In this way everyone who is party to the communication knows what a design factor (or factor of safety) of 2 means and adjusts, if necessary, the judgmental perspective.

6–17 Stochastic Analysis[28]

As already demonstrated in this chapter, there are a great many factors to consider in a fatigue analysis, much more so than in a static analysis. So far, each factor has been treated in a deterministic manner, and if not obvious, these factors are subject to variability and control the overall reliability of the results. When reliability is important, then fatigue testing must certainly be undertaken. There is no other way. Consequently, the methods of stochastic analysis presented here and in other sections of this book constitute guidelines that enable the designer to obtain a good understanding of the various issues involved and help in the development of a safe and reliable design.

In this section, key stochastic modifications to the deterministic features and equations described in earlier sections are provided in the same order of presentation.

Endurance Limit

To begin, a method for estimating endurance limits, the *tensile strength correlation method*, is presented. The ratio $\boldsymbol{\phi} = \mathbf{S}'_e/\bar{S}_{ut}$ is called the *fatigue ratio*.[29] For ferrous metals, most of which exhibit an endurance limit, the endurance limit is used as a numerator. For materials that do not show an endurance limit, an endurance strength at a specified number of cycles to failure is used and noted. Gough[30] reported the stochastic nature of the fatigue ratio $\boldsymbol{\phi}$ for several classes of metals, and this is shown in Fig. 6–36. The first item to note is that the coefficient of variation is of the order 0.10 to 0.15, and the distribution varies for classes of metals. The second item to note is that Gough's data include materials of no interest to engineers. In the absence of testing, engineers use the correlation that $\boldsymbol{\phi}$ represents to estimate the endurance limit $\mathbf{S}'_e$ from the mean ultimate strength $\bar{S}_{ut}$.

Gough's data are for ensembles of metals, some chosen for metallurgical interest, and include materials that are not commonly selected for machine parts. Mischke[31] analyzed data for 133 common steels and treatments in varying diameters in rotating bending,[32] and the result was

$$\boldsymbol{\phi} = 0.445 d^{-0.107} \mathbf{LN}(1, 0.138)$$

where d is the specimen diameter in inches and $\mathbf{LN}(1, 0.138)$ is a unit lognormal variate with a mean of 1 and a standard deviation (and coefficient of variation) of 0.138. For the standard R. R. Moore specimen,

$$\boldsymbol{\phi}_{0.30} = 0.445(0.30)^{-0.107} \mathbf{LN}(1, 0.138) = 0.506 \mathbf{LN}(1, 0.138)$$

[28]Review Chap. 20 before reading this section.

[29]From this point, since we will be dealing with statistical distributions in terms of means, standard deviations, etc. A key quantity, the ultimate strength, will here be presented by its mean value, $\bar{S}_{ut}$. This means that certain terms that were defined earlier in terms of the minimum value of S_{ut} will change slightly.

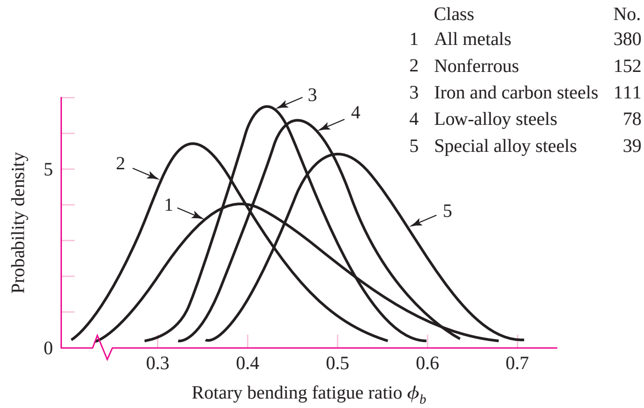
[30]In J. A. Pope, *Metal Fatigue,* Chapman and Hall, London, 1959.

[31]Charles R. Mischke, "Prediction of Stochastic Endurance Strength," *Trans. ASME, Journal of Vibration, Acoustics, Stress, and Reliability in Design,* vol. 109, no. 1, January 1987, pp. 113–122.

[32]Data from H. J. Grover, S. A. Gordon, and L. R. Jackson, *Fatigue of Metals and Structures,* Bureau of Naval Weapons, Document NAVWEPS 00-2500435, 1960.

Figure 6–36

The lognormal probability density PDF of the fatigue ratio ϕ_b of Gough.



Also, 25 plain carbon and low-alloy steels with $S_{ut} > 212$ kpsi are described by

$$\mathbf{S}'_e = 107\mathbf{LN}(1, 0.139) \text{ kpsi}$$

In summary, for the rotating-beam specimen,

$$\mathbf{S}'_e = \begin{cases} 0.506\bar{S}_{ut}\mathbf{LN}(1, 0.138) \text{ kpsi or MPa} & \bar{S}_{ut} \leq 212 \text{ kpsi (1460 MPa)} \\ 107\mathbf{LN}(1, 0.139) \text{ kpsi} & \bar{S}_{ut} > 212 \text{ kpsi} \\ 740\mathbf{LN}(1, 0.139) \text{ MPa} & \bar{S}_{ut} > 1460 \text{ MPa} \end{cases} \tag{6–70}$$

where $\bar{S}_{ut}$ is the *mean* ultimate tensile strength.

Equations (6–70) represent the state of information before an engineer has chosen a material. In choosing, the designer has made a random choice from the ensemble of possibilities, and the statistics can give the odds of disappointment. If the testing is limited to finding an estimate of the ultimate tensile strength mean $\bar{S}_{ut}$ with the chosen material, Eqs. (6–70) are directly helpful. If there is to be rotary-beam fatigue testing, then statistical information on the endurance limit is gathered and there is no need for the correlation above.

Table 6–9 compares approximate mean values of the fatigue ratio $\bar{\phi}_{0.30}$ for several classes of ferrous materials.

Endurance Limit Modifying Factors

A Marin equation can be written as

$$\mathbf{S}_e = \mathbf{k}_a k_b \mathbf{k}_c \mathbf{k}_d \mathbf{k}_f \mathbf{S}'_e \tag{6–71}$$

where the size factor k_b is deterministic and remains unchanged from that given in Sec. 6–9. Also, since we are performing a stochastic analysis, the "reliability factor" k_e is unnecessary here.

The surface factor $\mathbf{k}_a$ cited earlier in deterministic form as Eq. (6–20), p. 288, is now given in stochastic form by

$$\mathbf{k}_a = a\bar{S}_{ut}^{b}\mathbf{LN}(1, C) \qquad (\bar{S}_{ut} \text{ in kpsi or MPa}) \tag{6–72}$$

where Table 6–10 gives values of a, b, and C for various surface conditions.

Table 6–9

Comparison of Approximate Values of Mean Fatigue Ratio for Some Classes of Metals

Material Class	$\bar{\phi}_{0.30}$
Wrought steels	0.50
Cast steels	0.40
Powdered steels	0.38
Gray cast iron	0.35
Malleable cast iron	0.40
Normalized nodular cast iron	0.33

Table 6–10

Parameters in Marin Surface Condition Factor

$\mathbf{k}_a = aS_{ut}^b\,\mathbf{LN}(1, C)$				
	a			
Surface Finish	kpsi	MPa	*b*	Coefficient of Variation, *C*
Ground*	1.34	1.58	−0.086	0.120
Machined or Cold-rolled	2.67	4.45	−0.265	0.058
Hot-rolled	14.5	58.1	−0.719	0.110
As-forged	39.8	271	−0.995	0.145

*Due to the wide scatter in ground surface data, an alternate function is $\mathbf{k_a} = 0.878\mathbf{LN}(1, 0.120)$.
Note: S_{ut} in kpsi or MPa.

EXAMPLE 6–16 A steel has a mean ultimate strength of 520 MPa and a machined surface. Estimate $\mathbf{k}_a$.

Solution From Table 6–10,

$$\mathbf{k}_a = 4.45(520)^{-0.265}\mathbf{LN}(1, 0.058)$$

$$\bar{k}_a = 4.45(520)^{-0.265}(1) = 0.848$$

$$\hat{\sigma}_{ka} = C\bar{k}_a = (0.058)4.45(520)^{-0.265} = 0.049$$

Answer so $\mathbf{k}_a = \mathbf{LN}(0.848, 0.049)$.

The load factor $\mathbf{k}_c$ for axial and torsional loading is given by

$$(\mathbf{k}_c)_{\text{axial}} = 1.23\bar{S}_{ut}^{-0.0778}\mathbf{LN}(1, 0.125) \tag{6–73}$$

$$(\mathbf{k}_c)_{\text{torsion}} = 0.328\bar{S}_{ut}^{0.125}\mathbf{LN}(1, 0.125) \tag{6–74}$$

where $\bar{S}_{ut}$ is in kpsi. There are fewer data to study for axial fatigue. Equation (6–73) was deduced from the data of Landgraf and of Grover, Gordon, and Jackson (as cited earlier).

Torsional data are sparser, and Eq. (6–74) is deduced from data in Grover et al. Notice the mild sensitivity to strength in the axial and torsional load factor, so $\mathbf{k}_c$ in these cases is not constant. Average values are shown in the last column of Table 6–11, and as footnotes to Tables 6–12 and 6–13. Table 6–14 shows the influence of material classes on the load factor $\mathbf{k}_c$. Distortion energy theory predicts $(k_c)_{\text{torsion}} = 0.577$ for materials to which the distortion-energy theory applies. For bending, $\mathbf{k}_c = \mathbf{LN}(1, 0)$.

Table 6–11

Parameters in Marin Loading Factor

Mode of Loading	α kpsi	α MPa	β	C	Average k_c
	$k_c = \alpha \bar{S}_{ut}^{\beta}\,\mathbf{LN}(1, C)$				
Bending	1	1	0	0	1
Axial	1.23	1.43	−0.0778	0.125	0.85
Torsion	0.328	0.258	0.125	0.125	0.59

Table 6–12

Average Marin Loading Factor for Axial Load

$\bar{S}_{ut}$, kpsi	k_c^*
50	0.907
100	0.860
150	0.832
200	0.814

*Average entry 0.85.

Table 6–13

Average Marin Loading Factor for Torsional Load

$\bar{S}_{ut}$, kpsi	k_c^*
50	0.535
100	0.583
150	0.614
200	0.636

*Average entry 0.59.

Table 6–14

Average Marin Torsional Loading Factor k_c for Several Materials

Material	Range	n	$\bar{k}_c$	$\hat{\sigma}_{kc}$
Wrought steels	0.52–0.69	31	0.60	0.03
Wrought Al	0.43–0.74	13	0.55	0.09
Wrought Cu and alloy	0.41–0.67	7	0.56	0.10
Wrought Mg and alloy	0.49–0.60	2	0.54	0.08
Titanium	0.37–0.57	3	0.48	0.12
Cast iron	0.79–1.01	9	0.90	0.07
Cast Al, Mg, and alloy	0.71–0.91	5	0.85	0.09

Source: The table is an extension of P. G. Forrest, *Fatigue of Metals*, Pergamon Press, London, 1962, Table 17, p. 110, with standard deviations estimated from range and sample size using Table A–1 in J. B. Kennedy and A. M. Neville, *Basic Statistical Methods for Engineers and Scientists*, 3rd ed., Harper & Row, New York, 1986, pp. 54–55.

EXAMPLE 6–17 Estimate the Marin loading factor $\mathbf{k}_c$ for a 1–in-diameter bar that is used as follows.

(*a*) In bending. It is made of steel with $\mathbf{S}_{ut} = 100\mathbf{LN}(1, 0.035)$ kpsi, and the designer intends to use the correlation $\mathbf{S}'_e = \boldsymbol{\phi}_{0.30}\bar{S}_{ut}$ to predict $\mathbf{S}'_e$.

(*b*) In bending, but endurance testing gave $\mathbf{S}'_e = 55\mathbf{LN}(1, 0.081)$ kpsi.

(*c*) In push-pull (axial) fatigue, $\mathbf{S}_{ut} = \mathbf{LN}(86.2, 3.92)$ kpsi, and the designer intended to use the correlation $\mathbf{S}'_e = \boldsymbol{\phi}_{0.30}\bar{S}_{ut}$.

(*d*) In torsional fatigue. The material is cast iron, and $\mathbf{S}'_e$ is known by test.

Solution (*a*) Since the bar is in bending,

Answer
$$\mathbf{k}_c = (1, 0)$$

(*b*) Since the test is in bending and use is in bending,

Answer
$$\mathbf{k}_c = (1, 0)$$

(*c*) From Eq. (6–73),

Answer
$$(\mathbf{k}_c)_{ax} = 1.23(86.2)^{-0.0778}\mathbf{LN}(1, 0.125)$$
$$\bar{k}_c = 1.23(86.2)^{-0.0778}(1) = 0.870$$
$$\hat{\sigma}_{kc} = C\bar{k}_c = 0.125(0.870) = 0.109$$

(*d*) From Table 6–15, $\bar{k}_c = 0.90$, $\hat{\sigma}_{kc} = 0.07$, and

Answer
$$C_{kc} = \frac{0.07}{0.90} = 0.08$$

The temperature factor $\mathbf{k}_d$ is

$$\mathbf{k}_d = \bar{k}_d\mathbf{LN}(1, 0.11) \qquad (6\text{–}75)$$

where $\bar{k}_d = k_d$, given by Eq. (6–27), p. 291.

Finally, $\mathbf{k}_f$ is, as before, the miscellaneous factor that can come about from a great many considerations, as discussed in Sec. 6–9, where now statistical distributions, possibly from testing, are considered.

Stress Concentration and Notch Sensitivity

Notch sensitivity q was defined by Eq. (6–31), p. 295. The stochastic equivalent is

$$\mathbf{q} = \frac{\mathbf{K}_f - 1}{K_t - 1} \qquad (6\text{–}76)$$

where K_t is the theoretical (or geometric) stress-concentration factor, a deterministic quantity. A study of lines 3 and 4 of Table 20–6, will reveal that adding a scalar to (or subtracting one from) a variate $\mathbf{x}$ will affect only the mean. Also, multiplying (or dividing) by a scalar affects both the mean and standard deviation. With this in mind, we can

Table 6–15

Heywood's Parameter $\sqrt{a}$ and coefficients of variation C_{Kf} for steels

Notch Type	$\sqrt{a}(\sqrt{\text{in}})$, S_{ut} in kpsi	$\sqrt{a}(\sqrt{\text{mm}})$, S_{ut} in MPa	Coefficient of Variation C_{Kf}
Transverse hole	$5/S_{ut}$	$174/S_{ut}$	0.10
Shoulder	$4/S_{ut}$	$139/S_{ut}$	0.11
Groove	$3/S_{ut}$	$104/S_{ut}$	0.15

relate the statistical parameters of the fatigue stress-concentration factor $\mathbf{K}_f$ to those of notch sensitivity $\mathbf{q}$. It follows that

$$\mathbf{q} = \mathbf{LN}\left(\frac{\bar{K}_f - 1}{K_t - 1}, \frac{C\bar{K}_f}{K_t - 1}\right)$$

where $C = C_{Kf}$ and

$$\bar{q} = \frac{\bar{K}_f - 1}{K_t - 1}$$

$$\hat{\sigma}_q = \frac{C\bar{K}_f}{K_t - 1} \qquad (6\text{–}77)$$

$$C_q = \frac{C\bar{K}_f}{\bar{K}_f - 1}$$

The fatigue stress-concentration factor $\mathbf{K}_f$ has been investigated more in England than in the United States. For $\bar{K}_f$, consider a modified Neuber equation (after Heywood[33]), where the fatigue stress-concentration factor is given by

$$\bar{K}_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t}\dfrac{\sqrt{a}}{\sqrt{r}}} \qquad (6\text{–}78)$$

where Table 6–15 gives values of $\sqrt{a}$ and C_{Kf} for steels with transverse holes, shoulders, or grooves. Once $\mathbf{K}_f$ is described, $\mathbf{q}$ can also be quantified using the set Eqs. (6–77).

The modified Neuber equation gives the fatigue stress-concentration factor as

$$\mathbf{K}_f = \bar{K}_f \mathbf{LN}\left(1, C_{K_f}\right) \qquad (6\text{–}79)$$

EXAMPLE 6–18

Estimate $\mathbf{K}_f$ and $\mathbf{q}$ for the steel shaft given in Ex. 6–6, p. 296.

Solution

From Ex. 6–6, a steel shaft with $S_{ut} = 690$ MPa and a shoulder with a fillet of 3 mm was found to have a theoretical stress-concentration factor of $K_t \doteq 1.65$. From Table 6–15,

$$\sqrt{a} = \frac{139}{S_{ut}} = \frac{139}{690} = 0.2014\sqrt{\text{mm}}$$

[33] R. B. Heywood, *Designing Against Fatigue*, Chapman & Hall, London, 1962.

From Eq. (6–78),

$$K_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t}\dfrac{\sqrt{a}}{\sqrt{r}}} = \frac{1.65}{1 + \dfrac{2(1.65 - 1)}{1.65}\dfrac{0.2014}{\sqrt{3}}} = 1.51$$

which is 2.5 percent lower than what was found in Ex. 6–6.

From Table 6–15, $C_{Kf} = 0.11$. Thus from Eq. (6–79),

Answer
$$\mathbf{K}_f = 1.51\,\mathbf{LN}(1, 0.11)$$

From Eq. (6–77), with $K_t = 1.65$

$$\bar{q} = \frac{1.51 - 1}{1.65 - 1} = 0.785$$

$$C_q = \frac{C_{K_f}\bar{K}_f}{\bar{K}_f - 1} = \frac{0.11(1.51)}{1.51 - 1} = 0.326$$

$$\hat{\sigma}_q = C_q\bar{q} = 0.326(0.785) = 0.256$$

So,

Answer
$$\mathbf{q} = \mathbf{LN}(0.785, 0.256)$$

EXAMPLE 6–19 The bar shown in Fig. 6–37 is machined from a cold-rolled flat having an ultimate strength of $\mathbf{S}_{ut} = \mathbf{LN}(87.6, 5.74)$ kpsi. The axial load shown is completely reversed. The load amplitude is $\mathbf{F}_a = \mathbf{LN}(1000, 120)$ lbf.
(*a*) Estimate the reliability.
(*b*) Reestimate the reliability when a rotating bending endurance test shows that $\mathbf{S}'_e = \mathbf{LN}(40, 2)$ kpsi.

Solution
(*a*) From Eq. (6–70), $\mathbf{S}'_e = 0.506\bar{S}_{ut}\mathbf{LN}(1, 0.138) = 0.506(87.6)\mathbf{LN}(1, 0.138)$

$$= 44.3\mathbf{LN}(1, 0.138) \text{ kpsi}$$

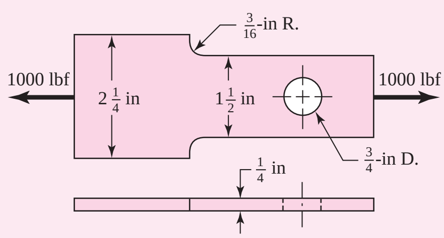
From Eq. (6–72) and Table 6–10,

$$\mathbf{k}_a = 2.67\bar{S}_{ut}^{-0.265}\mathbf{LN}(1, 0.058) = 2.67(87.6)^{-0.265}\mathbf{LN}(1, 0.058)$$

$$= 0.816\mathbf{LN}(1, 0.058)$$

$$k_b = 1 \qquad \text{(axial loading)}$$

Figure 6–37



From Eq. (6–73),

$$\mathbf{k}_c = 1.23\bar{S}_{ut}^{-0.0778}\mathbf{LN}(1, 0.125) = 1.23(87.6)^{-0.0778}\mathbf{LN}(1, 0.125)$$

$$= 0.869\mathbf{LN}(1, 0.125)$$

$$\mathbf{k}_d = \mathbf{k}_f = (1, 0)$$

The endurance strength, from Eq. (6–71), is

$$\mathbf{S}_e = \mathbf{k}_a k_b \mathbf{k}_c \mathbf{k}_d \mathbf{k}_f \mathbf{S}'_e$$

$$\mathbf{S}_e = 0.816\mathbf{LN}(1, 0.058)(1)0.869\mathbf{LN}(1, 0.125)(1)(1)44.3\mathbf{LN}(1, 0.138)$$

The parameters of $\mathbf{S}_e$ are

$$\bar{S}_e = 0.816(0.869)44.3 = 31.4 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.125^2 + 0.138^2)^{1/2} = 0.195$$

so $\mathbf{S}_e = 31.4\mathbf{LN}(1, 0.195)$ kpsi.

In computing the stress, the section at the hole governs. Using the terminology of Table A–15–1 we find $d/w = 0.50$, therefore $K_t \doteq 2.18$. From Table 6–15, $\sqrt{a} = 5/S_{ut} = 5/87.6 = 0.0571$ and $C_{kf} = 0.10$. From Eqs. (6–78) and (6–79) with $r = 0.375$ in,

$$\mathbf{K}_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t}\dfrac{\sqrt{a}}{\sqrt{r}}}\mathbf{LN}\left(1, C_{K_f}\right) = \frac{2.18}{1 + \dfrac{2(2.18 - 1)}{2.18}\dfrac{0.0571}{\sqrt{0.375}}}\mathbf{LN}(1, 0.10)$$

$$= 1.98\mathbf{LN}(1, 0.10)$$

The stress at the hole is

$$\boldsymbol{\sigma} = \mathbf{K}_f\frac{\mathbf{F}}{A} = 1.98\mathbf{LN}(1, 0.10)\frac{1000\mathbf{LN}(1, 0.12)}{0.25(0.75)}$$

$$\bar{\sigma} = 1.98\frac{1000}{0.25(0.75)}10^{-3} = 10.56 \text{ kpsi}$$

$$C_\sigma = (0.10^2 + 0.12^2)^{1/2} = 0.156$$

so stress can be expressed as $\boldsymbol{\sigma} = 10.56\mathbf{LN}(1, 0.156)$ kpsi.[34]

The endurance limit is considerably greater than the load-induced stress, indicating that finite life is not a problem. For interfering lognormal-lognormal distributions, Eq. (5–43), p. 250, gives

$$z = -\frac{\ln\left(\dfrac{\bar{S}_e}{\bar{\sigma}}\sqrt{\dfrac{1 + C_\sigma^2}{1 + C_{S_e}^2}}\right)}{\sqrt{\ln\left[\left(1 + C_{S_e}^2\right)\left(1 + C_\sigma^2\right)\right]}} = -\frac{\ln\left(\dfrac{31.4}{10.56}\sqrt{\dfrac{1 + 0.156^2}{1 + 0.195^2}}\right)}{\sqrt{\ln[(1 + 0.195^2)(1 + 0.156^2)]}} = -4.37$$

From Table A–10 the probability of failure $p_f = \Phi(-4.37) = .000\,006\,35$, and the reliability is

Answer $$R = 1 - 0.000\,006\,35 = 0.999\,993\,65$$

[34]Note that there is a simplification here. The area is *not* a deterministic quantity. It will have a statistical distribution also. However no information was given here, and so it was treated as being deterministic.

(*b*) The rotary endurance tests are described by $\mathbf{S}'_e = 40\mathbf{LN}(1, 0.05)$ kpsi whose mean is *less* than the predicted mean in part *a*. The mean endurance strength $\bar{S}_e$ is

$$\bar{S}_e = 0.816(0.869)40 = 28.4 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.125^2 + 0.05^2)^{1/2} = 0.147$$

so the endurance strength can be expressed as $\mathbf{S}_e = 28.3\mathbf{LN}(1, 0.147)$ kpsi. From Eq. (5–43),

$$z = -\frac{\ln\left(\dfrac{28.4}{10.56}\sqrt{\dfrac{1+0.156^2}{1+0.147^2}}\right)}{\sqrt{\ln[(1+0.147^2)(1+0.156^2)]}} = -4.65$$

Using Table A–10, we see the probability of failure $p_f = \Phi(-4.65) = 0.000\,001\,71$, and

$$R = 1 - 0.000\,001\,71 = 0.999\,998\,29$$

an increase! The reduction in the probability of failure is $(0.000\,001\,71 - 0.000\,006\,35)/0.000\,006\,35 = -0.73$, a reduction of 73 percent. We are analyzing an existing design, so in part (*a*) the factor of safety was $\bar{n} = \bar{S}/\bar{\sigma} = 31.4/10.56 = 2.97$. In part (*b*) $\bar{n} = 28.4/10.56 = 2.69$, a *decrease*. This example gives you the opportunity to see the role of the design factor. Given knowledge of $\bar{S}$, C_S, $\bar{\sigma}$, C_σ, and reliability (through z), the mean factor of safety (as a design factor) separates $\bar{S}$ and $\bar{\sigma}$ so that the reliability goal is achieved. Knowing $\bar{n}$ alone *says nothing about the probability of failure*. Looking at $\bar{n} = 2.97$ and $\bar{n} = 2.69$ says nothing about the respective probabilities of failure. The tests did not reduce $\bar{S}_e$ significantly, but reduced the variation C_S such that the reliability was *increased*.

When a mean design factor (or mean factor of safety) defined as $\bar{S}_e/\bar{\sigma}$ is said to be *silent* on matters of frequency of failures, it means that a scalar factor of safety by itself does not offer any information about probability of failure. Nevertheless, some engineers let the factor of safety speak up, and they can be wrong in their conclusions.

As revealing as Ex. 6–19 is concerning the meaning (and lack of meaning) of a design factor or factor of safety, let us remember that the rotary testing associated with part (*b*) changed *nothing* about the part, but only our knowledge about the part. The mean endurance limit was 40 kpsi all the time, and our adequacy assessment had to move with what was known.

Fluctuating Stresses

Deterministic failure curves that lie among the data are candidates for regression models. Included among these are the Gerber and ASME-elliptic for ductile materials, and, for brittle materials, Smith-Dolan models, which use mean values in their presentation. Just as the deterministic failure curves are located by endurance strength and ultimate tensile (or yield) strength, so too are stochastic failure curves located by $\mathbf{S}_e$ and by $\mathbf{S}_{ut}$ or $\mathbf{S}_y$. Figure 6–32, p. 320, shows a parabolic Gerber mean curve. We also need to establish a contour located one standard deviation from the mean. Since stochastic curves are most likely to be used with a radial load line we will use the equation given in Table 6–7, p. 307, expressed in terms of the strength means as

$$\bar{S}_a = \frac{r^2 \bar{S}_{ut}^2}{2\bar{S}_e}\left[-1 + \sqrt{1 + \left(\frac{2\bar{S}_e}{r\bar{S}_{ut}}\right)^2}\,\right] \tag{6–80}$$

Because of the positive correlation between $\mathbf{S}_e$ and $\mathbf{S}_{ut}$, we increment $\bar{S}_e$ by $C_{Se}\bar{S}_e$, $\bar{S}_{ut}$ by $C_{Sut}\bar{S}_{ut}$, and $\bar{S}_a$ by $C_{Sa}\bar{S}_a$, substitute into Eq. (6–80), and solve for C_{Sa} to obtain

$$C_{Sa} = \frac{(1+C_{Sut})^2}{1+C_{Se}} \frac{\left\{-1 + \sqrt{1 + \left[\dfrac{2\bar{S}_e(1+C_{Se})}{r\bar{S}_{ut}(1+C_{Sut})}\right]^2}\,\right\}}{\left[-1 + \sqrt{1 + \left(\dfrac{2\bar{S}_e}{r\bar{S}_{ut}}\right)^2}\,\right]} - 1 \tag{6–81}$$

Equation (6–81) can be viewed as an interpolation formula for C_{Sa}, which falls between C_{Se} and C_{Sut} depending on load line slope r. Note that $\mathbf{S}_a = \bar{S}_a \mathbf{LN}(1, C_{Sa})$.

Similarly, the ASME-elliptic criterion of Table 6–8, p. 308, expressed in terms of its means is

$$\bar{S}_a = \frac{r\bar{S}_y\bar{S}_e}{\sqrt{r^2\bar{S}_y^2 + \bar{S}_e^2}} \tag{6–82}$$

Similarly, we increment $\bar{S}_e$ by $C_{Se}\bar{S}_e$, $\bar{S}_y$ by $C_{Sy}\bar{S}_y$, and $\bar{S}_a$ by $C_{Sa}\bar{S}_a$, substitute into Eq. (6–82), and solve for C_{Sa}:

$$C_{Sa} = (1+C_{Sy})(1+C_{Se})\sqrt{\frac{r^2\bar{S}_y^2 + \bar{S}_e^2}{r^2\bar{S}_y^2(1+C_{Sy})^2 + \bar{S}_e^2(1+C_{Se})^2}} - 1 \tag{6–83}$$

Many *brittle* materials follow a Smith-Dolan failure criterion, written deterministically as

$$\frac{n\sigma_a}{S_e} = \frac{1 - n\sigma_m/S_{ut}}{1 + n\sigma_m/S_{ut}} \tag{6–84}$$

Expressed in terms of its means,

$$\frac{\bar{S}_a}{\bar{S}_e} = \frac{1 - \bar{S}_m/\bar{S}_{ut}}{1 + \bar{S}_m/\bar{S}_{ut}} \tag{6–85}$$

For a radial load line slope of r, we substitute $\bar{S}_a/r$ for $\bar{S}_m$ and solve for $\bar{S}_a$, obtaining

$$\bar{S}_a = \frac{r\bar{S}_{ut} + \bar{S}_e}{2}\left[-1 + \sqrt{1 + \frac{4r\bar{S}_{ut}\bar{S}_e}{(r\bar{S}_{ut} + \bar{S}_e)^2}}\,\right] \tag{6–86}$$

and the expression for C_{Sa} is

$$C_{Sa} = \frac{r\bar{S}_{ut}(1+C_{Sut}) + \bar{S}_e(1+C_{Se})}{2\bar{S}_a} \cdot \left\{-1 + \sqrt{1 + \frac{4r\bar{S}_{ut}\bar{S}_e(1+C_{Se})(1+C_{Sut})}{[r\bar{S}_{ut}(1+C_{Sut}) + \bar{S}_e(1+C_{Se})]^2}}\,\right\} - 1 \tag{6–87}$$

EXAMPLE 6–20

A rotating shaft experiences a steady torque $\mathbf{T} = 1360\mathbf{LN}(1, 0.05)$ lbf · in, and at a shoulder with a 1.1-in small diameter, a fatigue stress-concentration factor $\mathbf{K}_f = 1.50\mathbf{LN}(1, 0.11)$, $\mathbf{K}_{fs} = 1.28\mathbf{LN}(1, 0.11)$, and at that location a bending moment of $\mathbf{M} = 1260\mathbf{LN}(1, 0.05)$ lbf · in. The material of which the shaft is machined is hot-rolled 1035 with $\mathbf{S}_{ut} = 86.2\mathbf{LN}(1, 0.045)$ kpsi and $\mathbf{S}_y = 56.0\mathbf{LN}(1, 0.077)$ kpsi. Estimate the reliability using a stochastic Gerber failure zone.

Solution

Establish the endurance strength. From Eqs. (6–70) to (6–72) and Eq. (6–20), p. 288,

$$\mathbf{S}'_e = 0.506(86.2)\mathbf{LN}(1, 0.138) = 43.6\mathbf{LN}(1, 0.138) \text{ kpsi}$$

$$\mathbf{k}_a = 2.67(86.2)^{-0.265}\mathbf{LN}(1, 0.058) = 0.820\mathbf{LN}(1, 0.058)$$

$$k_b = (1.1/0.30)^{-0.107} = 0.870$$

$$\mathbf{k}_c = \mathbf{k}_d = \mathbf{k}_f = \mathbf{LN}(1, 0)$$

$$\mathbf{S}_e = 0.820\mathbf{LN}(1, 0.058)0.870(43.6)\mathbf{LN}(1, 0.138)$$

$$\bar{S}_e = 0.820(0.870)43.6 = 31.1 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.138^2)^{1/2} = 0.150$$

and so $\mathbf{S}_e = 31.1\mathbf{LN}(1, 0.150)$ kpsi.

Stress (in kpsi):

$$\sigma_a = \frac{32\mathbf{K}_f\mathbf{M}_a}{\pi d^3} = \frac{32(1.50)\mathbf{LN}(1, 0.11)1.26\mathbf{LN}(1, 0.05)}{\pi(1.1)^3}$$

$$\bar{\sigma}_a = \frac{32(1.50)1.26}{\pi(1.1)^3} = 14.5 \text{ kpsi}$$

$$C_{\sigma a} = (0.11^2 + 0.05^2)^{1/2} = 0.121$$

$$\boldsymbol{\tau}_m = \frac{16\mathbf{K}_{fs}\mathbf{T}_m}{\pi d^3} = \frac{16(1.28)\mathbf{LN}(1, 0.11)1.36\mathbf{LN}(1, 0.05)}{\pi(1.1)^3}$$

$$\bar{\tau}_m = \frac{16(1.28)1.36}{\pi(1.1)^3} = 6.66 \text{ kpsi}$$

$$C_{\tau m} = (0.11^2 + 0.05^2)^{1/2} = 0.121$$

$$\bar{\sigma}'_a = \left(\bar{\sigma}_a^2 + 3\bar{\tau}_a^2\right)^{1/2} = [14.5^2 + 3(0)^2]^{1/2} = 14.5 \text{ kpsi}$$

$$\bar{\sigma}'_m = \left(\bar{\sigma}_m^2 + 3\bar{\tau}_m^2\right)^{1/2} = [0 + 3(6.66)^2]^{1/2} = 11.54 \text{ kpsi}$$

$$r = \frac{\bar{\sigma}'_a}{\bar{\sigma}'_m} = \frac{14.5}{11.54} = 1.26$$

Strength: From Eqs. (6–80) and (6–81),

$$\bar{S}_a = \frac{1.26^2 86.2^2}{2(31.1)}\left\{-1 + \sqrt{1 + \left[\frac{2(31.1)}{1.26(86.2)}\right]^2}\right\} = 28.9 \text{ kpsi}$$

$$C_{Sa} = \frac{(1+0.045)^2}{1+0.150} \frac{-1+\sqrt{1+\left[\dfrac{2(31.1)(1+0.15)}{1.26(86.2)(1+0.045)}\right]^2}}{-1+\sqrt{1+\left[\dfrac{2(31.1)}{1.26(86.2)}\right]^2}} - 1 = 0.134$$

Reliability: Since $\mathbf{S}_a = 28.9\mathbf{LN}(1, 0.134)$ kpsi and $\boldsymbol{\sigma}'_a = 14.5\mathbf{LN}(1, 0.121)$ kpsi, Eq. (5–43), p. 250, gives

$$z = -\frac{\ln\left(\dfrac{\bar{S}_a}{\bar{\sigma}_a}\sqrt{\dfrac{1+C^2_{\sigma_a}}{1+C^2_{S_a}}}\right)}{\sqrt{\ln\left[\left(1+C^2_{S_a}\right)\left(1+C^2_{\sigma_a}\right)\right]}} = -\frac{\ln\left(\dfrac{28.9}{14.5}\sqrt{\dfrac{1+0.121^2}{1+0.134^2}}\right)}{\sqrt{\ln[(1+0.134^2)(1+0.121^2)]}} = -3.83$$

From Table A–10 the probability of failure is $p_f = 0.000\,065$, and the reliability is, against fatigue,

Answer

$$R = 1 - p_f = 1 - 0.000\,065 = 0.999\,935$$

The chance of first-cycle yielding is estimated by interfering $\mathbf{S}_y$ with $\boldsymbol{\sigma}'_{\max}$. The quantity $\boldsymbol{\sigma}'_{\max}$ is formed from $\boldsymbol{\sigma}'_a + \boldsymbol{\sigma}'_m$. The mean of $\boldsymbol{\sigma}'_{\max}$ is $\bar{\sigma}'_a + \bar{\sigma}'_m = 14.5 + 11.54 = 26.04$ kpsi. The coefficient of variation of the sum is 0.121, since both COVs are 0.121, thus $C_{\sigma\,\max} = 0.121$. We interfere $\mathbf{S}_y = 56\mathbf{LN}(1, 0.077)$ kpsi with $\boldsymbol{\sigma}'_{\max} = 26.04\mathbf{LN}(1, 0.121)$ kpsi. The corresponding z variable is

$$z = -\frac{\ln\left(\dfrac{56}{26.04}\sqrt{\dfrac{1+0.121^2}{1+0.077^2}}\right)}{\sqrt{\ln[(1+0.077^2)(1+0.121^2)]}} = -5.39$$

which represents, from Table A–10, a probability of failure of approximately 0.0^7358 [which represents $3.58(10^{-8})$] of first-cycle yield in the fillet.

The probability of observing a fatigue failure exceeds the probability of a yield failure, something a deterministic analysis does not foresee and in fact could lead one to expect a yield failure should a failure occur. Look at the $\boldsymbol{\sigma}'_a\mathbf{S}_a$ interference and the $\boldsymbol{\sigma}'_{\max}\mathbf{S}_y$ interference and examine the z expressions. These control the relative probabilities. A deterministic analysis is oblivious to this and can mislead. Check your statistics text for events that are not mutually exclusive, but are independent, to quantify the probability of failure:

$$\begin{aligned}
p_f &= p(\text{yield}) + p(\text{fatigue}) - p(\text{yield and fatigue})\\
&= p(\text{yield}) + p(\text{fatigue}) - p(\text{yield})\,p(\text{fatigue})\\
&= 0.358(10^{-7}) + 0.65(10^{-4}) - 0.358(10^{-7})0.65(10^{-4}) = 0.650(10^{-4})\\
R &= 1 - 0.650(10^{-4}) = 0.999\,935
\end{aligned}$$

against either or both modes of failure.

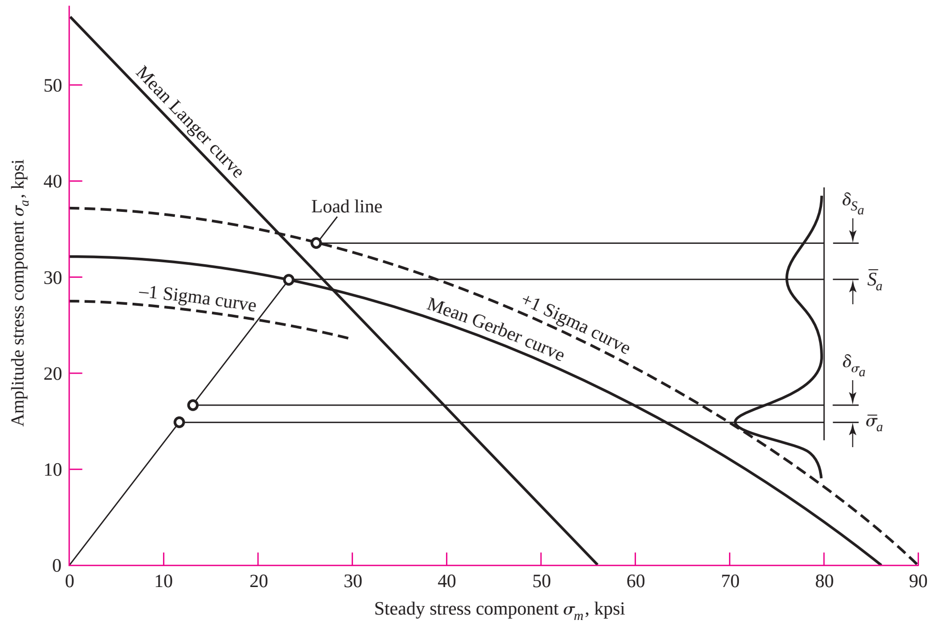


Figure 6–38

Designer's fatigue diagram for Ex. 6–20.

Examine Fig. 6–38, which depicts the results of Ex. 6–20. The problem distribution of $\mathbf{S}_e$ was compounded of historical experience with $\mathbf{S}'_e$ and the uncertainty manifestations due to features requiring Marin considerations. The Gerber "failure zone" displays this. The interference with load-induced stress predicts the risk of failure. If additional information is known (R. R. Moore testing, with or without Marin features), the stochastic Gerber can accommodate to the information. Usually, the accommodation to additional test information is movement and contraction of the failure zone. In its own way the stochastic failure model accomplishes more precisely what the deterministic models and conservative postures intend. Additionally, stochastic models can estimate the probability of failure, something a deterministic approach cannot address.

The Design Factor in Fatigue

The designer, in envisioning how to execute the geometry of a part subject to the imposed constraints, can begin making a priori decisions without realizing the impact on the design task. Now is the time to note how these things are related to the reliability goal.

The mean value of the design factor is given by Eq. (5–45), repeated here as

$$\bar{n} = \exp\left[-z\sqrt{\ln\left(1 + C_n^2\right)} + \ln\sqrt{1 + C_n^2}\right] \doteq \exp[C_n(-z + C_n/2)] \qquad (6\text{–}88)$$

in which, from Table 20–6 for the quotient $\mathbf{n} = \mathbf{S}/\boldsymbol{\sigma}$,

$$C_n = \sqrt{\frac{C_S^2 + C_\sigma^2}{1 + C_\sigma^2}}$$

where C_S is the COV of the significant strength and C_σ is the COV of the significant stress at the critical location. Note that $\bar{n}$ is a function of the reliability goal (through z) and the COVs of the strength and stress. There are no means present, just measures of variability. The nature of C_S in a fatigue situation may be C_{Se} for fully reversed loading, or C_{Sa} otherwise. Also, experience shows $C_{Se} > C_{Sa} > C_{Sut}$, so C_{Se} can be used as a conservative estimate of C_{Sa}. If the loading is bending or axial, the form of

$\boldsymbol{\sigma}'_a$ might be

$$\boldsymbol{\sigma}'_a = \mathbf{K}_f \frac{\mathbf{M}_a c}{I} \qquad \text{or} \qquad \boldsymbol{\sigma}'_a = \mathbf{K}_f \frac{\mathbf{F}}{A}$$

respectively. This makes the COV of $\boldsymbol{\sigma}'_a$, namely $C_{\sigma'_a}$, expressible as

$$C_{\sigma'_a} = \left(C_{Kf}^2 + C_F^2\right)^{1/2}$$

again a function of variabilities. The COV of $\mathbf{S}_e$, namely C_{Se}, is

$$C_{Se} = \left(C_{ka}^2 + C_{kc}^2 + C_{kd}^2 + C_{kf}^2 + C_{Se'}^2\right)^{1/2}$$

again, a function of variabilities. An example will be useful.

EXAMPLE 6–21

A strap to be made from a cold-drawn steel strip workpiece is to carry a fully reversed axial load $\mathbf{F} = \mathbf{LN}(1000, 120)$ lbf as shown in Fig. 6–39. Consideration of adjacent parts established the geometry as shown in the figure, except for the thickness t. Make a decision as to the magnitude of the design factor if the reliability goal is to be 0.999 95, then make a decision as to the workpiece thickness t.

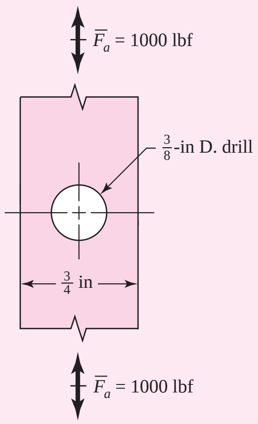


Figure 6–39

A strap with a thickness t is subjected to a fully reversed axial load of 1000 lbf. Example 6–21 considers the thickness necessary to attain a reliability of 0.999 95 against a fatigue failure.

Solution

Let us take each a priori decision and note the consequence:

A Priori Decision	Consequence
Use 1018 CD steel	$\bar{S}_{ut} = 87.6$ kpsi, $C_{Sut} = 0.0655$
Function:	
Carry axial load	$C_F = 0.12$, $C_{kc} = 0.125$
$R \ge 0.999\ 95$	$z = -3.891$
Machined surfaces	$C_{ka} = 0.058$
Hole critical	$C_{Kf} = 0.10$, $C_{\sigma'_a} = (0.10^2 + 0.12^2)^{1/2} = 0.156$
Ambient temperature	$C_{kd} = 0$
Correlation method	$C_{S'_e} = 0.138$
Hole drilled	$C_{Se} = (0.058^2 + 0.125^2 + 0.138^2)^{1/2} = 0.195$
	$C_n = \sqrt{\dfrac{C_{Se}^2 + C_{\sigma'_a}^2}{1 + C_{\sigma'_a}^2}} = \sqrt{\dfrac{0.195^2 + 0.156^2}{1 + 0.156^2}} = 0.2467$
	$\bar{n} = \exp\left[-(-3.891)\sqrt{\ln(1 + 0.2467^2)} + \ln\sqrt{1 + 0.2467^2}\right]$
	$= 2.65$

These eight a priori decisions have quantified the mean design factor as $\bar{n} = 2.65$. Proceeding deterministically hereafter we write

$$\sigma'_a = \frac{\bar{S}_e}{\bar{n}} = \bar{K}_f \frac{\bar{F}}{(w - d)t}$$

from which

$$t = \frac{\bar{K}_f \bar{n} \bar{F}}{(w - d)\bar{S}_e} \tag{1}$$

To evaluate the preceding equation we need $\bar{S}_e$ and $\bar{K}_f$. The Marin factors are

$$\mathbf{k}_a = 2.67\bar{S}_{ut}^{-0.265}\mathbf{LN}(1, 0.058) = 2.67(87.6)^{-0.265}\mathbf{LN}(1, 0.058)$$
$$\bar{k}_a = 0.816$$
$$k_b = 1$$
$$\mathbf{k}_c = 1.23\bar{S}_{ut}^{-0.078}\mathbf{LN}(1, 0.125) = 0.868\mathbf{LN}(1, 0.125)$$
$$\bar{k}_c = 0.868$$
$$\bar{k}_d = \bar{k}_f = 1$$

and the endurance strength is

$$\bar{S}_e = 0.816(1)(0.868)(1)(1)0.506(87.6) = 31.4 \text{ kpsi}$$

The hole governs. From Table A–15–1 we find $d/w = 0.50$, therefore $K_t = 2.18$. From Table 6–15 $\sqrt{a} = 5/\bar{S}_{ut} = 5/87.6 = 0.0571$, $r = 0.1875$ in. From Eq. (6–78) the fatigue stress-concentration factor is

$$\bar{K}_f = \frac{2.18}{1 + \dfrac{2(2.18 - 1)}{2.18}\dfrac{0.0571}{\sqrt{0.1875}}} = 1.91$$

The thickness t can now be determined from Eq. (1)

$$t \geq \frac{\bar{K}_f \bar{n} \bar{F}}{(w - d)S_e} = \frac{1.91(2.65)1000}{(0.75 - 0.375)31\,400} = 0.430 \text{ in}$$

Use $\frac{1}{2}$-in-thick strap for the workpiece. The $\frac{1}{2}$-in thickness attains and, in the rounding to available nominal size, exceeds the reliability goal.

The example demonstrates that, for a given reliability goal, the fatigue design factor that facilitates its attainment is decided by the variabilities of the situation. Furthermore, the necessary design factor is not a constant independent of the way the concept unfolds. Rather, it is a function of a number of seemingly unrelated a priori decisions that are made in giving definition to the concept. The involvement of stochastic methodology can be limited to defining the necessary design factor. In particular, in the example, the design factor is not a function of the design variable t; rather, t follows from the design factor.