Appendix **A**

Introduction to Vector and Matrix Algebra

In this appendix, our aim is to present definitions and elementary operations of vectors and matrices necessary for power system analysis.

A.1 Vectors

A vector x is defined as an ordered set of numbers (real or complex), i.e.

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{A.1}$$

$x_1, \ldots, x_n$ are known as the components of the vector x. Thus, the vector x is a n-dimensional column vector. Sometimes transposed form of (A.1) is found to be more convenient and is written as the row vector.

$$\boldsymbol{x}^T \triangleq [x_1, x_2, \ldots, x_n] \tag{A.2}$$

Some Special Vectors

The null vector **0** is one whose each component is zero, i.e.

$$\mathbf{0} \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The sum vector **1** has each of its components equal to unity, i.e.

$$\mathbf{1} \triangleq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The unit vector e_k is defined as the vector whose kth component is unity and the rest of the components are zero, i.e.

$$e_k \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} k\text{th component}$$

Some Fundamental Vector Operations

The two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are known as equal if, and only if $x_k = y_k$ for $k = 1, 2, ..., n$. Then we say

$$\boldsymbol{x} = \boldsymbol{y}$$

The product of a vector by a scalar is carried out by multiplying each component of the vector by that scalar, i.e.

$$\alpha\boldsymbol{x} = \boldsymbol{x}\alpha \triangleq \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

If a vector $\boldsymbol{y}$ is to be added to or subtracted from another vector $\boldsymbol{x}$ of the same dimension, then each component of the resulting vector will consist of addition or subtraction of the corresponding components of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, i.e.

$$\boldsymbol{x} \pm \boldsymbol{y} \triangleq \begin{bmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{bmatrix}$$

The following properties are applicable to the vector algebra:

$$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$$
$$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$$
$$\alpha_1(\alpha_2 \boldsymbol{x}) = (\alpha_1 \alpha_2)\boldsymbol{x}$$
$$(\alpha_1 + \alpha_2)\boldsymbol{x} = \alpha_1\boldsymbol{x} + \alpha_2\boldsymbol{x}$$
$$\alpha(\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y} + \alpha\boldsymbol{z}$$
$$\boldsymbol{0}\boldsymbol{x} = \boldsymbol{0}$$

The multiplication of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of same dimensions results in a very important product known as *inner* or *scalar* product*, i.e.

$$\boldsymbol{x}^T\boldsymbol{y} \triangleq \sum_{i=1}^{n} x_i y_i \triangleq \boldsymbol{y}^T\boldsymbol{x} \tag{A.3}$$

Also, it is interesting to note that

$$\boldsymbol{x}^T\boldsymbol{x} = |\,\boldsymbol{x}\,|^2 \tag{A.4}$$

$$\cos\phi \triangleq \frac{\boldsymbol{x}^T\boldsymbol{y}}{|\boldsymbol{x}||\boldsymbol{y}|} \tag{A.5}$$

where ϕ is angle between vectors, $|\,\boldsymbol{x}\,|$ and $|\,\boldsymbol{y}\,|$ are the geometric lengths of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Two non-zero vectors are said to be *orthogonal*, if

$$\boldsymbol{x}^T\boldsymbol{y} = 0 \tag{A.6}$$

A.2 Matrices

Definitions

Matrix

An $m \times n$ (or m, n) matrix is an ordered rectangular array of elements which may be real numbers, complex numbers, functions or operators.

$$\boldsymbol{A} \triangleq \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \tag{A.7}$$

The matrix is a rectangular array of mn elements.

a_{ij} denotes the (i, j)th element, i.e. the element located in the ith row and the jth column. The matrix $\boldsymbol{A}$ has m rows and n columns and is said to be of order $m \times n$.

When $m = n$, i.e. the number of rows is equal to that of columns, the matrix is said to be a *square matrix* of order n.

An $m \times 1$ matrix, i.e. a matrix having only one column is called a *column vector*. An $1 \times n$ matrix, i.e. a matrix having only one row is called a *row vector*.

Diagonal matrix

A diagonal matrix is a square matrix whose elements off the main diagonal are all zeros ($a_{ij} = 0$ for $i \neq j$).

Example

$$\boldsymbol{D} \triangleq \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

* Sometimes inner product is also represented by the following alternative forms $x \times y$, (x, y), $<x, y>$.

Null matrix

If all the elements of the square matrix are zero, the matrix is a *null* or *zero* matrix.

Example

$$\mathbf{0} \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ \ldots & \cdots & \cdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{A.8}$$

Unit (Identity) matrix

A *unit matrix* ***I*** is a diagonal matrix with all diagonal elements equal to unity. If a unit matrix is multiplied by a constant (λ), the resulting matrix is a diagonal matrix with all diagonal elements equal to λ. This matrix is known as a *scalar* matrix.

Example

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$= 4 \times 4$ unit matrix $\qquad = 3 \times 3$ scalar matrix

Determinant of a matrix

For each square matrix, there exists a determinant which is formed by taking the determinant of the elements of the matrix.

For example, if

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} \tag{A.9}$$

then

$$\det(\boldsymbol{A}) = |\boldsymbol{A}| = 2 \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 2(8) + (-6) + (-5) = 5 \tag{A.10}$$

Transpose of a matrix

The transpose of matrix ***A*** denoted by $\boldsymbol{A}^T$ is the matrix formed by interchanging the rows and columns of ***A***.

Note that

$$(\boldsymbol{A}^T)^T = \boldsymbol{A}$$

Symmetric matrix

A square matrix is symmetric, if it is equal to its transpose, i.e.

$$\boldsymbol{A}^T = \boldsymbol{A}$$

Notice that the matrix $\boldsymbol{A}$ of Eq. (A.9) is a symmetric matrix.

Minor

The minor M_{ij} of an $n \times n$ matrix is the determinant of $(n - 1) \times (n - 1)$ matrix formed by deleting the ith row and the jth column of the $n \times n$ matrix.

Cofactor

The cofactor A_{ij} of element a_{ij} of the matrix $\boldsymbol{A}$ is defined as

$$A_{ij} = (-1)^{i+j}\boldsymbol{M}_{ij}$$

Adjoint matrix

The adjoint matrix of a square matrix $\boldsymbol{A}$ is found by replacing each element a_{ij} of matrix $\boldsymbol{A}$ by its cofactor $\boldsymbol{A}_{ij}$ and then transposing.

For example, if $\boldsymbol{A}$ is given by Eq. (A.9), then

$$\text{adj}\,\boldsymbol{A} = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 8 & 6 & -5 \\ 6 & 7 & -5 \\ -5 & -5 & 5 \end{bmatrix}^T = \begin{bmatrix} 8 & 6 & -5 \\ 6 & 7 & -5 \\ -5 & -5 & 5 \end{bmatrix} \quad \text{(A.11)}$$

Singular and non-singular matrices

A square matrix is called singular, if its associated determinant is zero, and non-singular, if its associated determinant is non-zero.

A.3 Elementary Matrix Operations

Equality of matrices

Two matrices $\boldsymbol{A}(m \times n)$ and $\boldsymbol{B}(m \times n)$ are said to be equal, if and only if

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, ..., m, j = 1, 2, ..., n$$

Then we write

$$\boldsymbol{A} = \boldsymbol{B}$$

Multiplication of a matrix by a scalar

A matrix is multiplied by a scalar α if all the mn elements are multiplied by α, i.e.

$$\alpha \boldsymbol{A} = \boldsymbol{A}\alpha = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \cdots & \cdots & \cdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{bmatrix} \tag{A.12}$$

Addition (or subtraction) of matrices

To add (or subtract) two matrices of the same order (same number of rows, and same number of columns), simply add (or subtract) the corresponding elements of the two matrices, i.e. when two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same order are added, a new matrix C results such that

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B};$$

whose ijth element equals

$$c_{ij} = a_{ij} + b_{ij}$$

Example

Let

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

then

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}$$

Addition and subtraction are defined only for matrices of the same order.

The following laws hold for addition:

1. The *commutative law*: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
2. *The associative law*: $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$

Further

$$(\boldsymbol{A} \pm \boldsymbol{B})^T = \boldsymbol{A}^T \pm \boldsymbol{B}^T$$

Matrix Multiplication

The product of two matrices $\boldsymbol{A} \times \boldsymbol{B}$ is defined if $\boldsymbol{A}$ has the same number of columns as the number of rows in $\boldsymbol{B}$. The matrices are then said to be *conformable*. If a matrix $\boldsymbol{A}$ is of order $m \times n$ and $\boldsymbol{B}$ is an $n \times q$ matrix, the product $\boldsymbol{C} = \boldsymbol{AB}$ will be an $m \times q$ matrix. The element c_{ij} of the product is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{A.13}$$

Thus, the elements c_{ij} are obtained by multiplying the elements of the ith row of $\boldsymbol{A}$ with the corresponding elements of the jth column of $\boldsymbol{B}$ and then summing these elements products.

For example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where

$$c_{11} = a_{11}\, b_{11} + a_{12}\, b_{21}$$
$$c_{12} = a_{11}\, b_{12} + a_{12}\, b_{22}$$
$$c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21}$$
$$c_{22} = a_{21}\, b_{12} + a_{22}\, b_{22}$$

If the product $\boldsymbol{AB}$ is defined, the product $\boldsymbol{BA}$ may or may not be defined. Even if $\boldsymbol{BA}$ is defined, the resulting products of $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are not, in general, equal. Thus, it is important to note that in general matrix multiplication is not commutative, i.e.

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

The associative and distributive laws hold for matrix multiplication (when the appropriate operations are defined), i.e.

Associative law: $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{ABC}$

Distributive law: $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$

Example A.1 Given the two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

Find $\boldsymbol{AB}$ and $\boldsymbol{BA}$

Solution

$\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable ($\boldsymbol{A}$ has two columns and $\boldsymbol{B}$ has two rows); thus, we have

$$\boldsymbol{AB} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 9 \\ 0 & 2 & 1 \end{bmatrix}; \boldsymbol{BA} = \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$$

A matrix remains unaffected, if a null matrix, defined by Eq. (A.8) is added to it, i.e.

$$\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$$

If a null matrix is multiplied to another matrix $\boldsymbol{A}$, the result is a null matrix

$$\boldsymbol{A0} = \boldsymbol{0A} = \boldsymbol{0}$$

Also

$$A - A = 0$$

Note that equation $\boldsymbol{AB} = \mathbf{0}$ does not mean that either $\boldsymbol{A}$ or $\boldsymbol{B}$ necessarily has to be a null matrix, e.g.

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiplication of any matrix by a unit matrix results in the original matrix, i.e.

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$

The transpose of the product of two matrices is the product of their transposes in reverse order, i.e.

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$$

The concept of matrix multiplication assists in the solution of simultaneous linear algebraic equations. Consider such a set of equations

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= c_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= c_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= c_m \end{aligned} \tag{A.14}$$

or

$$\sum_{i=1}^{n} a_{ij} x_j = c_i; \quad i = 1, 2, \ldots, m$$

Using the rules of matrix multiplication defined above, Eqs (A.14) can be written in the compact notation as

$$\boldsymbol{Ax} = \boldsymbol{c} \tag{A.15}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

It is clear that the *vector-matrix* Eq. (A.15) is a useful shorthand representation of the set of linear algebraic equations (A.14).

Matrix Inversion

Division does not exist as such in matrix algebra. However, if $\boldsymbol{A}$ is a square non-singular matrix, its inverse ($\boldsymbol{A}^{-1}$) is defined by the relation

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \tag{A.16}$$

The conventional method for obtaining an inverse is to use the following relation

$$\boldsymbol{A}^{-1} = \frac{\text{adj } \boldsymbol{A}}{\det \boldsymbol{A}} \tag{A.17}$$

It is easy to prove that the inverse is unique

The following are the important properties characterising the inverse:

$$\begin{aligned} (\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \\ (\boldsymbol{A}^{-1})^T &= (\boldsymbol{A}^T)^{-1} \\ (\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \end{aligned} \tag{A.18}$$

Example If $\boldsymbol{A}$ is given by Eq. (A.9), then from Eqs (A.10), (A.11), (A.17), we get

$$\boldsymbol{A}^{-1} = \frac{\text{adj } \boldsymbol{A}}{\det \boldsymbol{A}} = \frac{1}{5}\begin{bmatrix} 8 & 6 & -5 \\ 6 & 7 & -5 \\ -5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} 8/5 & 6/5 & -1 \\ 6/5 & 7/5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

A.4 Scalar and Vector Functions

A scalar function of n scalar variables is defined as

$$y \triangleq f(x_1, x_2, \ldots, x_n) \tag{A.19}$$

It can be written as a scalar function of a vector variable $\boldsymbol{x}$, i.e.

$$y = f(\boldsymbol{x}) \tag{A.20}$$

where $\boldsymbol{x}$ is an n-dimension vector,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In general, a scalar function could be a function of several vector variables, e.g.

$$y = f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{A.21}$$

where $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{p}$ are vectors of various dimensions.

A vector function is defined as

$$\boldsymbol{y} = \boldsymbol{f}(x) \triangleq \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix} \tag{A.22}$$

In general, a vector function is a function of several vector variables, e.g.

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{A.23}$$

A.5 Derivatives of Scalar and Vector Functions

A derivative of a scalar function (Eq. A.20) with respect to a vector variable $\boldsymbol{x}$ is defined as

$$\frac{\partial f}{\partial \boldsymbol{x}} \triangleq \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix} \tag{A.24}$$

It may be noted that the derivative of a scalar function with respect to a vector of dimension n is a vector of the same dimension.

The derivative of a vector function (Eq. A.22) with respect to a vector variable $\boldsymbol{x}$ is defined as

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \triangleq \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{A.25}$$

$$= \begin{bmatrix} \left[\dfrac{\partial f_1}{\partial \boldsymbol{x}}\right]^T \\ \left[\dfrac{\partial f_2}{\partial \boldsymbol{x}}\right]^T \\ \vdots \\ \left[\dfrac{\partial f_m}{\partial \boldsymbol{x}}\right]^T \end{bmatrix} \tag{A.26}$$

Consider now a scalar function defined as

$$\boldsymbol{s} = \lambda^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{A.27}$$

$$= \lambda_1 f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \lambda_2 f_2(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \cdot + \lambda_m f_m(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{A.28}$$

Let us find $\frac{\partial s}{\partial \lambda}$. According to Eq. (A.24), we can write

$$\frac{\partial s}{\partial \lambda} = \begin{bmatrix} f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \\ f_2(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \\ \vdots \\ f_m(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{A.29}$$

Let us now find $\frac{\partial s}{\partial \boldsymbol{x}}$. According to Eq. (A.24), we can write

$$\frac{\partial s}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial s}{\partial x_1} \\ \frac{\partial s}{\partial x_2} \\ \vdots \\ \frac{\partial s}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \ldots + \lambda_m \frac{\partial f_m}{\partial x_1} \\ \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} + \ldots + \lambda_m \frac{\partial f_m}{\partial x_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1 \frac{\partial f_1}{\partial x_n} + \lambda_2 \frac{\partial f_2}{\partial x_n} + \ldots + \lambda_m \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial s}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$= \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]^T \lambda \tag{A.30}$$

References

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