SUPPLEMENT TO CHAPTER 18

Stochastic Periodic-Review Models

Sections 18.6 and 18.7 present stochastic inventory models for analyzing inventory systems where there is considerable uncertainty about future demands. Section 18.6 considers a *continuous-review* inventory system where the inventory level of a *stable product* (one that will remain salable indefinitely) is being monitored on a continuous basis. Section 18.7 describes a single-period model for a *perishable product* that will remain salable for only the one period.

We now return to considering a stable product that will remain salable indefinitely. We again assume that the demand is uncertain so that a stochastic model is needed. However, in contrast to the continuous-review inventory system considered in Sec. 18.6, we now assume that the system is only being monitored periodically. At the end of each period, when the current inventory level is determined, a decision is made on how much to order (if any) to replenish inventory for the next period. Each of these decisions takes into account the planning for multiple periods into the future.

We begin with the simplest case where the planning is only being done for the next two periods and no setup cost is incurred when placing an order to replenish inventory.

A Stochastic Two-Period Model with No Setup Cost

One option with a stochastic periodic-review inventory system is to plan ahead only one period at a time, using the stochastic single-period model from Sec. 18.7 to make the ordering decision each time. However, this approach would only provide a relatively crude approximation. If the probability distribution of demand in each period can be forecasted multiple periods into the future, better decisions can be made by coordinating the plans for all these periods than by planning ahead just one period at a time. This can be quite difficult for many periods but is considerably less difficult when considering only two periods at a time.

Even for a planning horizon of two periods, using the optimal one-period solution twice is not generally the optimal policy for the two-period problem. Smaller costs can usually be achieved by viewing the problem from a two-period viewpoint and then using the methods of probabilistic dynamic programming introduced in Sec. 11.4 to obtain the best inventory policy.

Assumptions. Except for having two periods, the assumptions for this model are basically the same as for the one-period model presented in the preceding section, as summarized below.

1. Each application involves a single stable product.
2. Planning is being done for two periods, where unsatisfied demand in period 1 is backlogged to be met in period 2, but there is no backlogging of unsatisfied demand in period 2.
3. The demands D_1 and D_2 for periods 1 and 2 are *independent and identically distributed* random variables. Their common probability distribution has probability density function $f(x)$ and cumulative distribution function $F(d)$.
4. The initial inventory level (before replenishing) at the beginning of period 1 is I_1 ($I_1 \geq 0$).
5. The decisions to be made are S_1 and S_2, the inventory levels to reach by replenishing (if needed) at the beginning of period 1 and period 2, respectively.
6. The objective is to *minimize the expected total cost for both periods,* where the cost components for each period are

   c = unit cost for purchasing or producing each unit,
   h = holding cost per unit remaining at the end of each period,
   p = shortage cost per unit of unsatisfied demand at the end of each period.

For simplicity, we are assuming that the demand distributions for the two periods are the same and that the values of the above cost components also are the same for the two periods. In many applications, there will be differences between the periods that should be incorporated into the analysis. For example, because of assumption 2, the value of p may well be different for the two periods. Such extensions of the model can be incorporated into the dynamic programming analysis presented below, but we will not delve into these extensions.

Analysis. To begin the analysis, let

S_i^* = optimal value of S_i, for $i = 1, 2$,
$C_1(I_1)$ = expected total cost for both periods when following an optimal policy given that I_1 is the initial inventory level (before replenishing) at the beginning of period 1,
$C_2(I_2)$ = expected total cost for just period 2 when following an optimal policy given that I_2 is the inventory level (before replenishing) at the beginning of period 2.

To use the dynamic programming approach, we begin by solving for $C_2(I_2)$ and S_2^*, where there is just one period to go. Then we will use these results to find $C_1(I_1)$ and S_1^*.

From the results for the single-period model, S_2^* is found by solving the equation

$$F(S_2^*) = \frac{p - c}{p + h}.$$

Given I_2, the resulting optimal policy then is the following:

Optimal Inventory Policy for Period 2

If $I_2 < S_2^*$, order $S_2^* - I_2$ to bring the inventory level up to S_2^*.
If $I_2 \geq S_2^*$, do not order.

The cost of this optimal policy can be expressed as

$$C_2(I_2) = \begin{cases} c(S_2^* - I_2) + L(S_2^*) & \text{if } I_2 < S_2^* \\ L(I_2) & \text{if } I_2 \geq S_2^*, \end{cases}$$

where $L(I)$ is the expected shortage plus holding cost for a single period when the inventory level (after replenishing) is I. $L(I)$ can be expressed as

$$L(I) = \int_I^\infty p(x - I)f(x)dx + \int_0^I h(I - x)f(x)dx.$$

When both periods 1 and 2 are considered, the costs incurred consist of the ordering cost $c(S_1 - I_1)$, the expected shortage plus holding cost $L(S_1)$, and the costs associated with following an optimal policy during the second period. Thus, the expected cost of following the optimal policy for two periods is given by

$$C_1(I_1) = \min_{S_1 \geq I_1} \{c(S_1 - I_1) + L(S_1) + E[C_2(I_2)]\},$$

where $E[C_2(I_2)]$ is obtained as follows. Note that

$$I_2 = S_1 - D_1,$$

so I_2 is a random variable when beginning period 1. Thus,

$$C_2(I_2) = C_2(S_1 - D_1) = \begin{cases} c(S_2^* - S_1 + D_1) + L(S_2^*) & \text{if } S_1 - D_1 < S_2^* \\ L(S_1 - D_1) & \text{if } S_1 - D_1 \geq S_2^*. \end{cases}$$

Hence, $C_2(I_2)$ is a random variable, and its expected value is given by

$$\begin{aligned} E[C_2(I_2)] &= \int_0^\infty C_2(S_1 - x)f(x)\,dx \\ &= \int_0^{S_1 - S_2^*} L(S_1 - x)f(x)\,dx \\ &\quad + \int_{S_1 - S_2^*}^\infty [c(S_2^* - S_1 + x) + L(S_2^*)]f(x)\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} C_1(I_1) = \min_{S_1 \geq I_1} \Bigg\{ & c(S_1 - I_1) + L(S_1) + \int_0^{S_1 - S_2^*} L(S_1 - x)f(x)\,dx \\ & + \int_{S_1 - S_2^*}^\infty [(S_2^* - S_1 + x) + L(S_2^*)]f(x)\,dx \Bigg\}. \end{aligned}$$

It can be shown that $C_1(I_1)$ has a unique minimum and that the optimal value of S_1, denoted by S_1^*, satisfies the equation

$$\begin{aligned} -p + (p + h)F(S_1^*) + (c - p)\,F(S_1^* - S_2^*) \\ + (p + h)\int_0^{S_1^* - S_2^*} F(S_1^* - x)f(x) = 0. \end{aligned}$$

The resulting optimal policy for period 1 then is the following:

Optimal Inventory Policy for Period 1

If $I_1^* < S_1^*$, order $S_1^* - I_1$ to bring the inventory level up to S_1^*.
If $I_1 \geq S_1^*$, do not order.

The procedure for finding S_1^* reduces to a simpler result for certain demand distributions. We summarize two such cases next.

Suppose that the demand in each period has a *uniform distribution* over the range 0 to t, that is,

$$f(x) = \begin{cases} \dfrac{1}{t} & \text{if } 0 \leq x \leq t \\ 0 & \text{otherwise.} \end{cases}$$

Then S_1^* can be obtained from the expression

$$S_1^* = \sqrt{(S_2^*)^2 + \frac{2t(c-p)}{p+h} S_2^* + \frac{t^2[2p(p+h)+(h+c)^2]}{(p+h)^2}} - \frac{t(h+c)}{p+h}.$$

Now suppose that the demand in each period has an *exponential distribution,* i.e.,

$$f(x) = \alpha e^{-\alpha x}, \qquad \text{for } x \geq 0.$$

Then S_1^* satisfies the relationship

$$(h+c)e^{-\alpha(S_1^*-S_2^*)} + (p+h)e^{-\alpha S_1^*} + \alpha(p+h)(S_1^* - S_2^*)e^{-\alpha S_1^*} = 2h + c.$$

An alternative way of finding S_1^* is to let t denote $\alpha(S_1^* - S_2^*)$. Then t satisfies the relationship

$$e^{-t}[(h+c) + (p+h)e^{-\alpha S_2^*} + t(p+h)e^{-\alpha S_2^*}] = 2h + c,$$

and

$$S_1^* = \frac{1}{\alpha}t + S_2^*.$$

When the demand has either a uniform or an exponential distribution, an automatic procedure is available in your IOR Tutorial for calculating S_1^* and S_2^*.

Example. Consider a two-period problem where

$$c = 10, \qquad h = 10, \qquad p = 15,$$

and where the probability density function of the demand in each period is given by

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } 0 \leq x \leq 10 \\ 0 & \text{otherwise,} \end{cases}$$

so that the cumulative distribution function of demand is

$$F(d) = \begin{cases} 0 & \text{if } d < 0 \\ \frac{d}{10} & \text{if } 0 \leq d \leq 10 \\ 1 & \text{if } d > 10. \end{cases}$$

We find S_2^* from the equation

$$F(S_2^*) = \frac{p-c}{p+h} = \frac{15-10}{15+10} = \frac{1}{5},$$

so that

$$S_2^* = 2.$$

To find S_1^*, we plug into the expression given for S_1^* for the case of a *uniform* demand distribution, and we obtain

$$S_1^* = \sqrt{2^2 + \frac{2(10)(10-15)}{15+10}(2) + 10^2 \frac{2(15)(15+10)+(10+10)^2}{(15+10)^2}}$$
$$- \frac{10(10+10)}{15+10}$$
$$= \sqrt{4 - 8 + 184} - 8 = 13.42 - 8 = 5.42,$$

where 5.42 now needs to be rounded to an integer.

Substituting $S_1^* = 5$ and $S_1^* = 6$ into $C_1(I_1)$ leads to a smaller value with $S_1^* = 5$. Thus, the optimal policy can be described as follows:

If $I_1 < 5$, order $5 - I_1$ to bring the inventory level up to 5.
If $I_1 \geq 5$, do not order in period 1.
If $I_2 < 2$, order $2 - I_2$ to bring the inventory level up to 2.
If $I_2 \geq 2$, do not order in period 2.

Since unsatisfied demand in period 1 is backlogged to be met in period 2, $I_2 = 5 - D$ can turn out to be either positive or negative.

Stochastic Multiperiod Models—An Overview

The two-period model can be extended to several periods or to an infinite number of periods. This section presents a summary of multiperiod results that have practical importance.

Multiperiod Model with No Setup Cost. Consider the direct extension of the above two-period model to n periods ($n > 2$) with the identical assumptions. The only difference is that a *discount factor* α (described in Sec. 18.2), with $0 < \alpha < 1$, now will be used in calculating the expected total cost for n periods. (Although the symbol α has been used elsewhere to denote the parameter for the exponential distribution, it will instead be used here to denote the discount factor for the remainder of this supplement.) The problem still is to find the critical numbers $S_1^*, S_2^*, \ldots, S_n^*$ that describe the optimal inventory policy. As in the two-period model, these values are difficult to obtain numerically, but it can be shown[1] that the optimal policy has the following form.

Optimal Inventory Policy

For each period i ($i = 1, 2, \ldots, n$), with I_i as the inventory level entering that period (before replenishing), do the following:

If $I_i < S_i^*$, order $S_i^* - I_i$ to bring the inventory level up to S_i^*.
If $I_i \geq S_i^*$, do not order.

Furthermore,

$$S_n^* \leq S_{n-1}^* \leq \cdots \leq S_2^* \leq S_1^*.$$

For the *infinite-period* case (where $n = \infty$), all these critical numbers $S_1^*, S_2^*, \ldots$ are *equal*. Let S^* denote this constant value. It can be shown that S^* satisfies the equation

$$F(S^*) = \frac{p - c(1 - \alpha)}{p + h}.$$

When the demand has either a uniform or an exponential distribution, an automatic procedure is available in your IOR Tutorial for calculating S^*.

A Variation of the Multiperiod Inventory Model with No Setup Cost. These results for the infinite-period case (all the critical numbers equal the same value S^* and S^* satisfies the above equation) also apply when n is finite if two new assumptions are made about

[1]See Theorem 4 in R. Bellman, I. Glicksberg, and O. Gross, "On the Optimal Inventory Equation," *Management Science,* **2:** 83–104, 1955. Also see p. 163 in K. J. Arrow, S. Karlin, and H. Scarf (eds.), *Studies in the Mathematical Theory of Inventory and Production,* Stanford University Press, Stanford, CA, 1958.

what happens at the end of the last period. One new assumption is that each unit left over at the end of the final period can be salvaged with a return of the initial purchase cost c. Similarly, if there is a shortage at this time, assume that the shortage is met by an emergency shipment with the same unit purchase cost c.

Example. Consider again the bicycle example as it was introduced in Example 2 of Sec. 18.1. The cost estimates given there imply that

$$c = 35, \qquad h = 1, \qquad p = 15.$$

Suppose now that the distributor places an order with the manufacturer for various bicycle models on the first working day of each month. Because of this routine, she is willing to assume that the marginal setup cost is zero for including an order for the bicycle model under consideration. The appropriate discount factor is $\alpha = 0.995$. From past history, the distribution of demand can be approximated by a uniform distribution with the probability density function

$$f(x) = \begin{cases} \dfrac{1}{800} & \text{if } 0 \le x \le 800 \\ 0 & \text{otherwise,} \end{cases}$$

so the cumulative distribution function over this interval is

$$F(d) = \frac{1}{800}\,d, \qquad \text{if } 0 \le d \le 800.$$

The distributor expects to stock this model indefinitely, so the *infinite-period model with no setup cost* is appropriate.

For this model, the critical number S^* for every period satisfies the equation

$$F(S^*) = \frac{p - c(1 - \alpha)}{p + h},$$

so

$$\frac{S^*}{800} = \frac{15 - 35(1 - 0.995)}{15 + 1} = 0.9266,$$

which yields $S^* = 741$. Thus, if the number of bicycles on hand I at the first of each month is fewer than 741, the optimal policy calls for bringing the inventory level up to 741 (ordering $741 - I$ bicycles). Otherwise, no order is placed.

Multiperiod Model with Setup Cost. The introduction of a fixed setup cost K that is incurred when ordering (whether through purchasing or producing) often adds more realism to the model. For the *single-period model with a setup cost* described in Sec. 18.7, we found that an (s, S) policy is optimal, so that the two critical numbers s^* and S^* indicate *when* to order (namely, if the inventory level is less than s^*) and *how much* to order (bring the inventory level up to S^*). Now with multiple periods, an (s, S) policy again is optimal, but the value of each critical number may be different in different periods. Let s_i^* and S_i^* denote these critical numbers for period *i,* and again let I_i be the inventory level (before replenishing) at the beginning of period *i*.

Optimal Inventory Policy

The optimal policy is to do the following at the beginning of each period i ($i = 1, 2, \ldots, n$):

If $I_i < S_i^*$, order $S_i^* - I_i$ to bring the inventory level up to S_i^*.
If $I_i \geq S_i^*$, do not order.

Unfortunately, computing exact values of the s_i^* and S_i^* is extremely difficult.

A Multiperiod Model with Batch Orders and No Setup Cost. In the preceding models, *any integer quantity* could be ordered (or produced) at the beginning of each period. However, in some applications, the product may come in a standard batch size, e.g., a case or a truckload. Let Q be the number of units in each batch. In our current model, we assume that the number of units ordered must be a *nonnegative integer multiple* of Q.

This model makes the same assumptions about what happens at the end of the last period as the variation of the multiperiod model with no setup cost presented earlier. Thus, we assume that each unit left over at the end of the final period can be salvaged with a return of the initial purchase cost c. Similarly, if there is a shortage at this time, we assume that the shortage is met by an emergency shipment with the same unit purchase cost c.

Otherwise, the assumptions are the same as for our standard multiperiod model with no setup cost.

The optimal policy for this model is known as a **(*k*, *Q*) policy** because it uses a critical number k and the quantity Q as described below.

> If at the beginning of a period the inventory level (before replenishing) is less than k, an order should be placed for the smallest integer multiple of Q that will bring the inventory level up to at least k (and probably higher). Otherwise, an order should not be placed. The same critical number k is used in each period.

The critical number k is chosen as follows. Plot the function

$$G(S) = (1 - \alpha)cS + h \int_0^S (S - x)f(x)\,dx + p \int_S^\infty (x - S)\,f(x)\,dx,$$

as shown in Fig. 1. This function necessarily has the convex shape shown in the figure. As before, the minimizing value S^* satisfies the equation

$$F(S^*) = \frac{p - c(1 - \alpha)}{p + h}.$$

As shown in this figure, if a "ruler" of length Q is placed horizontally into the "valley," k is that value of the abscissa to the left of S^* where the ruler intersects the valley. If the inventory level lies in R_1, then Q is ordered; if it lies in R_2, then $2Q$ is ordered; and so on. However, if the inventory level is at least $k,$ then no order should be placed.

These results hold regardless of whether the number of periods n is finite or infinite.

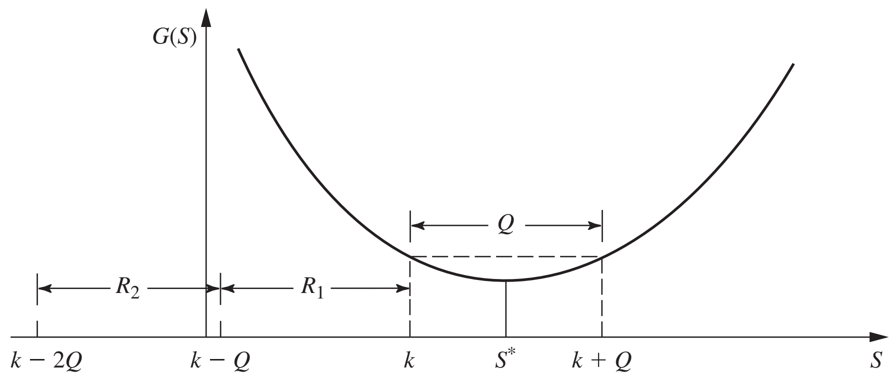

FIGURE 1
Plot of the $G(S)$ function for the stochastic multiperiod model with batch orders and no setup cost.

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Automatic Procedures in IOR Tutorial:

Stochastic Two-Period Model, No Setup Cost
Stochastic Infinite-Period Model, No Setup Cost

PROBLEMS

To the left of each of the following problems, we have inserted an A whenever one of the automatic procedures listed above can be helpful.

A **18S-1.** Consider the following inventory situation. Demands in different periods are independent but with a common probability density function given by

$$f(x) = \begin{cases} \dfrac{e^{-x/25}}{25} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Orders may be placed at the start of each period without setup cost at a unit cost of $c = 10$. There are a holding cost of 6 per unit remaining in stock at the end of each period and a shortage cost of 15 per unit of unsatisfied demand at the end of each period (with backlogging except for the final period).
(a) Find the optimal one-period policy.
(b) Find the optimal two-period policy.

A **18S-2.** Consider the following inventory situation. Demands in different periods are independent but with a common probability density function $f(x) = \frac{1}{50}$ for $0 \leq x \leq 50$. Orders may be placed at the start of each period without setup cost at a unit cost of $c = 10$. There are a holding cost of 8 per unit remaining in stock at the end of each period and a penalty cost of 15 per unit of unsatisfied demand at the end of each period (with backlogging except for the final period).
(a) Find the optimal one-period policy.
(b) Find the optimal two-period policy.

A **18S-3.** Find the optimal inventory policy for the following two-period model by using a discount factor of $\alpha = 0.9$. The demand D has the probability density function

$$f(x) = \begin{cases} \dfrac{1}{25}e^{-x/25} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the costs are

Holding cost = \$0.25 per item,
Shortage cost = \$2 per item,
Purchase price = \$1 per item.

Stock left over at the end of the final period is salvaged for \$1 per item, and shortages remaining at this time are met by purchasing the needed items at \$1 per item.

A **18S-4.** Solve Prob. 18S-3 for a two-period model, assuming no salvage value, no backlogging at the end of the second period, and no discounting.

A **18S-5.** Solve Prob. 18S-3 for an infinite-period model.

A **18S-6.** Determine the optimal inventory policy when the goods are to be ordered at the end of every month from now on. The cost of bringing the inventory level up to S when I already is available is given by $2(S - I)$. Similarly, the cost of having the monthly demand D exceed S is given by $5(D - S)$. The probability density function for D is given by $f(x) = e^{-x}$. The holding cost when S exceeds D is given by $S - D$. A monthly discount factor of 0.95 is used.

A **18S-7.** Solve the inventory problem given in Prob. 18S-6, but assume that the policy is to be used for only 1 year (a 12-period model). Shortages are backlogged each month, except that any shortages remaining at the end of the year are made up by purchasing similar items at a unit cost of \$2. Any remaining inventory at the end of the year can be sold at a unit price of \$2.

A **18S-8.** A supplier of high-fidelity receiver kits is interested in using an optimal inventory policy. The distribution of demand per month is uniform between 2,000 and 3,000 kits. The supplier's cost for each kit is \$150. The holding cost is estimated to be \$2 per kit remaining at the end of a month, and the shortage cost is \$30 per kit of unsatisfied demand at the end of a month. Using a monthly discount factor of $\alpha = 0.99$, find the optimal inventory policy for this infinite-period problem.

A **18S-9.** The weekly demand for a certain type of electronic calculator is estimated to be

$$f(x) = \begin{cases} \dfrac{1}{1{,}000} e^{-x/1{,}000} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The unit cost of these calculators is \$80. The holding cost is \$0.70 per calculator remaining at the end of a week. The shortage cost is \$2 per calculator of unsatisfied demand at the end of a week. Using a weekly discount factor of $\alpha = 0.998$, find the optimal inventory policy for this infinite-period problem.

18S-10. Consider a one-period model where the only two costs are the holding cost, given by

$$h(S - D) = \frac{3}{10}(S - D), \qquad \text{for } S \geq D,$$

and the shortage cost, given by

$$p(D - S) = 2.5(D - S), \qquad \text{for } D \geq S.$$

The probability density function for demand is given by

$$f(x) = \begin{cases} \dfrac{e^{-x/25}}{25} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If you order, you must order an *integer* number of *batches* of 100 units each, and this quantity is delivered immediately. Let $G(S)$ denote the total expected cost when there are S units available for the period (after ordering).

(a) Write the expression for $G(S)$.
(b) What is the optimal ordering policy?

18S-11. Find the optimal (k, Q) policy for Prob. 18S-10 for an infinite-period model with a discount factor of $\alpha = 0.90$.

18S-12. For the infinite-period model with no setup cost, show that the value of S^* that satisfies

$$F(S^*) = \frac{p - c(1 - \alpha)}{p + h}$$

is equivalent to the value of S that satisfies

$$\frac{dL(S)}{dS} + c(1 - \alpha) = 0.$$

where $L(S)$, the expected shortage plus holding cost, is given by

$$L(S) = \int_S^\infty p(x - S) f(x)\, dx + \int_0^S h(S - x) f(x)\, dx.$$