APPENDIX 3A: Duration and Immunization

In the body of the chapter, you learned how to calculate duration and came to understand that the duration measure has economic meaning because it indicates the price sensitivity or elasticity of an asset or liability's value to small changes in interest rates. For FIs, the major relevance of duration is as a measure for managing interest rate risk exposure. Also important is duration's role in allowing an FI to hedge or immunize its balance sheet or some subset on that balance sheet against interest rate risk. The following section considers an example of an FI's use of the duration measure for immunization purposes. That is, we look at how duration can be used by insurance company and pension fund managers to help meet promised cash flow payments to policyholders or beneficiaries at a particular time in the future.

Duration and Immunizing Future Payments

Frequently, pension fund and life insurance company managers face the problem of structuring their asset investments so they can pay a given cash amount to policyholders in some future period. The classic example of this is an insurance policy that pays the holder some lump sum when the holder reaches retirement age. The risk to the life insurance company manager is that interest rates on the funds generated from investing the holder's premiums could fall. Thus, the accumulated returns on the premiums invested may not meet the target or promised amount. In effect, the insurance company would be forced to draw down its reserves and net worth to meet its payout commitments. (See Chapter 15 for a discussion of this risk.)

Suppose that it is 2015 and the insurer must make a guaranteed payment to an investor in five years, 2020. For simplicity, we assume that this target guaranteed payment is \$1,469, a lump-sum policy payout on retirement, equivalent to investing \$1,000 at an annually compounded rate of 8 percent over five years. Of course, realistically, this payment would be much larger, but the underlying principles of the example do not change by scaling up or down the payout amount.

To immunize or protect itself against interest rate risk, the insurer needs to determine which investments would produce a cash flow of exactly \$1,469 in five years, regardless of what happens to interest rates in the immediate future. By investing either in a five-year maturity and duration zero-coupon bond or a coupon bond with a five-year duration, the FI would produce a \$1,469 cash flow in five years, no matter what happens to interest rates in the immediate future. Next we consider the two strategies: buying five-year zero-coupon bonds and buying five-year duration coupon bonds.

Buy Five-Year Zero-Coupon Bonds. Given a \$1,000 face value and an 8 percent yield and assuming annual compounding, the current price per five-year zero-coupon bond is \$680.58 per bond

$$P = 680.58 = \frac{1,000}{(1.08)^5}$$

If the insurer buys 1.469 of these bonds at a total cost of \$1,000 in 2015, these investments would produce \$1,469 on maturity in five years. The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability to its policyholders. Intuitively, since the issuer of the zero-coupon bonds pays no intervening cash flows or coupons, future changes in interest rates have no reinvestment income effect. Thus, the return would be unaffected by intervening interest rate changes.

Buy Five-Year Duration Coupon Bonds. Suppose that no five-year zero-coupon bonds exist. In this case, the portfolio manager may seek to invest in appropriate duration coupon bonds to hedge interest rate risk. In this example, the appropriate investment is in five-year

TABLE 3–13 The Duration of a Six-Year Bond with 8 Percent Coupon Paid Annually and an 8 Percent Rate of Return

t	CF_t	$\frac{1}{(1+8\%)^t}$	$\frac{CF_t}{(1+8\%)^t}$	$\frac{CF_t * t}{(1+8\%)^t}$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1,080	0.6302	680.58	4,083.48
			1,000.00	4,992.71

$D = 4{,}992.71/1{,}000.00 = 4.993$ years

duration coupon bonds. Consider a six-year maturity bond with an 8 percent coupon paid annually, an 8 percent rate of return and $1,000 face value. The duration of this six-year maturity bond is computed as 4.993 years, or approximately 5 years (see Table 3–13). By buying this six-year maturity, five-year duration bond in 2015 and holding it for five years until 2020, the term exactly matches the insurer's target horizon. We show in the next set of examples that the cash flow generated at the end of five years is $1,469 whether the rate of return stays at 8 percent or instantaneously (immediately) rises to 9 percent or falls to 7 percent. Thus, buying a coupon bond whose duration exactly matches the investment time horizon of the insurer also immunizes the insurer against interest rate changes.

EXAMPLE 3–16 Rate of Return Remains at 8 Percent

Assuming the rate of return on the bond stays at 8 percent throughout the five years, the cash flows received by the insurer on the bond are as follows:

1. Coupons, 5 × $80	$ 400
2. Reinvestment income	69
3. Proceeds from sale of bond at end of the fifth year	1,000
	$1,469

We calculate each of the three components of the insurer's income from the bond investment as follows:

1. *Coupons.* The $400 from coupons is simply the annual coupon of $80 received in each of the five years.

2. *Reinvestment income.* Because the coupons are received annually, they can be reinvested at 8 percent as they are received, generating an additional cash flow of $69. To understand this, consider the coupon payments as an annuity stream of $80 invested at 8 percent at the end of each year for five years. The future value of the annuity stream is calculated as $\$80\ \{[(1+0.08)^5 - 1]/0.08]\} = 80(5.867) = \$469$. Subtracting the $400 of invested coupon payments leaves $69 of reinvestment income.

3. *Bond sale proceeds.* The proceeds from the sale are calculated by recognizing that the six-year bond has just one year left to maturity when the insurance company sells it at the end of the fifth year (i.e., year 2020). That is:

↓ **Sell**	**$1,080**
Year 5 (2020)	Year 6 (2021)

What price can the insurer expect to receive upon selling the bond at the end of the fifth year with one year left to maturity? A buyer would be willing to pay the present value of the $1,080—final coupon plus face value—to be received at the end of the one remaining year, or:

$$P_5 = \frac{1{,}080}{1.08} = \$1{,}000$$

Thus, the insurer would be able to sell the one remaining cash flow of $1,080, to be received in the bond's final year, for $1,000.

Next we show that since this bond has a duration of five years, matching the insurer's target period, even if the rate of return were to instantaneously fall to 7 percent or rise to 9 percent, the expected cash flows from the bond still would sum exactly to $1,469. That is, the coupons plus reinvestment income plus principal received at the end of the fifth year would be immunized. In other words, the cash flows on the bond are protected against interest rate changes.

EXAMPLE 3–17 Rate of Return Falls to 7 Percent

In this example, if the rate of return on the bond falls, the cash flows over the five years are as follows:

1. Coupons, 5 × $80	$ 400
2. Reinvestment income	60
3. Bond sale proceeds	1,009
	$1,469

Thus, the amount of the total proceeds over the five years is unchanged from proceeds generated when the rate of return was 8 percent. To see why this occurs, consider what happens to the three parts of the cash flow when rates fall to 7 percent:

1. *Coupons.* These are unchanged, since the insurer still receives five annual coupons of $80 ($400).
2. *Reinvestment income.* The coupons can now be reinvested only at the lower rate of 7 percent. Thus, at the end of five years $\$80\,\{[(1.07)^5 - 1]/0.07]\} = 80(5.751) = \$460$. Subtracting the $400 in original coupon payments leaves $60. Because interest rates have fallen, the investor has $9 less in reinvestment income at the end of the five-year planning horizon.
3. *Bond sale proceeds.* When the six-year maturity bond is sold at the end of the fifth year with one cash flow of $1,080 remaining, investors would be willing to pay more:

$$P_5 = \frac{1{,}080}{1.07} = \$1{,}009$$

That is, the bond can be sold for $9 more than when rates were 8 percent. The reason is that investors can get only 7 percent on newly issued bonds, but this older bond was issued with a higher coupon of 8 percent.

A comparison of reinvestment income with bond sale proceeds indicates that the decrease in rates has produced a *gain* of $9 on the bond sale proceeds. This offsets the loss of reinvestment income of $9 as a result of reinvesting at a lower interest rate. Thus, total cash flows remain unchanged at $1,469.

EXAMPLE 3–18 **Rate of Return Rises to 9 Percent**

In this example, if the rate of return on the bond rises, the proceeds from the bond investment are as follows:

1. Coupons, $5 \times \$80$	\$ 400
2. Reinvestment income 80 ($\{[(1.09)^5 - 1]/0.09\} - 400$)	78
3. Bond sale proceeds (1,080/1.09)	991
	\$1,469

Notice that the rise in the rate of return on the bond from 8 to 9 percent leaves the final terminal cash flow unaffected at \$1,469. The rise in rates has generated \$9 extra reinvestment income (\$78 − \$69), but the price at which the bond can be sold at the end of the fifth year has declined from \$1,000 to \$991, equal to a capital loss of \$9. Thus, the gain in reinvestment income is exactly offset by the capital loss on the sale of the bond.

The preceding examples demonstrate that matching the duration of a coupon bond to the FI's target or investment horizon *immunizes* it against instantaneous shocks to interest rates. The gains or losses on reinvestment income that result from a change in the rate of return on the bond are exactly offset by losses or gains from the bond proceeds on sale.

APPENDIX 3B: More on Convexity

In the main text of this chapter, we explained why convexity is a desirable feature for assets. In this appendix we then ask: Can we measure convexity? And can we incorporate this measurement in the duration model to adjust for or offset the error in prediction due to its presence? The answer to both questions is yes.

Theoretically speaking, duration is the slope of the price–interest rate curve, and convexity, or curvature, is the change in the slope of the price–interest rate curve. Consider the total effect of a change in interest rates on a bond's price as being broken into a number of separate effects. The precise mathematical derivation of these separate effects is based on a Taylor series expansion that you might remember from your math classes. Essentially, the first-order effect (dP/dr) of an interest rate change on the bond's price is the price–interest rate curve slope effect, which is measured by duration. The second-order effect (dP^2/d^2r) measures the change in the slope of the price–interest rate curve. This is the curvature or convexity effect. There are also third-, fourth-, and higher-order effects from the Taylor series expansion, but for all practical purposes these effects can be ignored.

We have noted that overlooking the curvature of the price–interest rate curve may cause errors in predicting the price sensitivity of a portfolio of assets and liabilities to changes in interest rates, especially when interest rates change by large amounts. We can adjust for this by explicitly recognizing the second-order effect of interest rate changes by measuring the change in the slope of the price–interest rate curve around a given point. Just as D (duration) measures the slope effect (dP/dr), we introduce a new parameter (CX) to measure the curvature effect (dP^2/d^2r) of the price–interest rate curve.

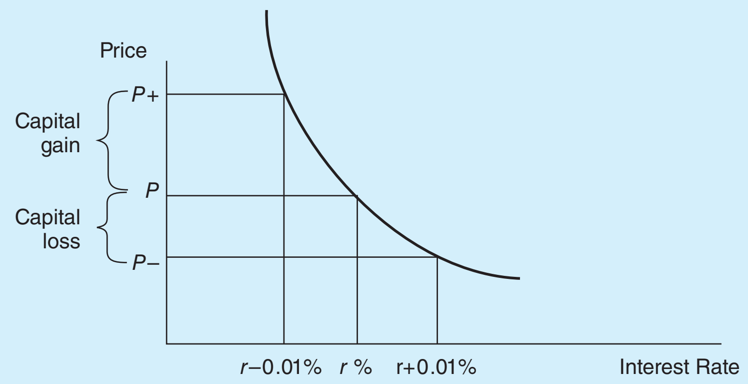
The resulting equation, predicting the change in a security's price ($\Delta P/P$), is:

$$\frac{\Delta P}{P} = -D\frac{\Delta r}{(1 + r)} + \frac{1}{2}CX(\Delta r)^2 \quad \textbf{(1)}$$

or:

$$\frac{\Delta P}{P} = -MD\Delta r + \frac{1}{2}CX(\Delta r)^2 \quad \textbf{(2)}$$

Figure 3–10 **Convexity and the Price–Interest Rate Curve**



The first term in Equation 1 is the simple duration model that over- or underpredicts price changes for large changes in interest rates, and the second term is the second-order effect of interest rate changes, that is, the convexity or curvature adjustment. In Equation 1, the first term D can be divided by $1 + r$ to produce what we called earlier *modified duration (MD)*. You can see this in Equation 2. In the convexity term, the numbers 1/2 and $(\Delta r)^2$ result from the fact that the convexity effect is the second-order effect of interest rate changes, while duration is the first-order effect. The parameter CX reflects the degree of curvature in the price–interest rate curve at the current interest rate level, that is, the degree to which the *capital gain effect* exceeds the *capital loss effect* for an equal change in rates up or down. At best, the FI manager can only approximate the curvature effect by using a parametric measure of CX. Even though calculus is based on infinitesimally small changes, in financial markets the smallest change in interest rates normally observed is one basis point, or a 1/100 of 1 percent change. One possible way to measure CX is introduced next.

As just discussed, the convexity effect is the degree to which the capital gain effect more than offsets the capital loss effect for an equal increase and decrease in interest rates at the current interest rate level. In Figure 3–10 we depict rates changing upward by one basis point ($r + 0.01\%$) and downward by one basis point ($r - 0.01\%$). Because convexity measures the curvature of the price–interest rate curve around the rate level r percent, it intuitively measures the degree to which the capital gain effect of a small rate decrease exceeds the capital loss effect of a small rate increase. By definition, the CX parameter equals:

$$CX = \begin{matrix}\text{Scaling}\\ \text{factor}\end{matrix}\left[\begin{matrix}\text{The capital loss}\\ \text{from a one-basis-point}\\ \text{rise in rates}\\ \text{(negative effect)}\end{matrix} + \begin{matrix}\text{The capital gain}\\ \text{from a one-basis-point}\\ \text{fall in rates}\\ \text{(positive effect)}\end{matrix}\right]$$

The sum of the two terms in the brackets reflects the degree to which the capital gain effect exceeds the capital loss effect for a small one-basis-point interest rate change down and up. The scaling factor normalizes this measure to account for a larger 1 percent change in rates. Remember, when interest rates change by a large amount, the convexity effect is important to measure. A commonly used scaling factor is 10^8 so that:[17]

$$CX = 10^8\left[\frac{\Delta P-}{P} + \frac{\Delta P+}{P}\right]$$

17. This is consistent with the effect of a 1 percent (100 basis points) change in rates.

Calculation of CX. To calculate the convexity of an 8 percent coupon, 8 percent rate of return six-year maturity Eurobond that has a price of $1,000:[18]

$$CX = 10^8\left[\underbrace{\frac{999.53785 - 1{,}000}{1{,}000}}_{\substack{\text{Capital loss from}\\ \text{a one-basis-point}\\ \text{increase in rates}}} + \underbrace{\frac{1{,}000.46243 - 1{,}000}{1{,}000}}_{\substack{\text{Capital gain from}\\ \text{a one-basis-point}\\ \text{decrease in rates}}}\right]$$

$$CX = 10^8\,[0.00000028]$$
$$CX = 28$$

This value for CX can be inserted into the bond price prediction Equation (2) with the convexity adjustment:

$$\frac{\Delta P}{P} = -MD\Delta r_b + \frac{1}{2}(28)\Delta r_b^2$$

Assuming a 2 percent increase in r_b (from 8 percent to 10 percent):

$$\frac{\Delta P}{P} = -\left[\frac{4.993}{1.08}\right]0.02 + \frac{1}{2}(28)(0.02)^2$$
$$= -0.0925 + 0.0056 = -0.0869 \text{ or } -8.69\%$$

The simple duration model (the first term) predicts that a 2 percent rise in interest rates will cause the bond's price to fall 9.25 percent. However, for large changes in rates, the duration model overpredicts the price fall. The duration model with the second-order convexity adjustment predicts a price fall of 8.69 percent; it adds back 0.56 percent due to the convexity effect. This is much closer to the true fall in the six-year, 8 percent coupon bond's price if we calculate this using time value of money formulas and a 10 percent rate of return to discount the coupon and face value cash flows on the bond. The true value of the bond price fall is 8.71 percent. That is, using the convexity adjustment reduces the error between the predicted value and true value to just a few basis points.[19]

In Table 3–14 we calculate various properties of convexity, where:

T = Time to maturity
r_b = Required rate of return or yield
C = Annual coupon
D = Duration
CX = Convexity

Part 1 of Table 3–14 shows that as the bond's maturity (T) increases, so does its convexity (CX). As a result, long-term bonds have more convexity—which is a desirable property—than do short-term bonds. This property is similar to that possessed by duration.[20] Part 2 of Table 3–14 shows that coupon bonds of the same maturity (T) have less convexity than do zero-coupon bonds. However, for coupon bonds and zero-coupon bonds of the same duration, part 3 of the table shows that the coupon bond has more convexity. We depict the convexity of both in Figure 3–11.

18. You can easily check that $999.53785 is the price of the six-year bond when rates are 8.01 percent and $1,000.46243 is the price of the bond when rates fall to 7.99 percent. Since we are dealing in small numbers and convexity is sensitive to the number of decimal places assumed, use at least five decimal places in calculating the capital gain or loss. In fact, the more decimal places used, the greater the accuracy of the *CX* measure.

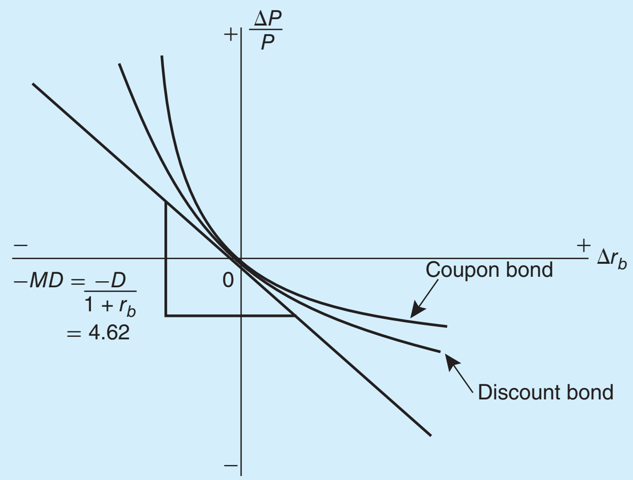
19. It is possible to use the third moment of the Taylor series expansion to reduce this small error (8.71 percent versus 8.69 percent) even further. In practice, few people do this.

20. Note that the *CX* measure differs according to the level of interest rates. For example, we are measuring *CX* in Table 3–14 when rates are 8 percent. If rates were 12 percent, the *CX* number would change. This is intuitively reasonable, as the curvature of the price–interest rate curve differs at each point on the price–interest rate curve. Note that duration also changes with the level of interest rates.

TABLE 3–14 Properties of Convexity

1. Convexity Increases with Bond Maturity			2. Convexity Varies with Coupon		3. For Same Duration, Zero-Coupon Bonds Are Less Convex than Coupon Bonds	
Example			Example		Example	
A	B	C	A	B	A	B
$T = 6$	$T = 18$	$T = \infty$	$T = 6$	$T = 6$	$T = 6$	$T = 5$
$r_b = 8\%$	$r_b = 8\%$	$r_b = 8\%$	$r_b = 8\%$	$r_b = 8\%$	$r_b = 8\%$	$r_b = 8\%$
$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 0\%$	$C = 8\%$	$C = 0\%$
$D = 5$	$D = 10.12$	$D = 13.5$	$D = 5$	$D = 6$	$D = 5$	$D = 5$
$CX = 28$	$CX = 130$	$CX = 312$	$CX = 28$	$CX = 36$	$CX = 28$	$CX = 25.72$

Figure 3–11 Convexity of a Coupon versus a Zero-Coupon Bond with the Same Duration



Finally, before leaving convexity, we might look at one important use of the concept by managers of insurance companies, pension funds, and mutual funds. Remembering that convexity is a desirable form of interest rate risk insurance, FI managers could structure an asset portfolio to maximize its desirable effects. As an example, consider a pension fund manager with a 15-year payout horizon. To immunize the risk of interest rate changes, the manager purchases bonds with a 15-year duration. Consider two alternative strategies to achieve this:

Strategy 1: Invest 100 percent of resources in a 15-year zero-coupon bond with an 8 percent rate of return.

Strategy 2: Invest 50 percent in the very short-term money market (federal funds) and 50 percent in 30-year zero-coupon bonds with an 8 percent rate of return.

The duration (D) and convexities (CX) of these two asset portfolios are:

Strategy 1: $D = 15$, $CX = 206$

Strategy 2:[21] $D = ½(0) + ½(30) = 15$, $CX = ½(0) + ½(797) = 398.5$

21. The duration and convexity of one-day federal funds are approximately zero (see Chapters 4 and 5).

Figure 3–12 Barbell Strategy

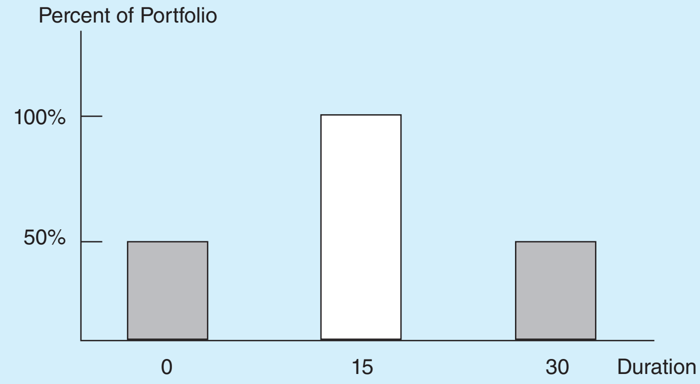
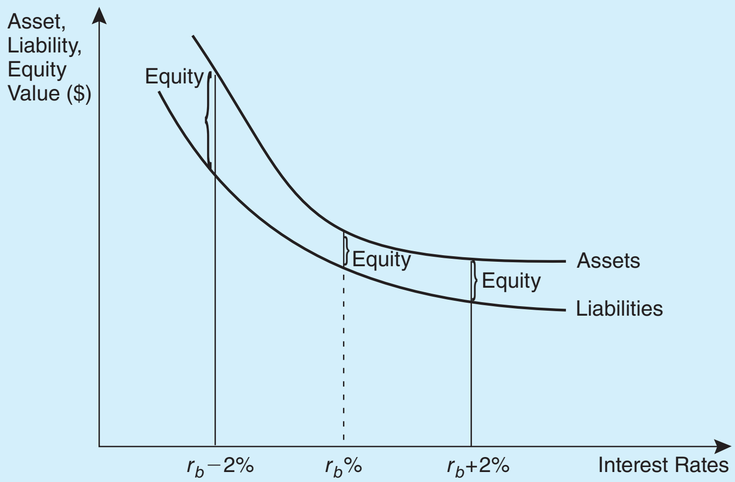


Figure 3–13 Assets Are More Convex than Liabilities



Strategies 1 and 2 have the same durations, but strategy 2 has a greater convexity. Strategy 2 is often called a barbell portfolio, as shown in Figure 3–12 by the shaded bars.[22] Strategy 1 is the unshaded bar. To the extent that the market does not price (or fully price) convexity, the barbell strategy dominates the direct duration matching strategy (strategy 1).[23]

More generally, an FI manager may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as shown in Figure 3–13. As a result, both positive and negative shocks to interest rates would have beneficial effects on the FI's net worth.

22. This is called a barbell because the weights are equally loaded at the extreme ends of the duration range or bar as in weight lifting.

23. In a world in which convexity is priced, the long-term 30-year bond's price would rise to reflect the competition among buyers to include this more convex bond in their barbell asset portfolios. Thus, buying bond insurance—in the form of the barbell portfolio—would involve an additional cost to the FI manager. In addition, to be hedged in both a duration sense and a convexity sense, the manager should not choose the convexity of the asset portfolio without seeking to match it to the convexity of its liability portfolio. For further discussion of the convexity "trap" that results when an FI mismatches its asset and liability convexities, see J. H. Gilkeson and S. D. Smith, "The Convexity Trap: Pitfalls in Financing Mortgage Portfolios and Related Securities," Federal Reserve Bank of Atlanta, *Economic Review,* November-December 1992, pp. 17–27.