Matrix Algebra

A matrix is a rectangular array of scalars, also called as *elements* or *entries*. The general notation of a matrix is exemplified as follows:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

We generally use a bold upper-case Roman letter for the matrix and the corresponding lower case letter case for its scalar elements. Each of the elements is represented by means of subscripts denoting the row and column to which it belongs. The number of rows and the number of columns determines the order (or dimension) of a matrix. The matrix represented here has an order $m \times n$: a total of m rows and n columns. To illustrate, suppose that a chain has four stores in New Delhi, each of which deals in five products. The stock (in '000 Rs) of each of the products in each of the stores as on December 31, 2016 may be shown below as matrix S.

$$\begin{array}{ccc} & & \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \boldsymbol{S} = & \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} & \begin{bmatrix} 25 & 12 & 23 & 32 & 14 \\ 41 & 84 & 25 & 63 & 19 \\ 10 & 47 & 26 & 85 & 58 \\ 8 & 24 & 12 & 58 & 41 \end{bmatrix} \end{array}$$

This matrix is of the order 4×5, wherein stores S_1, S_2, S_3 and S_4 are represented in rows and the products P_1, P_2, P_3, P_4 and P_5 in columns. Here different elements may be expressed as s_{ij}. For example, $s_{23} = 25$ indicates that in the store S_2, the stock of the product P_3 is equal to Rs 25 thousand.

When the number of rows and the number of columns are equal so that $m = n$, the matrix is called a *square matrix* and when $m \neq n$, the matrix is *rectangular*. Further, a matrix for which m = 1 is a single row of n elements is known as a *row vector* while a matrix with a single column, which has m elements in one column is a termed as a *column vector*.

Transpose of a Matrix If A is a matrix of the order $m \times n$, then the transpose of A, denoted as A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns of matrix A. Thus, to write the transpose of a matrix, the first row elements are written as column 1, the second row elements are written as column 2, ... and so on. The transpose of a given matrix may be expressed as $A^T = [a_{ij}]^T = [a_{ji}]_{n\times m}$. To illustrate, the transpose of matrix S considered earlier, S^T, is given here:

$$\boldsymbol{S^T} = \begin{bmatrix} 25 & 41 & 10 & 8 \\ 12 & 84 & 47 & 24 \\ 23 & 25 & 26 & 12 \\ 32 & 63 & 85 & 58 \\ 14 & 19 & 58 & 41 \end{bmatrix}$$

Matrix Operations

The various operations that may be performed on matrices are discussed here.

Matrix Addition Two matrices A and B can be added to obtain matrix C when all the three of them are of the same order and $c_{ij} = a_{ij} + b_{ij}$, for all ij. Thus, corresponding elements are added in the addition of matrices. Further, matrix addition is both commutative and associative so that $A + B = B + A$, and $(A + B) + C = A + (B + C)$.

Matrix Subtraction Like for addition, the conformability requirement for matrix subtraction is that the matrices involved are of the same order. Thus, we have $A – B = D$ if and only if A, B, and D are conformable and $d_{ij} = a_{ij} – b_{ij}$.

Example 1 For a company with several outlets selling different products, the following opening stock situation is given (the rows indicate outlets and the columns show products):

$$A = \begin{bmatrix} 120 & 110 & 90 & 150 \\ 200 & 180 & 210 & 110 \\ 175 & 190 & 160 & 80 \\ 140 & 170 & 180 & 140 \end{bmatrix}$$

Deliveries, D are made to the outlets during the month are shown below:

$$D = \begin{bmatrix} 140 & 120 & 150 & 110 \\ 125 & 130 & 110 & 160 \\ 115 & 100 & 140 & 170 \\ 160 & 140 & 110 & 150 \end{bmatrix}$$

A monthly sales report for the company is:

$$S = \begin{bmatrix} 170 & 225 & 140 & 250 \\ 310 & 290 & 315 & 260 \\ 280 & 230 & 280 & 220 \\ 300 & 300 & 275 & 205 \end{bmatrix}$$

Obtain the total stock, T after deliveries, and (b) closing stock, C.

(a) The matrix T would be obtained by adding the matrices A and D. The result is:

$$T = A + D = \begin{bmatrix} 120 & 110 & 90 & 150 \\ 200 & 180 & 210 & 110 \\ 175 & 190 & 160 & 80 \\ 140 & 170 & 180 & 140 \end{bmatrix} + \begin{bmatrix} 140 & 120 & 150 & 110 \\ 125 & 130 & 110 & 160 \\ 115 & 100 & 140 & 170 \\ 160 & 140 & 110 & 150 \end{bmatrix} = \begin{bmatrix} 260 & 230 & 240 & 260 \\ 325 & 310 & 320 & 270 \\ 290 & 290 & 300 & 250 \\ 300 & 310 & 290 & 290 \end{bmatrix}$$

The elements in matrix T are obtained by adding corresponding elements of A and D. For example, 120 + 140 = 260, 110 + 120 = 230, etc.

(b) Similarly, matrix C would be obtained by subtracting matrix S from matrix T as shown here:

$$C = T - S = \begin{bmatrix} 260 & 230 & 240 & 260 \\ 325 & 310 & 320 & 270 \\ 290 & 290 & 300 & 250 \\ 300 & 310 & 290 & 290 \end{bmatrix} - \begin{bmatrix} 170 & 225 & 140 & 250 \\ 310 & 290 & 315 & 260 \\ 280 & 230 & 280 & 220 \\ 300 & 300 & 275 & 205 \end{bmatrix} = \begin{bmatrix} 90 & 5 & 100 & 10 \\ 15 & 20 & 5 & 10 \\ 10 & 60 & 20 & 30 \\ 0 & 10 & 15 & 85 \end{bmatrix}$$

The elements in matrix C are obtained by subtracting corresponding elements of S from corresponding elements T.

Matrix Multiplication Two matrices *A* and *B* can be multiplied as *AB* = *C*, if and only if the number of columns in matrix *A* is equal to the number of rows in matrix *B*. The resulting matrix *C* would have number of rows as of matrix *A* and the number of columns as in matrix *B*, so that if matrix *A* is of the order $m \times n$ and matrix *B* of $n \times p$, then the matrix *C* would be of the order $m \times p$. From this requirement of conformability in multiplication it is evident that the order of multiplication is important in matrix multiplication. It may be observed that matrices *A*, $[a_{ij}]_{m\times n}$, and *B*, $[b_{ij}]_{n\times p}$ can be multiplied as *AB* but not as *BA*. Thus, *BA* is not defined in this case. It may be noted that even if two matrices *A* and *B* are conformable for multiplication both as *AB* and *BA* (this would happen when both of them are square matrices of the same order), in general $AB \neq BA$. Therefore, in multiplication of matrices, it is useful to distinguish between pre- and post- multiplication of one matrix by another. In the matrix multiplication *AB* = *C*, we say that *B* is pre-multiplied by *A* or that *A* is post-multiplied by *B* to form matrix *C*.

Once the conformability for multiplication is established, the element in a given row and column of the resulting matrix is determined as the sum of products of successive elements in the corresponding row of the first matrix and the corresponding column of the second matrix. To illustrate, in *AB* = *C*, the element c_{23} would be obtained by multiplying the second row elements of A by the elements in the third column of B (first with first, second with second etc) and adding those products up.

Example 2 Obtain the multiplication AB given the following matrices:

$$A = \begin{pmatrix} 4 & 7 \\ 9 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 8 & 5 \\ 2 & 6 & 7 \end{pmatrix}$$

Here matrix *A* is of the order 2 × 2 while *B* is of the order 2 × 3. Hence they are conformable for multiplication as *AB*. Different elements of the matrix *C* are obtained as:

$$\mathrm{C} = \begin{pmatrix} 4\times 3 + 7\times 2 & 4\times 8 + 7\times 6 & 4\times 5 + 7\times 7 \\ 9\times 3 + 1\times 2 & 9\times 8 + 1\times 6 & 9\times 5 + 1\times 7 \end{pmatrix}$$

Accordingly,

$$\boldsymbol{C} = \begin{pmatrix} 26 & 74 & 69 \\ 29 & 78 & 52 \end{pmatrix}$$

Simultaneous Equations in Matrix Notation We can use matrix multiplication to express a set of equations in matrix notation. To illustrate, let us consider a set of three equations in three unknown variables as given here:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These can be written as *AX* = *B* as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Determinants

We now consider the concept of a determinant that is used in the solution to the systems of linear equations. A determinant is a function that associates a real number with a given square matrix. Let us consider how this number may be calculated.

For a 2×2 matrix, the determinant is given by the product of elements on the principal diagonal *minus* the products of elements on the other diagonal. Accordingly, suppose the following matrix is given.

$$A = \begin{pmatrix} 8 & 4 \\ 5 & 3 \end{pmatrix}$$

We have its determinant, $|A|$, equal to 4 obtained as: $|A| = 8 \times 3 - 5 \times 4 = 4$.

For a square matrix with $n = 3$, we have

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example 3 Consider the following matrix and obtain its determinant.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix}$$

We have,

$$|A| = 3 \times 2 \times 2 - 3 \times 4 \times 3 - 5 \times 1 \times 2 + 5 \times 4 \times 4 + 1 \times 1 \times 3 - 1 \times 2 \times 4$$

= 12 – 36 – 10 + 80 + 3 – 8 = 41

Solution to Simultaneous Equations using Determinants

Solution to a set of simultaneous equations can be obtained with the help of determinants. The method based on determinants is known as Cramer's rule. The steps involved in this method are as follows:

(a) Calculate determinant of matrix *A*, $|A|$.
(b) Replace first column of matrix *A* by elements of vector *B* and call the matrix A_1, and calculate the determinant of A_1, $|A_1|$.
(c) Replace second column of matrix *A* by elements of vector *B* and call the matrix A_2, and calculate the determinant of A_2, $|A_2|$.
(d) In case of a three-variable problem, replace third column of matrix *A* by elements of vector *B* and call the matrix A_3, and calculate the determinant of A_3, $|A_3|$.
(e) Calculate the values of x_1, x_2 etc as:

$$x_1 = \frac{|A_1|}{|A|}, \qquad x_2 = \frac{|A_2|}{|A|} \ and \ x_3 = \frac{|A_3|}{|A|}$$

Example 4 Solve the following equations for x_1 and x_2:

$$9x_1 + 5x_2 = 43$$
$$7x_1 + 3x_2 = 29$$

In matrix notation,

$$\begin{pmatrix} 9 & 5 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 43 \\ 29 \end{pmatrix}$$

Here,

$$|A| = 9 \times 3 - 7 \times 5 = -8$$
$$A_1 = \begin{pmatrix} 43 & 5 \\ 29 & 3 \end{pmatrix}, \quad |A_1| = 43 \times 3 - 29 \times 5 = -16$$
$$A_2 = \begin{pmatrix} 9 & 43 \\ 7 & 29 \end{pmatrix}, \quad |A_2| = 9 \times 29 - 7 \times 43 = -40$$

Accordingly,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-16}{-8} = 2, \qquad and \quad x_2 = \frac{|A_2|}{|A|} = \frac{-40}{-8} = 5$$

Example 5 Using Cramer's rule, solve for the unknowns in the system of linear equations given below:

$$3x_1 + 5x_2 + x_3 = 46$$
$$x_1 + 2x_2 + 4x_3 = 36$$
$$4x_1 + 3x_2 + 2x_3 = 39$$

Writing the given equations in the form $AX = B$ we have:

$$\begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 46 \\ 36 \\ 39 \end{bmatrix}$$

Here we consider matrix A and matrices A_1, A_2, and A_3 created by replacing respectively the first, second and third columns of A by vector B. Also, we calculate their determinants.

$$A = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{pmatrix}$$
$$|A| = 3 \times 2 \times 2 - 3 \times 4 \times 3 - 5 \times 1 \times 2 + 5 \times 4 \times 4 + 1 \times 1 \times 3 - 1 \times 2 \times 4$$
$= 12 – 36 – 10 + 80 + 3 – 8 = 41$

$$A_1 = \begin{pmatrix} 46 & 5 & 1 \\ 36 & 2 & 4 \\ 39 & 3 & 2 \end{pmatrix}$$
$$|A_1| = 46 \times 2 \times 2 - 46 \times 4 \times 3 - 5 \times 36 \times 2 + 5 \times 4 \times 39 + 1 \times 36 \times 3 - 1 \times 2 \times 39$$
$= 184 – 552 – 360 + 780 + 108 – 78 = 82$

$$A_2 = \begin{pmatrix} 3 & 46 & 1 \\ 1 & 36 & 4 \\ 4 & 39 & 2 \end{pmatrix}$$

$|A_2| = 3 \times 36 \times 2 - 3 \times 4 \times 39 - 46 \times 1 \times 2 + 46 \times 4 \times 4 + 1 \times 1 \times 39 - 1 \times 36 \times 4$

= 216 – 468 – 92 + 736 + 39 – 144 = 287

$$A_3 = \begin{pmatrix} 3 & 5 & 46 \\ 1 & 2 & 36 \\ 4 & 3 & 39 \end{pmatrix}$$

$|A_3| = 3 \times 2 \times 39 - 3 \times 36 \times 3 - 5 \times 1 \times 39 + 5 \times 36 \times 4 + 46 \times 1 \times 3 - 46 \times 2 \times 4$

= 234 – 324 – 195 + 720 + 138 – 368 = 205

From these values,

$$x_1 = \frac{|A_1|}{|A|} = \frac{82}{41} = 2, \quad x_2 = \frac{|A_2|}{|A|} = \frac{287}{41} = 7 \quad and \quad x_3 = \frac{|A_3|}{|A|} = \frac{205}{41} = 5$$