$$

2. At least two of the following three inequalities holds:

$$2x_1 + 5x_2 - x_3 + x_4 \leq 30$$
$$-x_1 + 3x_2 + 5x_3 + x_4 \leq 40$$
$$3x_1 - x_2 + 3x_3 - x_4 \leq 60.$$

Show how to reformulate these restrictions to fit an MIP model.

12S-4. Reconsider Prob. 1. Follow the instructions for that problem after imposing one new restriction. To avoid doubling the start-up costs, just one factory would be used, where the choice would be based on maximizing profit. The same factory would be used for both new toys if both are produced.

12S-5. Reconsider the Fly-Right Airplane Co. problem introduced in Prob. 12.3-2. A more detailed analysis of the various cost and revenue factors now has revealed that the potential profit from producing airplanes for each customer cannot be expressed simply in terms of a *start-up cost* and a fixed *marginal net revenue* per airplane produced. Instead, the profits are given by the following table.

Airplanes Produced	Profit from Customer		
	1	2	3
0	0	0	0
1	−$1 million	$1 million	$1 million
2	$2 million	$5 million	$3 million
3	$4 million		$5 million
4			$6 million
5			$7 million

(a) Formulate a BIP model for this problem that includes constraints for *mutually exclusive alternatives.*

C **(b)** Use the computer to solve the model formulated in part (*a*). Then use this optimal solution to identify the optimal number of airplanes to produce for each customer.

(c) Formulate another BIP model for this model that includes constraints for *contingent decisions.*

C **(d)** Repeat part (*b*) for the model formulated in part (*c*).

12S-6. Reconsider the Wyndor Glass Co. problem presented in Sec. 3.1. Management now has decided that only one of the two new products should be produced, and the choice is to be made on the basis of maximizing profit. Introduce *auxiliary binary variables* to formulate an MIP model for this new version of the problem.

12S-7. Reconsider Prob. 3.1-11, where the management of the Omega Manufacturing Company is considering devoting excess production capacity to one or more of three products. Management now has decided to add the restriction that no more than two of the three prospective products should be produced.

(a) Introduce *auxiliary binary variables* to formulate an MIP model for this new version of the problem.

C **(b)** Use the computer to solve this model.

12S-8. Consider the following integer nonlinear programming problem:

$$\text{Maximize} \quad Z = 4x_1^2 - x_1^3 + 10x_2^2 - x_2^4,$$

subject to

$$x_1 + x_2 \leq 3$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0$$

x_1 and x_2 are integers.

This problem can be reformulated in two different ways as an equivalent pure BIP problem (with a linear objective function) with six binary variables (y_{1j} and y_{2j} for $j = 1, 2, 3$), depending on the interpretation given the binary variables.

(a) Formulate a BIP model for this problem where the binary variables have the interpretation,

$$y_{ij} = \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{otherwise.} \end{cases}$$

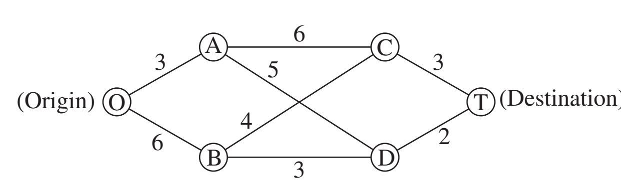
C **(b)** Use the computer to solve the model formulated in part (*a*), and thereby identify an optimal solution for (x_1, x_2) for the original problem.

(c) Formulate a BIP model for this problem where the binary variables have the interpretation,

$$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

C **(d)** Use the computer to solve the model formulated in part (*c*), and thereby identify an optimal solution for (x_1, x_2) for the original problem.

12S-9. Consider the following special type of *shortest-path problem* (see Sec. 10.3) where the nodes are in columns and the only paths considered always move forward one column at a time.



The numbers along the links represent distances, and the objective is to find the shortest path from the origin to the destination.

This problem also can be formulated as a BIP model involving both mutually exclusive alternatives and contingent decisions.

(a) Formulate this model. Identify the constraints that are for mutually exclusive alternatives and that are for contingent decisions.

C **(b)** Use the computer to solve this problem.